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Unit III

5. Z- transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \rightarrow \text{Z-transform}$$

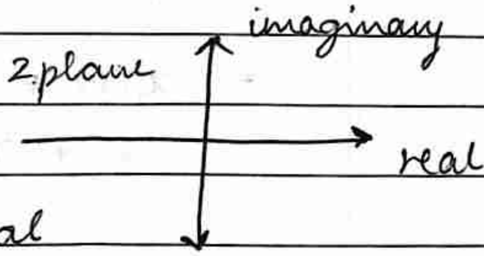
$$X(j\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n} \rightarrow \text{DTFT}$$

$$X(z) = X(e^{j\Omega}) \Big|_{z = e^{j\Omega}}$$

Set of finite values is called region of convergen-ce (ROC)

The counter part for continuous signal - Laplace & for discrete - Z transform

If ROC = 1 → unit circle



Z transform is used to represent discrete time signal into corresponding complex sinusoidal

DTFT is also used for same purpose, but the constraint on existence of FT is no more

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considered in Z-transform. Hence, it gives us the broader spectrum of the system characteris-ation.

Inverse Z transform is represented using partial fraction or power series expansion.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The infinite series of $X(z)$ is not convergent for all values of z. The set of z values, for which series is converging is called ROC (region of convergence)

- If $x_1(n) = [1, 2, 3, 4, 5, 6]$

$x_2(n) = [1, 2, 3, \underset{\uparrow}{4}, 5, 6]$

$$x_1(z) = \sum_{n=0}^{5} x_1(n) z^{-n}.$$

$$= x_1(0) + x_1(1) z^{-1} + x_1(2) z^{-2} + x_1(3) z^{-3} + x_1(4) z^{-4} + x_1(5) z^{-5}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4} + 6z^{-5}$$

$$= 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \frac{5}{z^4} + \frac{6}{z^5}$$

ROC = Entire z plane (for $z \neq 0$)

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$x_2(n) = [1, 2, 3, \underset{\uparrow}{4}, 5, 6]$

$$X_2(z) = \sum_{n=-3}^{2} x_2(n)\, z^{-n}$$

$$= x_2(-3)\, z^3 + x_2(-2)\, z^2 + x_2(-1)\, z^1 + x_2(0) + x_2(1)\, z^{-1} + x_2(2)\, z^{-2}$$

$$= z^3 + 2z^2 + 3z + 4 + \frac{5}{z} + \frac{6}{z^2}$$

ROC = z plane except $z \neq (0 \& \infty)$.

Right sided Sequence

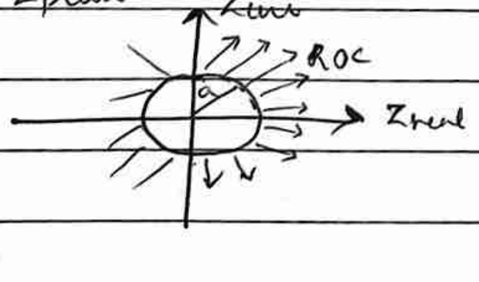
The signal exists for positive values of n.

$$x(n) = \begin{cases} a^n \cancel{x(n)} & , n \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{\infty} a^n \cancel{x(n)}\, z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{1}{1-a/z}$$

zplane



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Left Sided Sequence

$$x(n) = \begin{cases} -a^n u(n-1) & n<0 \\ 0 & \text{otherwise.} \end{cases}$$

Left handed sequence exist for negative values of n

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

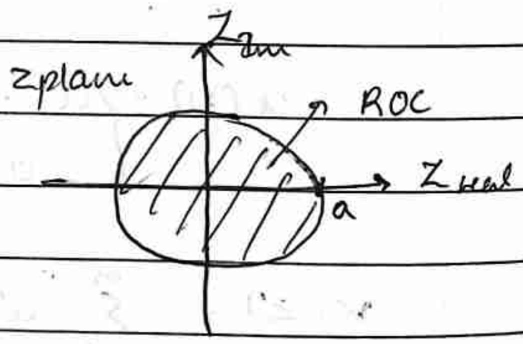
$$= -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=+\infty}^{1} (a^{-1}z)^m$$

$$X(z) = -[(a^{-1}z) + (a^{-1}z)^2 + \dots]$$

$$= -[(a^{-1}z)[1 + a^{-1}z + \dots]$$

$$= \frac{-a^{-1}z}{1-a^{-1}z}$$

$$= \frac{1}{1-a^{-1}z}$$



X(z) will converge

$(a^{-1}z)<1$ i.e $|z|<a$.

- Z transform of unit step $x(n) = u(n) = 1^n$ $(a=1)$

→ Right Sided Sequence.

$$X(z) = \sum_{n=0}^{\infty} 1^n \cdot z^{-n}.$$

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$$= \frac{1}{1-(1/2)},$$

$x(n) = \delta(n)$ (for $n=0$).

$= 1$.

Linearity Properties:

$x_1(n) \longrightarrow X_1(z)$

$x_2(n) \longrightarrow X_2(z)$.

$a x_1(n) + b x_2(n) \longrightarrow a X_1(z) + b X_2(z)$

Time Shifting Property:

$x(n) \longleftrightarrow X(z)$

$x(n-k) \longrightarrow z^{-k} X(z)$.

$y(n) = x(n-k)$

$$Y(z) = \sum_{n=-\infty}^{\infty} x(n-k)\, z^{-n}.$$

$n-k = m$.

$n = m+k$.

$$= \sum_{m=-\infty}^{\infty} x(m)\, z^{-(m+k)}$$

$$= z^{-k} \sum_{m=-\infty}^{\infty} x(m)\, z^{-m}.$$

$$= z^{-k} X(z).$$

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3. Convolution property:

$x_1(n) \rightarrow X_1(z)$

$x_2(n) \rightarrow X_2(z)$

$y(n) = x_1(n) * x_2(n) \longrightarrow X_1(z) * X_2(z).$

$$y(z) = \sum_{n=-\infty}^{\infty} x_1(n) * x_2(n) \; z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} x_1(n) \sum_{l=-\infty}^{\infty} x_2(l-n) \; z^{-n}.$$

$l - n = m.$

$$= \sum_{n=-\infty}^{\infty} x_1(n) \sum_{l=-\infty}^{\infty} x_2(m) \; z^{-(m+l)}$$

$$= \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} \sum_{l=-\infty}^{\infty} x_2(m) \; z^{-m}$$

$$= X_1(z) * X_2(z)$$

15/11/18 Properties of ROC.

i) ROC consist of a ring in z plane centered at origin

ii) ROC doesnot contain any poles.

iii) If $x(n)$ is of finite duration, ROC is entire ~~ROC~~ Z plane except $z=0$ & $z=\infty$.

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iv) If $x(n)$ is right sided sequence ROC is the region in the z-plane outside the outermost pole i.e outside ~~the~~ circle of radius equal to the largest magnitude of pole

v) If $x(n)$ is left sided sequence, ROC is the region in the z plane inside the innermost pole i.e inside the circle of radius equal to smallest magnitude of pole of $X(z)$

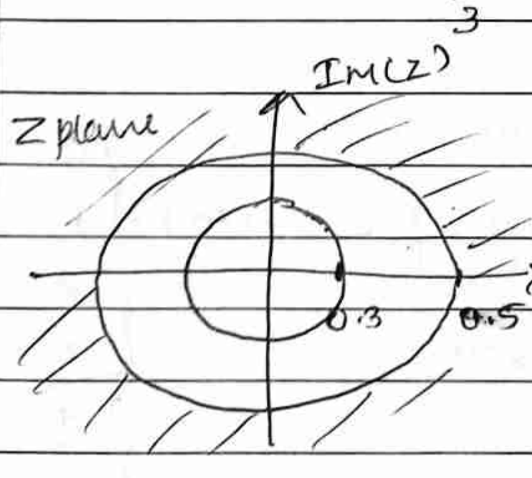
vi) If $x(n)$ is two sided sequence the ROC is concentric circle in a plane.

o $x(n) = 7\left(\frac{1}{3}\right)^n u(n) - 6\left(\frac{1}{2}\right)^n u(n)$.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} 7\left(\frac{1}{3}\right)^n z^{-n} - \sum_{n=0}^{\infty} 6\left(\frac{1}{2}\right)^n z^{-n}$$

$$= 7 \frac{1}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}}$$



~~Z=0~~
$|z| > 1/3$.
$|z| > 1/2$

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Determine ZT & mark ROC of

$$x(n) = -u(-n-1) + \left(\frac{1}{2}\right)^n u(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

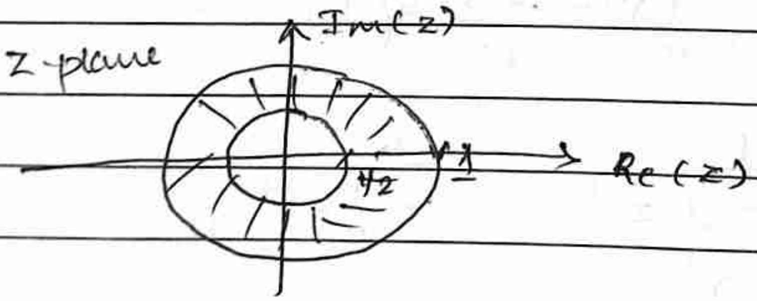
$$= \sum_{n=-\infty}^{\infty} \left[-u(-n-1) + \left(\frac{1}{2}\right)^n u(n) \right]$$

$$= \sum_{n=-\infty}^{-1} (-1) z^{-n} + \sum_{n=0}^{\infty} (1/2)^n z^{-n}$$

$$= (-1) \sum_{n=1}^{\infty} z^{n} + \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

$$= \frac{-z}{1-z} + \frac{1}{1-\frac{1}{2} z^{-1}}$$

$|z| < 1$, $\quad |\frac{1}{2} z^{-1}| < 1 \rightarrow |z| > \frac{1}{2}$.



- Determine the z transform for $x(n) = \begin{cases} 0 & \text{otherwise} \\ 1 & n=-1 \\ 2 & n=0 \\ -1 & n=1 \\ 1 & n=2 \end{cases}$

~~x(n)~~ $X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

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$$= \sum_{n=-1}^{2} x(n) z^{-n}.$$

$$= x(1) z^{-1} + x(2) z^{-2} + x(-1) z + x(0).$$

$$= \frac{-1}{z} + \frac{1}{z^2} + z + 2.$$

ZT exist for all z plane except $z = 0$ & ∞.

- $x(n) = 0 . \quad n < 0$

  $= 1 \quad 0 \le n \le 9$

  $= 0 \quad n > 9.$

- It is unit step sequence ~~upt~~ until 9.

$$x(z) = \sum_{n=0}^{9} x(n) z^{-n}.$$

$$= \frac{1-(z)^{10}}{1-z^{-1}} = \frac{1-(z^{-1})^{10}}{1-z^{-1}} \quad \forall z \text{ except at } 0.$$

- Find z transform for.

$$x(n) = \left[3\left(\frac{4}{5}\right)^n - \left(\frac{2}{3}\right)^{2n}\right] u(n).$$

$$= 3\left[\left(\frac{4}{5}\right)^n - \left(\frac{4}{9}\right)^n\right] u(n).$$

$$x(z) = \sum_{n=0}^{\infty} 3\left(\frac{4}{5}\right)^n z^{-n} - \sum_{n=0}^{\infty} 3\left(\frac{4}{9}\right)^n z^{-n}.$$

$$= \frac{3.}{1-\frac{4}{5}z^{-1}} - \frac{3}{1-\frac{4}{9}z^{-1}}.$$

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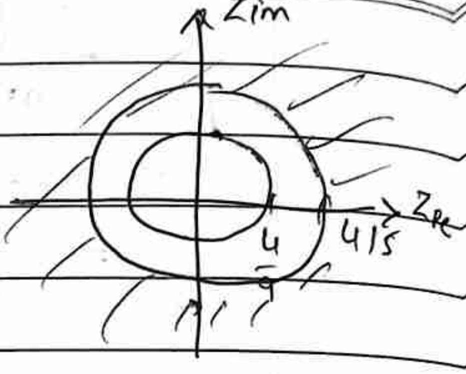
$\left|\frac{4}{5} z^{-1}\right| < 1 \Rightarrow |z| > \frac{4}{5}.$

$\left|\frac{4}{9} z^{-1}\right| < 1 \Rightarrow |z| > \frac{4}{9}.$

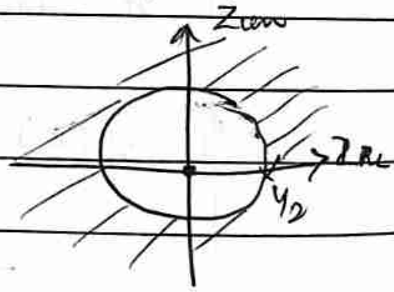


- $x(n) = \left(\frac{1}{2}\right)^n u(n-2).$

$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$

$= \sum_{n=2}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n = \frac{((1/2) z^{-1})^2}{1 - 1/2\, z^{-1}}$

$\left| 1/2\, z^{-1} \right| > 1$ ROC $|z| > 1/2.$



Initial Value Theorem.

For a Causal Sequence

$x(n) \xrightarrow{ZT} X(z)$

$\lim_{z \to \infty} X(z) = x(0).$

WKT, $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

$= x(0) + x(1) z^{-1} + x(2) z^{-2} \ldots$

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for $\lim_{z \to \infty} X(z) = x(0)$.

Inverse Z transform

It is used to transform $X(z)$ to its corresponding time domain signal $x(n)$.

Two methods: i) Partial fraction expansion
ii) Power series expansion.

Partial fraction expansion:

$$X(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{M} b_k z^{-n}}{\sum_{k=0}^{N} a_k z^{-n}}$$

i) If $M < N$, we can do P.F.E directly by factorising the denominator of polynomial of $X(z)$

ii) If $M \geq N$, then by long division method,

$$X(z) = \sum_{k=0}^{M-N} C_k z^{-k} + \frac{\bar{B} z}{A(z)}$$

where numerator $(\bar{B} z)$ having lesser order than $A(z)$

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- Find IZT for $X(z) = \dfrac{-1 + 5z^{-1}}{\left(1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}\right)}$

Here $M=1$, $N=2$ $(M<N)$

$$= \frac{A}{\left(1 - \frac{1}{2}z^{-1}\right)} + \frac{B}{(1 - z^{-1})}$$

$$-1 + 5z^{-1} = A(1 - z^{-1}) + B(1 - \tfrac{1}{2}z^{-1}).$$

$z^{-1} = 1$

$-1+5 = B(1/2)$

$2 \times 4 = B \Rightarrow \underline{B = 8.}$

$z^{-1} = 2.$

$9 = A(1-2)$

$\underline{A = -9.}$

$$= \frac{-9}{1 - 1/2\, z^{-1}} + \frac{8}{1 - z^{-1}}$$

$x(n) = -9 \cdot \left(\frac{1}{2}\right)^n u(n) + 8\, u(n).$

- $X(z) = \dfrac{1/4\, z^{-1}}{(1 - 1/2\, z^{-1})(1 - 1/4\, z^{-1})}$ $|z| > 1/2$ $|z| < 1/4$

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$\Rightarrow \frac{1}{4} z^{-1} = A(1 - \frac{1}{4} z^{-1}) + B(1 - \frac{1}{2} z^{-1})$.

$z^{-1} = 4$.

$1 = A(0) + B(-1)$

$\underline{B = -1}$.

$z^{-1} = 2$.

$\frac{1}{2} = \frac{1}{2} A \Rightarrow \underline{A = 1}$.

$= \frac{1}{1 - \frac{1}{2} z^{-1}} - \frac{1}{1 - \frac{1}{4} z^{-1}}$

for $|z| > \frac{1}{2}$

$x(n) = (\frac{1}{2})^n u(n) + [-(\frac{1}{4})^n u(n)]$

for $|z| < \frac{1}{4}$, $x(n) = -(\frac{1}{2})^n u(-n-1) + (\frac{1}{4})^n u(-n-1)$

for $\frac{1}{4} < |z| < \frac{1}{2}$, $x(n) = -(\frac{1}{2})^n u(-n-1) - (\frac{1}{4})^n u(n)$.

- Find the IZT of foll using p.f.m with $|z| > 2$.

$X(z) = \frac{1 - 2z^{-1} + z^{-2}}{1 - \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2}}$ $(m = n)$

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$$\tfrac{1}{2}z^{-2} - \tfrac{3}{2}z^{-1} + 1 \;\Big)\; z^{-2} - 2z^{-1} + 1 \;\Big(\; 2$$

$$\underline{z^{-2} - 3z^{-1} + 2}$$

$$z^{-1} - 1.$$

$$x(z) = 2 + \frac{z^{-1} - 1}{\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1.}$$

$$= 2 + \frac{z^{-1} - 1}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}$$

$$= 2 + \frac{(-9)}{(1 - \frac{1}{2}z^{-1})} + \frac{8}{(1 - z^{-1})}$$

$$x(n) = 2\delta(n) + (-9)\left(\frac{1}{2}\right)^n u(n) + 8\,u(n).$$

- Find the IZT of $X(z) = \dfrac{z^2 - 3z}{z^2 + \frac{3}{2}z - 1}$ $\quad \frac{1}{2} < |z| < 2$

$$z^2 + \tfrac{3}{2}z - 1 \;\Big)\; z^2 - 3z$$

$$x(z) = \frac{1 - 3z^{-1}}{1 + \frac{3}{2}z^{-1} - z^{-2}}$$

$$1 - 3z^{-1} = \frac{A}{(1 - \frac{1}{2}z^{-1})} + \frac{B}{(1 + 2z^{-1})}$$

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$$1 - 3z^{-1} = A(1+2z^{-1}) + B(1 - \tfrac{1}{2}z^{-1}).$$

$$z^{-1} = -\tfrac{1}{2}.$$

$$+\tfrac{5}{2} = B(1 + \tfrac{1}{4})$$

$$B = 2/\cancel{5}.$$

$$\cancel{A} = 2$$

put $z^{-1} = 2.$

$$-5 = A(1+4)$$

$$A = -1.$$

$$X(z) = \frac{2}{(1+2z^{-1})} + \frac{(-1)}{(1-\tfrac{1}{2}z^{-1})}.$$

$$x(n) \Rightarrow -2(-2)^n u(-n-1) - \left(\tfrac{1}{2}\right)^n u(n).$$

- $$X(z) = \frac{4 - 3z^{-1} + 3z^{-2}}{(1+2z^{-1})(1-3z^{-1})^2} \qquad |z| > 3.$$

$$= \frac{A}{(1+2z^{-1})} + \frac{B}{(1-3z^{-1})} + \frac{C}{(1-3z^{-1})^2}.$$

$$4 - 3z^{-1} + 3z^{-2} = A(1-3z^{-1})^2 + B(1+2z^{-1})(1-3z^{-1}) + C(1+2z^{-1})\cancel{\varepsilon}.$$

$$z^{-1} = +\tfrac{1}{3}.$$

$$\frac{10}{3} = C\left(1 + 2\left(\tfrac{1}{3}\right)\right) \Rightarrow \frac{\cancel{10}^{2}}{\cancel{3}} = C\frac{\cancel{5}}{\cancel{3}} \Rightarrow C = 2$$

$$z^{-1} = -\tfrac{1}{2}. \Rightarrow \frac{\cancel{13}^{25/4}}{\cancel{4}} = \frac{25}{\cancel{4}}A \Rightarrow A = \cancel{13/25}\ 1.$$

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$z^{-1} = 0$, $B = 1$.

$$= \frac{1}{1+2z^{-1}} + \frac{1}{1-3z^{-1}} + \frac{2}{(1-3z^{-1})^2}$$

$$\boxed{\text{Property: } \quad ZT\{n\alpha^n\} = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}}$$

for 3rd term, divide & multiply by 3z

$$(-2)^n u(n) + (3)^n u(n) + 2\,\frac{3z^{-1}z}{3(1-3z^{-1})^2}$$

$$= (-2)^n u(n) + (3)^n u(n) + \frac{2}{3}(n+1)\,3^{n+1}\,u(n+1)$$

2] Power Series Expansion

- Find Inverse of ZT for $x(z) = \frac{1}{1+\frac{1}{2}z^{-1}}$ $|z| > \frac{1}{2}$

$x(z) = \frac{1}{1+\frac{1}{2}z^{-1}}$ $|z| < \frac{1}{2}$. using P.S.E.M.

Since ROC is outside the circle radius ½, its corresponding time domain signal x(n) would be right sided seq. write the divisor & dividend for division method such that

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quotient is of neg power of z.

$$1+\tfrac{1}{2}z^{-1} \overline{)\;1\;} \quad 1-\tfrac{1}{2}z^{-1}+\tfrac{1}{4}z^{-2}-\tfrac{1}{8}z^{-3}$$

$$\underline{1+\tfrac{1}{2}z^{-1}}$$

$$-\tfrac{1}{2}z^{-1}$$

$$\underline{\overset{(+)}{-}\tfrac{1}{2}z^{-1} \overset{(+)}{-} \tfrac{1}{4}z^{-2}}$$

$$\tfrac{1}{4}z^{-2}$$

$$\underline{\tfrac{1}{4}z^{-2}-\tfrac{1}{8}z^{-3}}$$

$$x(n) = \delta(n) - \frac{1}{2}\delta(n-1) + \frac{1}{4}\delta(n-2) - \frac{1}{8}\delta(n-3)$$

$$x(n) = [1, -\tfrac{1}{2}, \tfrac{1}{4}, -\tfrac{1}{8} \ldots]$$

$$x(n) = (-\tfrac{1}{2})^n u(n)$$

For next part, $\frac{1}{2}z^{-1}+1 \overline{)\;1\;} \quad 2z-4z^2+8z^3$

$$\underline{(-)\,1+2z}$$

$$-2z$$

$$\underline{(+)\,-2z-4z^2}$$

$$\underline{(+)\,4z^2+8z^3}$$

$$x(n) = 2\delta(n+1) - 4\delta(n+2) + 8\delta(n+3)$$

$$= -(\tfrac{1}{2})^n u(-n-1)$$