Chapter 2

Time Domain Representations for LTI Systems

2.1 INTRODUCTION

In chapter 1, we discussed a number of basic system properties. The systems which satisfy linearity and time invariance are called *linear-time invariant* (LTI) systems. In this chapter, we will describe the different time-domain representations for LTI systems i.e., the representations that relate the output signal to the input signal when both the signals are represented as a function of time.

One of the primary reasons for LTI systems are easy to analyse is that they exhibit superposition property. Therefore if we can represent the input to an LTI systems in terms of a linear combination of a set of basic signals, then we can find the output of that system in terms of the responses to these basic signals.

There are different methods for representing an LTI system in time domain. Few of them are,

(i) In terms of impulse response.
(ii) Difference/differential equations.
(iii) Block diagram representation. etc.

2.2 IMPULSE RESPONSE REPRESENTATIONS FOR LTI SYSTEMS

A complete characterization of any LTI system can be represented in terms of its response to an unit impulse, which is known as *impulse response* of the system. Alternatively, the impulse response is the output of a LTI system due to an impulse input applied at $t=0$ or $n=0$.

If the input to a linear system is expressed as a weighted superposition of time-shifted impulses, then the output is a weighted superposition of the system responses to each time-shifted impulse. If the system is also time-invariant, then the system response to a time-shifted impulse is a time-shifted version of the system response to an impulse. Hence the output of a LTI system is given by a weighted superposition of time-shifted impulse responses. This weighted superposition is known as the *convolution sum* for discrete-time systems and the *convolution integral* for continuous-time systems.

2.2.1 The representation for discrete-time LTI systems in terms of Impulse Response

In this section, initially we will discuss to construct any discrete-time signal $x(n)$ in terms of discrete-time shifted unit impulses.

Consider a discrete-time signal $x(n)$ as shown in Fig. 2.1. For simplicity, we have taken 5 non-zero samples only.

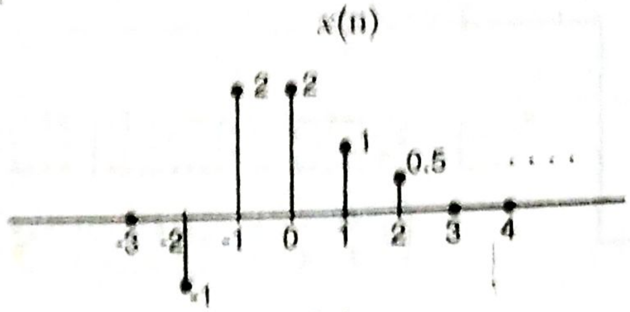
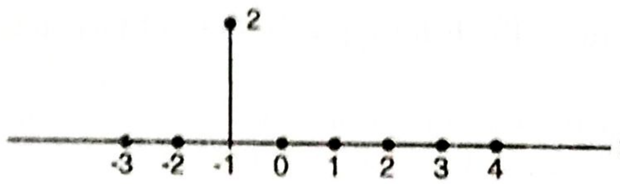


Fig. 2.1

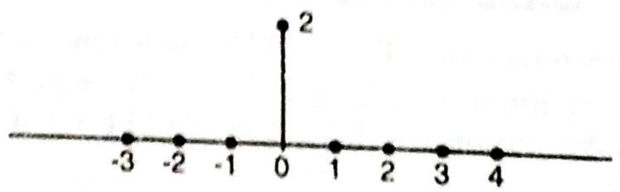
This signal $x(n)$ shown in Fig 2.1 can be splitted as shown below in Fig. 2.2.

$$x_1(n) = x(-2)\,\delta(n+2)$$

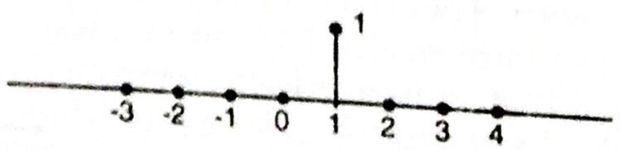
$$x_2(n) = x(-1)\,\delta(n+1)$$



$$x_3(n) = x(0)\,\delta(n)$$



$$x_4(n) = x(1)\,\delta(n-1)$$



$$x_5(n) = x(2)\,\delta(n-2)$$

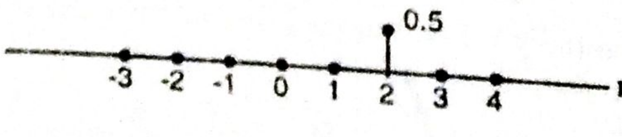


Fig. 2.2

By observing Fig 2.1 and Fig. 2.2 we get,

$$x(n) = x(-2)\,\delta(n+2) + x(-1)\,\delta(n+1) + x(0)\,\delta(n) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2)$$

$$\therefore x(n) = \sum_{k=-2}^{2} x(k)\,\delta(n-k)$$

But in general, any discrete-time signal $x(n)$ can be expressed as,

$$\therefore x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \qquad \ldots\ldots \quad (2.1)$$

In eqn. 2.1, we expressed $x(n)$ as the weighted sum of time-shifted impulses. Now consider a discrete-time LTI system as shown in Fig. 2.3.

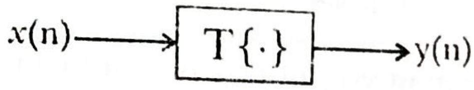


Fig. 2.3

where $x(n)$ is input to the system.
$y(n)$ is output of the system.
$T\{\cdot\}$ is system operator.

$\therefore$ We have,

$$y(n) = T\{x(n)\} \qquad \ldots\ldots \quad (2.2)$$

Substituting eqn. 2.1 in eqn. 2.2 we get,

$$\therefore y(n) = T\left\{\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)\right\}$$

Using linearity property we get,

$$\therefore \quad y(n) = \sum_{k=-\infty}^{\infty} x(k)\,T\{\delta(n-k)\} \qquad \ldots\ldots \quad (2.3)$$

In eqn. 2.3 $T\{\delta(n-k)\}$ corresponds to the operation of the system performed on time-shifted impulse $\delta(n-k)$.

$$\therefore T\{\delta(n-k)\} = h(n-k) \qquad \ldots\ldots \quad (2.4)$$

where $h(n)$ is the *impulse response* of the system.

Substituting eqn. 2.4 in eqn. 2.3 we get,

$$\therefore y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \qquad \ldots\ldots \quad (2.5)$$

$$\therefore \quad \boxed{y(n) = x(n) * h(n)} \qquad \ldots\ldots \quad (2.6)$$

From eqn. 2.5, we can say that the output of an LTI system is given by a weighted sum of time-shifted impulse responses. The eqn. 2.5 is known as *convolution sum.*

This concept is better understood by the following example. Consider an LTI system having impulse response h(n) as shown in Fig. 2.4.

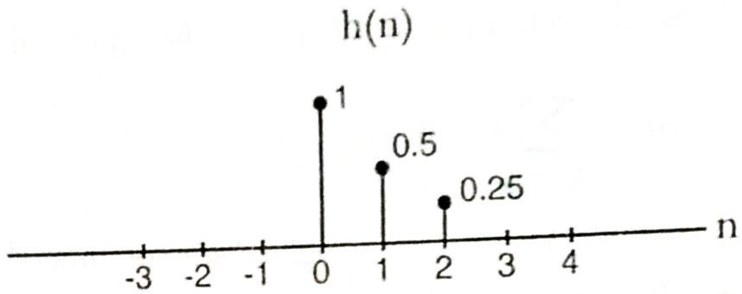


Fig. 2.4

Now say, we want to find the output y(n) of this system to an input $x(n)$ shown in Fig. 2.5.

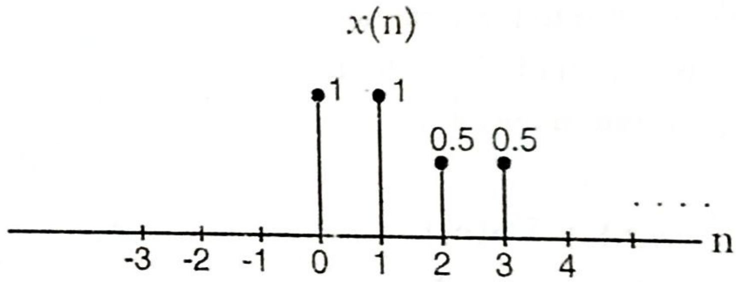
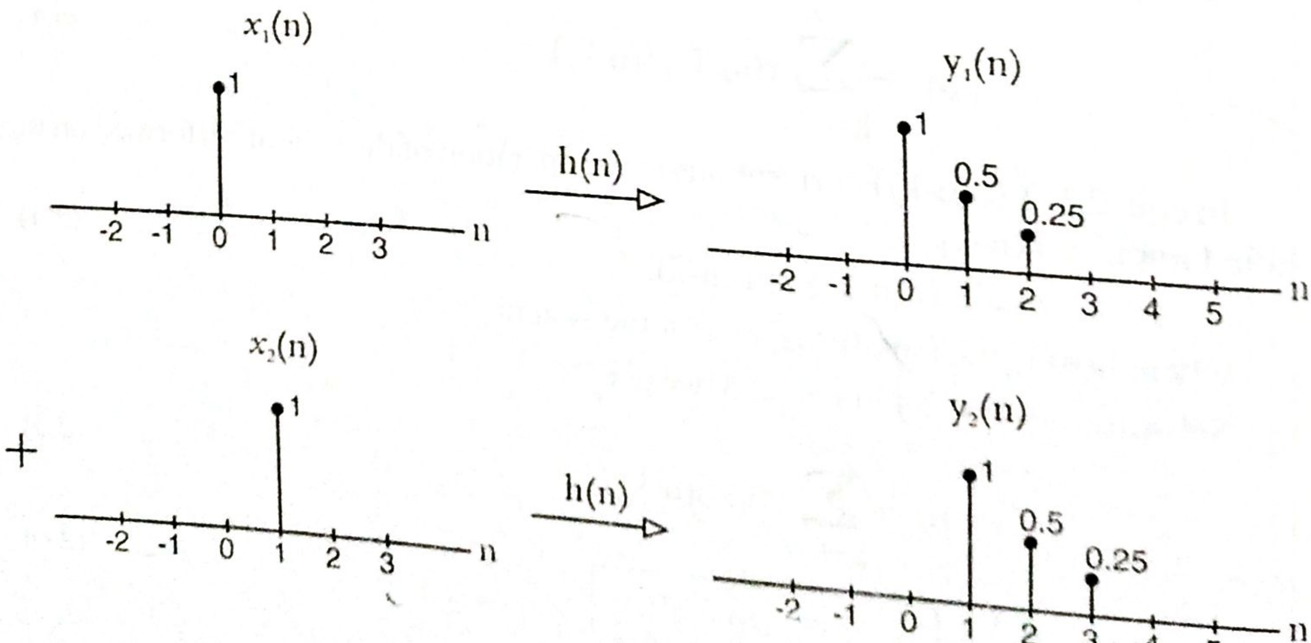


Fig. 2.5

The output y(n) is obtained by adding the responses due to each individual sample of the input as shown in Fig. 2.6.



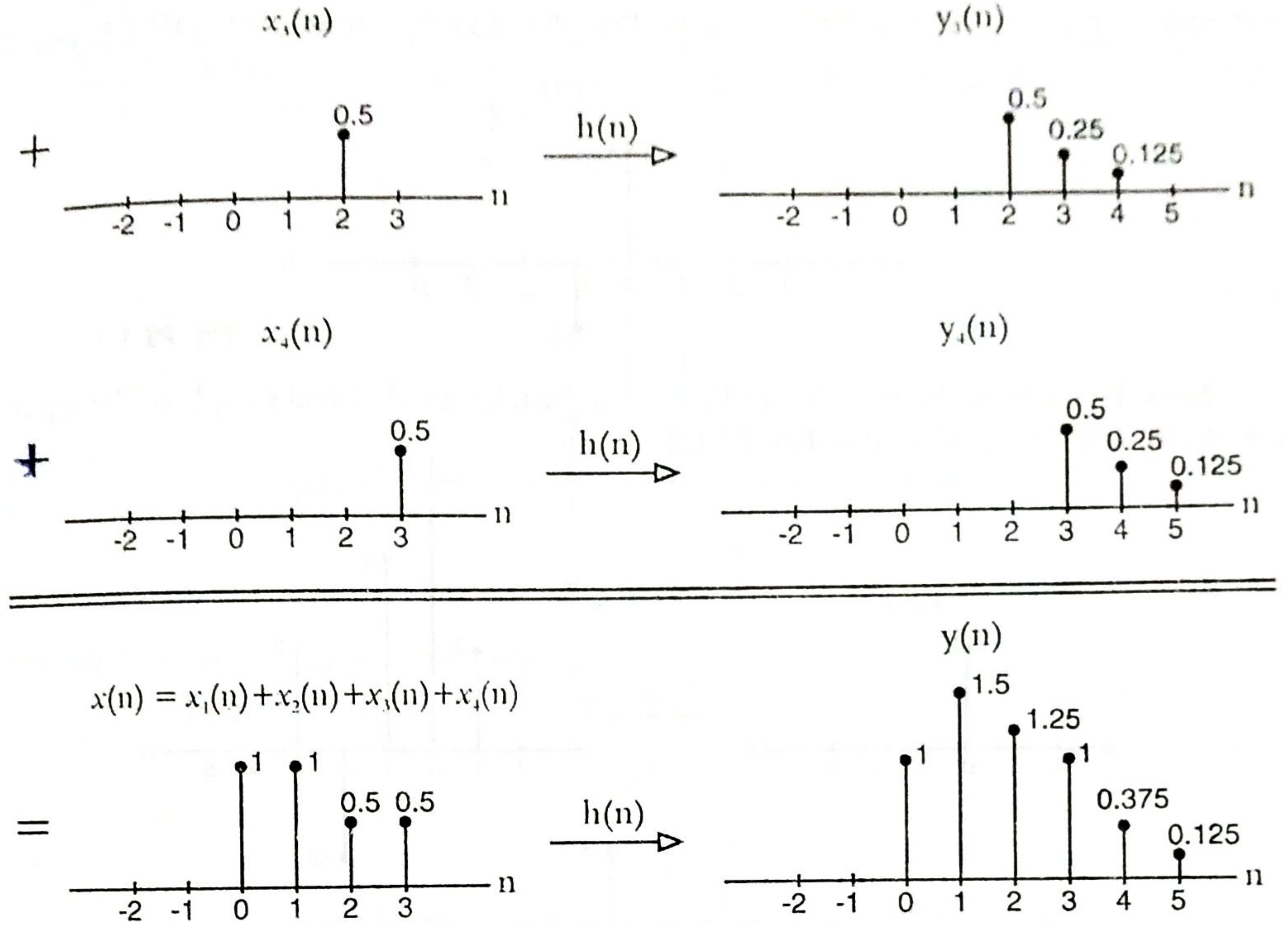


Fig. 2.6

But the method illustrated in Fig. 2.6 is advisable only if the number of samples in both the input $x(n)$ and the impulse response h(n) are finite and less.

Examples

Example 2.1 A discrete-time LTI system has impulse response h(n) as shown in Fig. P2.1. Using linearity and time invariance property determine the system output y(n) if the input $x(n)$ is given by,

$$x(n) = 2\delta(n) - \delta(n-1)$$

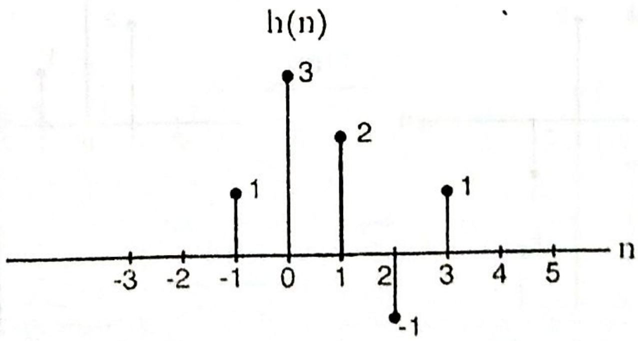


Fig P2.1

Solution : Given : $x(n) = 2\delta(n) - \delta(n-1)$. The plot of $x(n)$ is shown in Fig P2.1.1

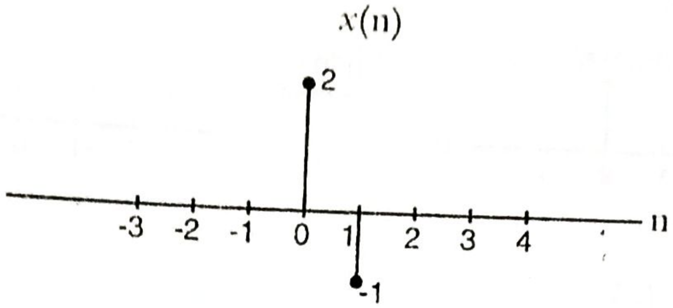


Fig. P2.1.1

Now, let us draw the response of the system by taking individual sample of the input signal $x(n)$ at a time as shown in Fig. P2.1.2.

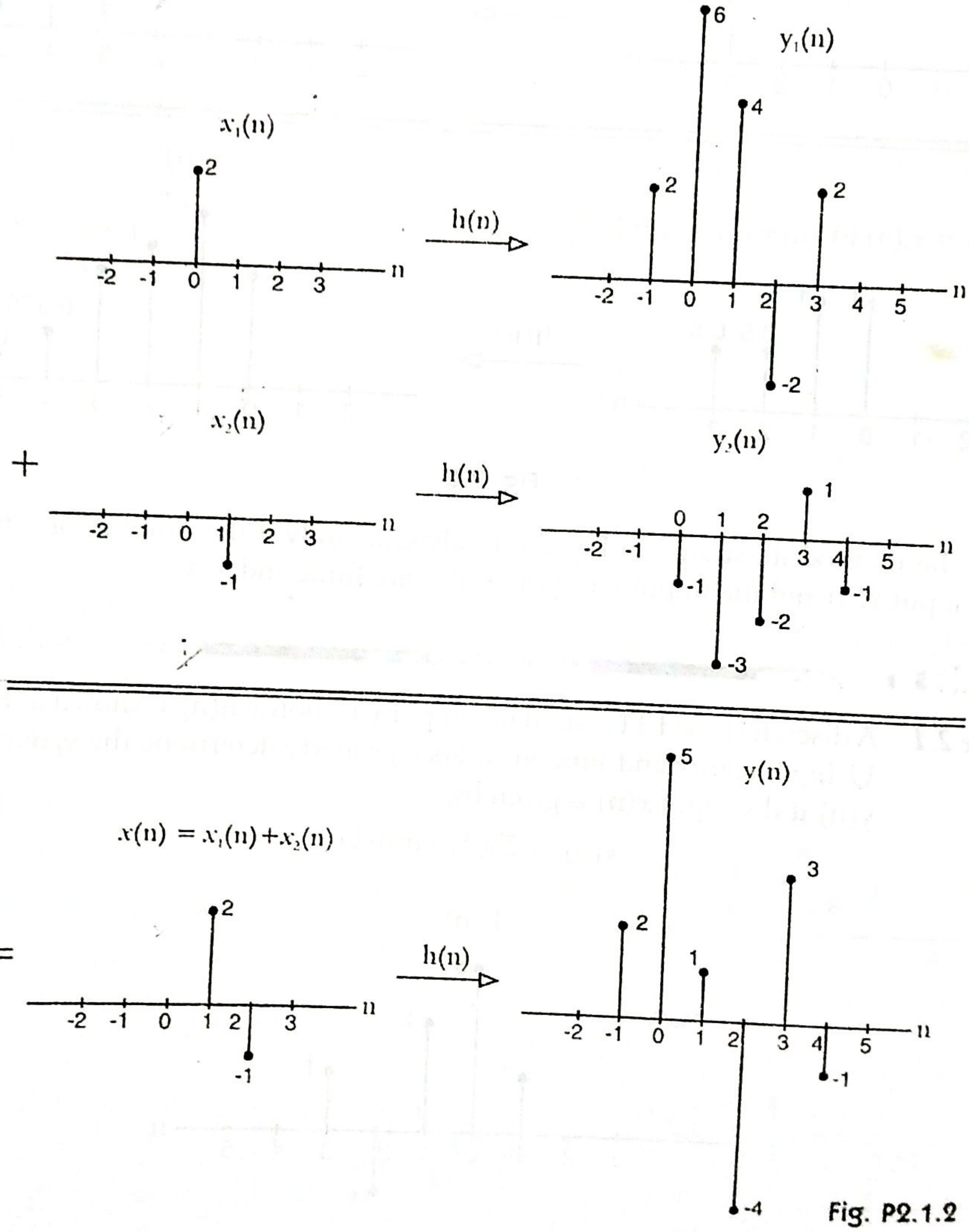


Fig. P2.1.2

$$\therefore\ y(n) = 2\delta(n+1) + 5\delta(n) + \delta(n-1) - 4\delta(n-2) + 3\delta(n-3) - \delta(n-4)$$

Example 2.2 A discrete-time LTI system has impulse response h(n) as shown in Fig P 2.2. Using linearity and time invariance property, determine the system output y(n) if the input $x(n)$ is,

$$x(n) = u(n) - u(n-3)$$

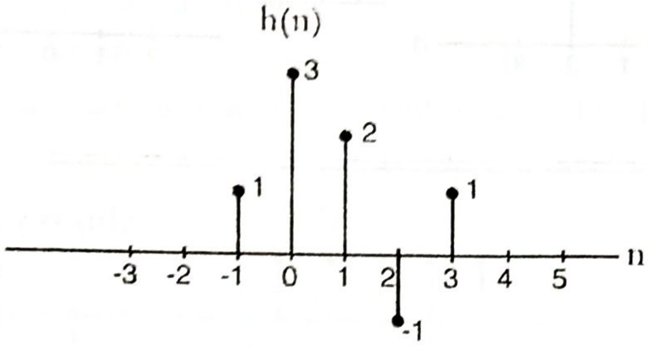


Fig. P2.2

Solution : Given : $x(n) = u(n) - u(n-3)$

The plot of $x(n)$ is shown in Fig. P2.2.1.

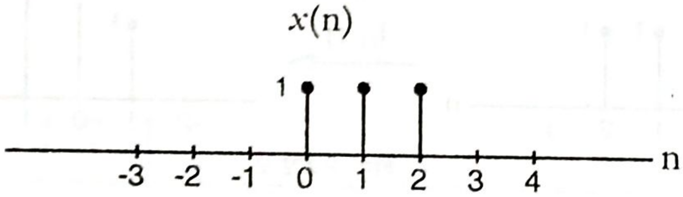
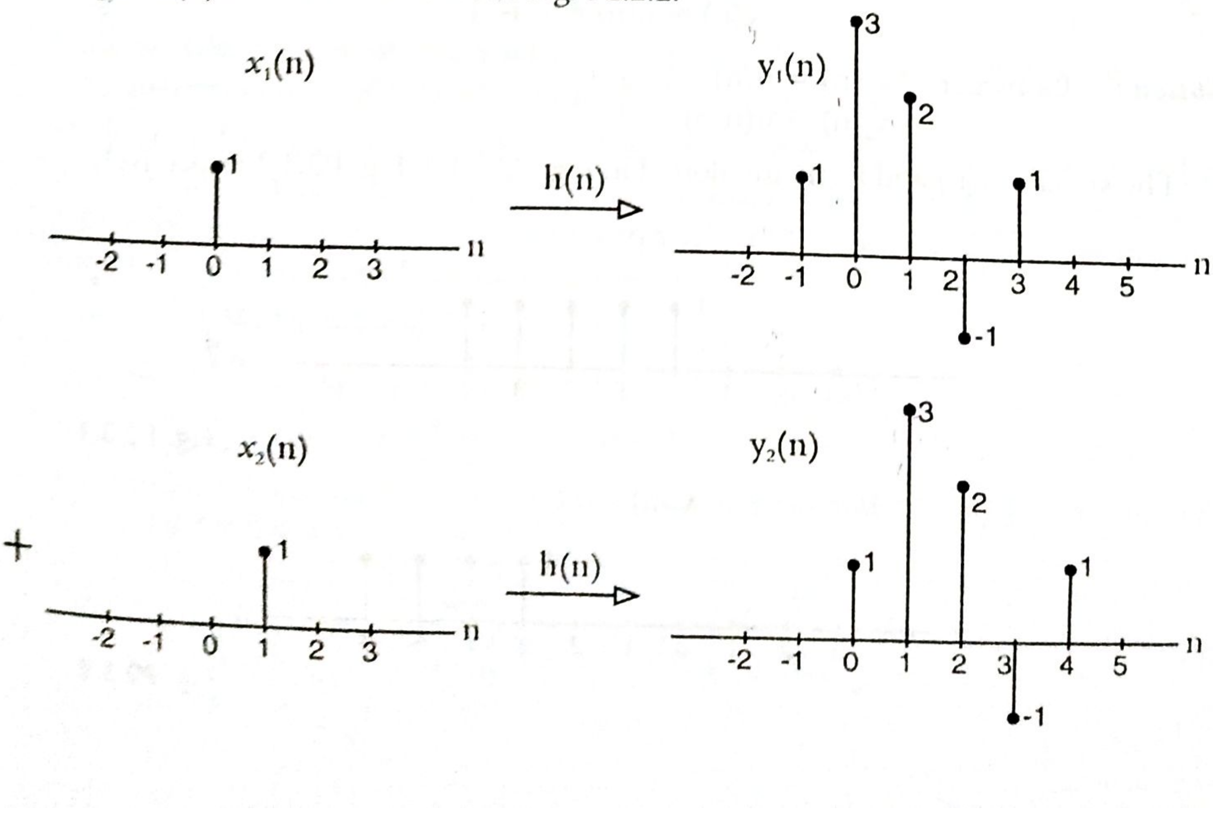


Fig. P2.2.1

Now, let us draw the response of the system by taking individual sample of the input signal $x(n)$ at a time as shown in Fig. P2.2.2.



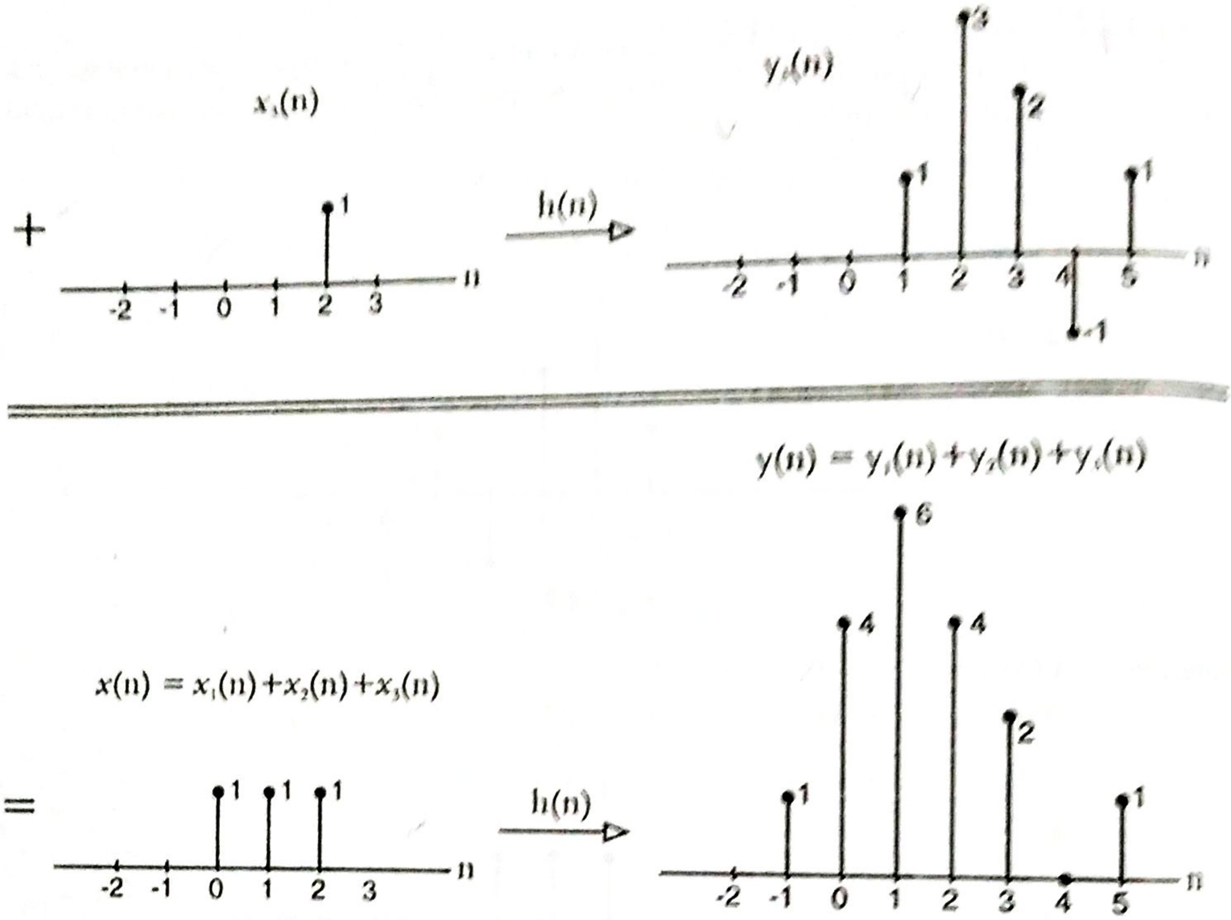


Fig. P2.2.2

$$\therefore\ y(n) = \delta(n+1) + 4\delta(n) + 6\delta(n-1) + 4\delta(n-2) + 2\delta(n-3) + \delta(n-5)$$

Example 2.3 Evaluate the discrete-time convolution sum given below.

$$y(n) = u(n) * u(n-3)$$

Solution : Consider $x_1(n) = u(n)$

$x_2(n) = u(n-3)$

The signals $x_1(n)$ and $x_2(n)$ are plotted in Fig. P2.3.1 & Fig. P2.3.2 respectively.

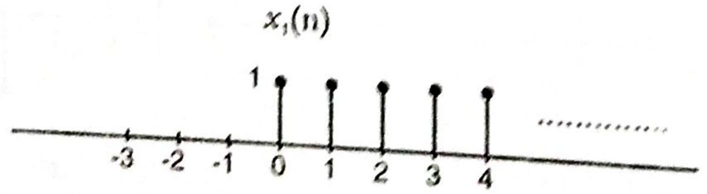


Fig. P2.3.1

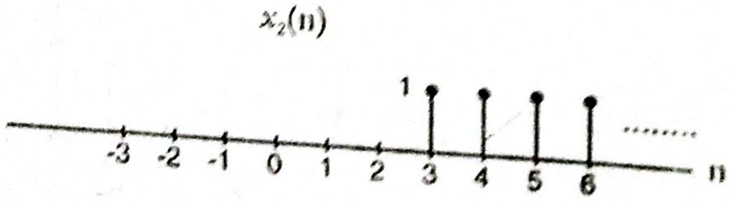


Fig. P2.3.2

We have to obtain,

$$y(n) = u(n) * u(n-3)$$
$$= x_1(n) * x_2(n)$$

$$\therefore y(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \qquad \text{....... P2.3.1}$$

Let us draw $x_1(k)$. It is obtained by replacing 'n' by 'k' in Fig. P2.3.1 as shown in Fig. P2.3.3 below.

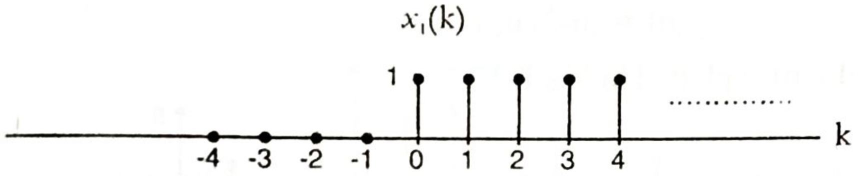


Fig. P2.3.3

Similarly obtain $x_2(k)$. Then obtain $x_2(n-k)$ with n=0 [i.e., $x_2(-k)$] by taking the mirror image of $x_2(k)$ as shown in Fig. P2.3.4.

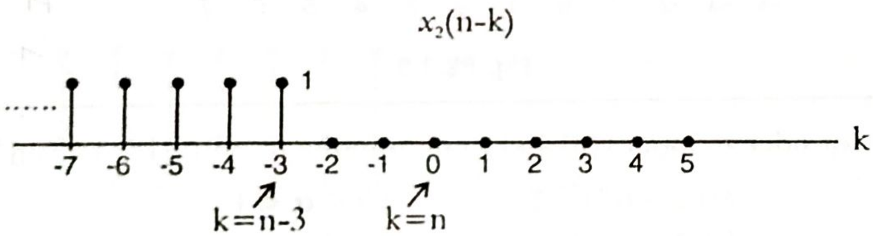


Fig. P2.3.4

Once we take the mirror image whatever we get at k=0, consider it as k=n. So the last non-zero sample of $x_2(n-k)$ with n=0 occurs at k=n-3.

Now according to eqn. P2.3.1, we have to multiply $x_1(k)$ and $x_2(n-k)$, then summation must be carried out for the entire k-axis. i.e., $-\infty < k < \infty$. Now if 'n' is positive integer it corresponds to forward shift of $x_2(n-k)$ and if 'n' is negative integer it corresponds to backward shift of $x_2(n-k)$ along the k-axis.

$\therefore$ When $n-3 < 0$ (i.e., $n < 3$)

$$x_1(k) \cdot x_2(n-k) = 0$$

$$\therefore y(n) = 0 \qquad ; n < 3$$

When $n-3 \geq 0$ (i.e., $n \geq 3$), then multiplication of $x_1(k)$ and $x_2(n-k)$ yields non-zero value for $0 \leq k \leq n-3$.

$$\therefore y(n) = \sum_{k=0}^{n-3} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=0}^{n-3} 1.1 \qquad \left[\because \sum_{n=0}^{N-1} 1 = N\right]$$

$$y(n) = n-2 \qquad ; n \geq 3$$

$$\therefore y(n) = \begin{cases} 0 & ; n < 3 \\ n-2 & ; n \geq 3 \end{cases}$$

Alternatively,

$$y(n) = (n-2)\, u(n-3)$$

The signal y(n) is plotted in Fig. P 2.3.5.

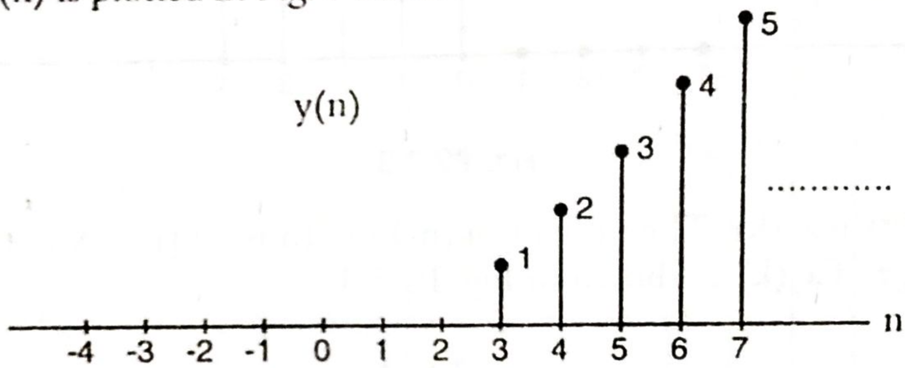


Fig. P2.3.5

Example 2.4 Consider an input $x(n)$ and a unit impulse response h(n) given by

$$x(n) = \alpha^n u(n) \qquad ; 0 < \alpha < 1$$
$$h(n) = u(n)$$

Evaluate and plot the output signal y(n).

Solution : We know that,

$$y(n) = x(n) * h(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$x(n) = \alpha^n u(n)$$

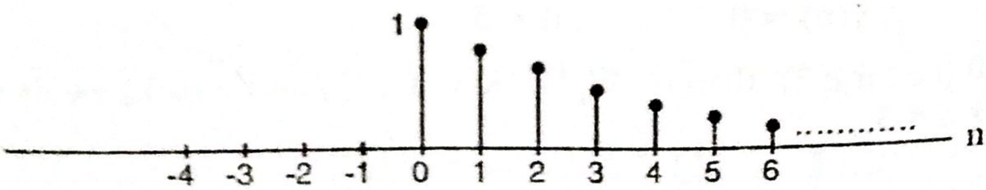


Fig. P2.4.1

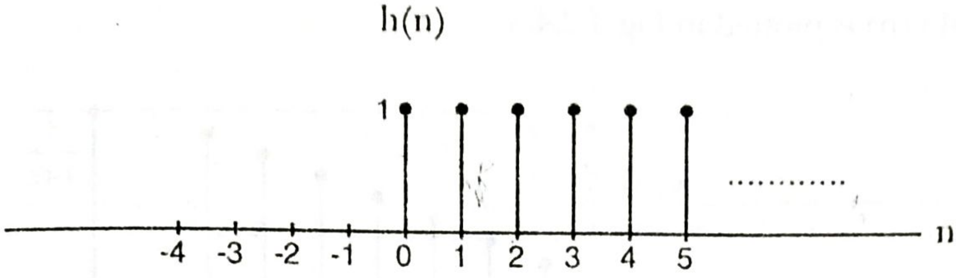


Fig. P2.4.2

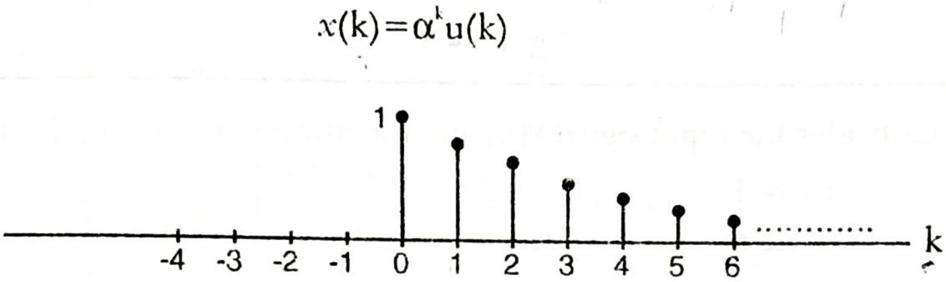


Fig. P2.4.3

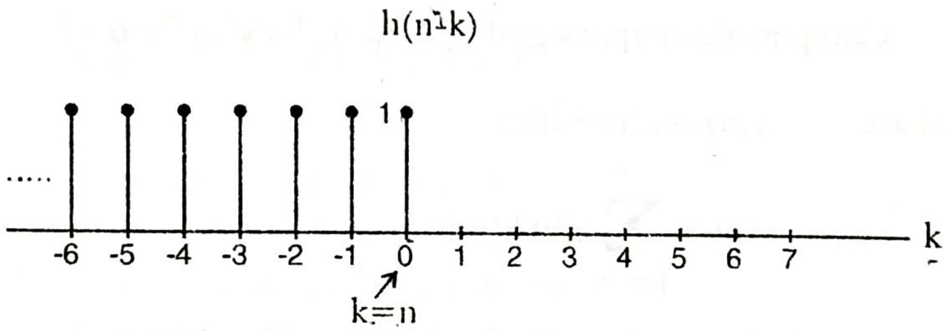


Fig. P2.4.4

When $n < 0$; $\quad x(k)\, h(n-k) = 0$

$$\therefore\; y(n) = 0 \qquad ; n < 0$$

When $n \geq 0$; $\quad$ multiplication of $x(k)$ and $h(n-k)$ yields non-zero value for $0 \leq k \leq n$.

$$y(n) = \sum_{k=0}^{n} x(k)\, h(n-k)$$

$$= \sum_{k=0}^{n} \alpha^k . 1$$

$$\left\{ \because \sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha} \quad ; \alpha \neq 1 \right.$$
$$\qquad\qquad = N \quad ; \alpha = 1$$

$$y(n) = \frac{1-\alpha^{n+1}}{1-\alpha} \qquad ; n \geq 0$$

$$\therefore\; y(n) = \left[\frac{1-\alpha^{n+1}}{1-\alpha}\right] u(n)$$

The signal y(n) is plotted in Fig. P2.4.5.

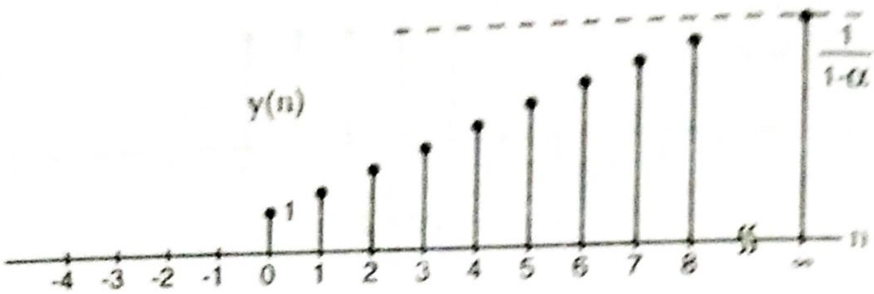


Fig. P2.4.5

Example 2.5 Consider the input signal $x(n)$ and the impulse response $h(n)$ given below

$$x(n) = 1 \quad ; 0 \le n \le 4$$
$$= 0 \quad ; \text{otherwise}$$

$$h(n) = \alpha^n \quad ; 0 \le n \le 6 \quad ; \alpha > 1$$
$$= 0 \quad ; \text{otherwise}$$

Compute the output signal y(n). Also plot y(n) for $\alpha = 2$.

Solution : We have $y(n) = x(n) * h(n)$

i.e., $$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

The signals $x(n)$ and $h(n)$ are plotted in Fig. P2.5.1 and Fig. P2.5.2 respectively.

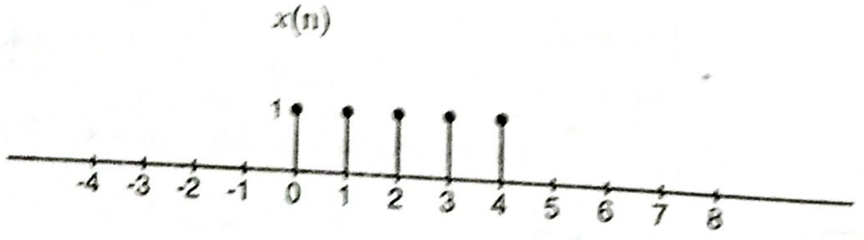


Fig. P2.5.1

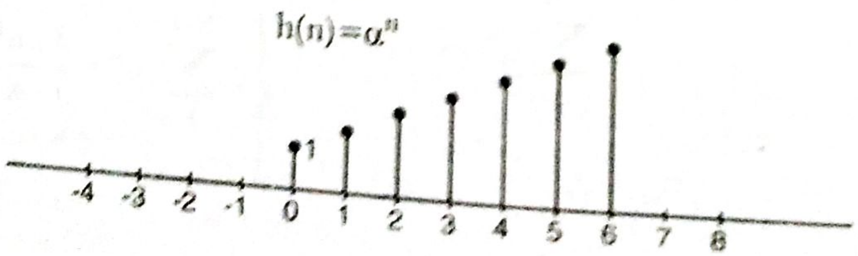


Fig. P2.5.2

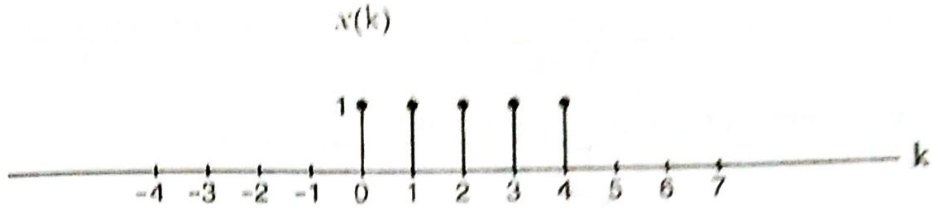


Fig. P2.5.3

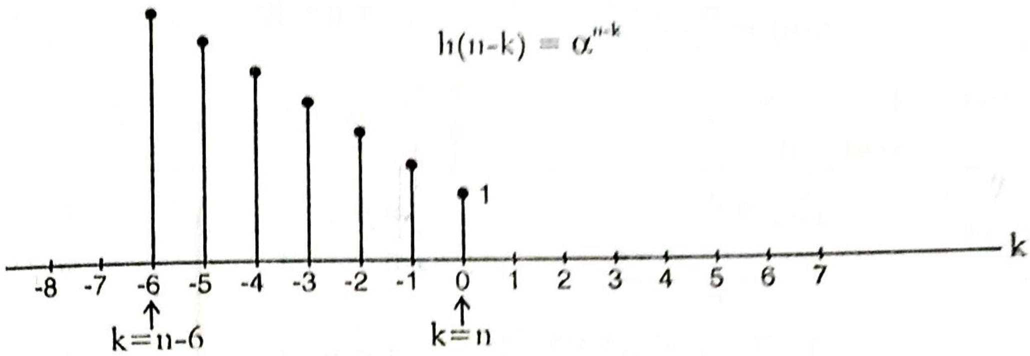


Fig. P2.5.4

When $n < 0$; $x(k)\,h(n-k) = 0$,

$$\therefore\ y(n) = 0 \qquad ;\ n < 0$$

When $n \geq 0$ and $n \leq 4$ (i.e., $0 \leq n \leq 4$)

$$y(n) = \sum_{k=0}^{n} x(k)\,h(n-k)$$

$$= \sum_{k=0}^{n} \alpha^{n-k}\,.\,1$$

$$y(n) = \frac{1-\alpha^{n+1}}{1-\alpha} \qquad ;\ 0 \leq n \leq 4$$

When $n > 4$ and $n-6 \leq 0$, (i.e., $4 < n \leq 6$)

$$y(n) = \sum_{k=0}^{4} \alpha^{n-k}$$

$$= \alpha^{n} \sum_{k=0}^{4} (\alpha^{-1})^{k}$$

$$= \alpha^{n}\,.\,\frac{1-(\alpha^{-1})^{5}}{1-(\alpha^{-1})}$$

$$y(n) = \frac{\alpha^{n-4} - \alpha^{n+1}}{1 - \alpha} \qquad ; 4 < n \le 6$$

When $n > 6$ and $n - 6 \le 4$, (i.e., $6 < n \le 10$)

$$y(n) = \sum_{k=n-6}^{4} \alpha^{n-k}$$

$$y(n) = \frac{\alpha^{n-4} - \alpha^{7}}{1 - \alpha} \qquad ; 6 < n \le 10$$

When $n - 6 > 4$ (i.e., $n > 10$)

$$y(n) = 0$$

$$\therefore y(n) = \begin{cases} 0 & ; n < 0 \\ \dfrac{1 - \alpha^{n+1}}{1 - \alpha} & ; 0 \le n \le 4 \\ \dfrac{\alpha^{n-4} - \alpha^{n+1}}{1 - \alpha} & ; 4 < n \le 6 \\ \dfrac{\alpha^{n-4} - \alpha^{7}}{1 - \alpha} & ; 6 < n \le 10 \\ 0 & ; n > 10 \end{cases}$$

The signal y(n) is plotted in Fig. P2.5.5. (for $\alpha = 2$; not to the scale)

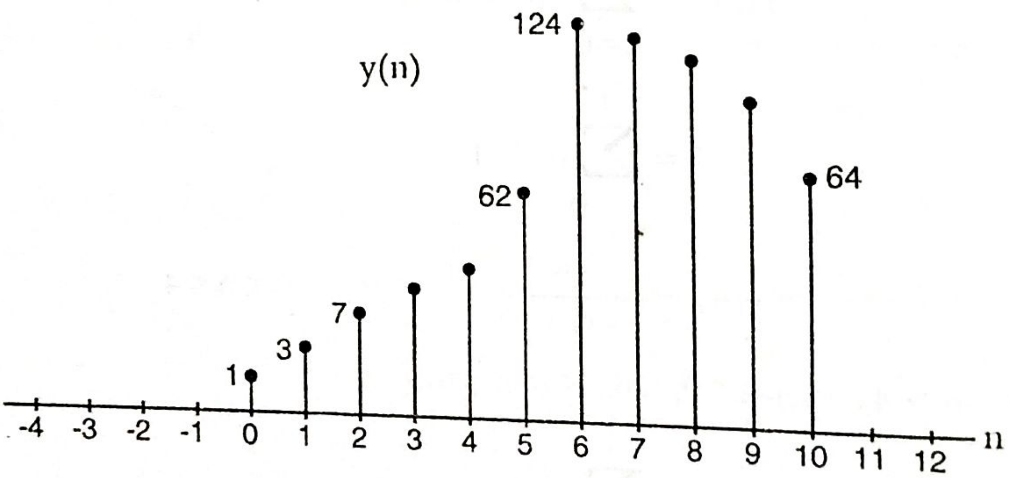


Fig. P2.5.5

Example 2.6 Consider an LTI system with input $x(n)$ and unit impulse response $h(n)$ given below.

Compute and plot the output signal $y(n)$.

$$x(n) = 2^{n}\, u(-n)$$

$$h(n) = u(n)$$

Solution : We have $y(n) = x(n) * h(n)$

i.e., $$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

The signals $x(n)$ and $h(n)$ are plotted in Fig. P2.6.1 and Fig. P2.6.2 respectively.

$x(n)=2^n u(-n)$

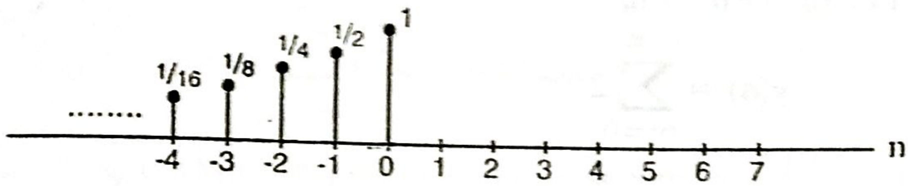


Fig. P2.6.1

$h(n)$

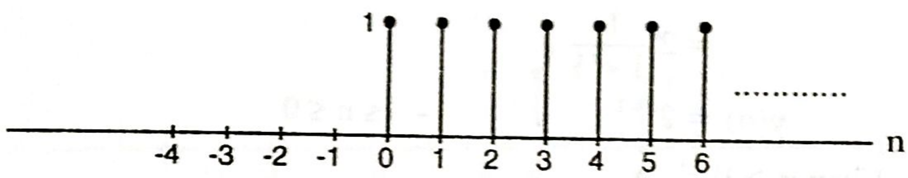


Fig. P2.6.2

$x(k)=2^k u(-k)$

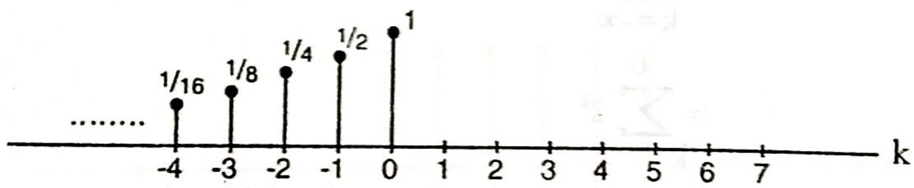


Fig. P2.6.3

$h(n-k)$

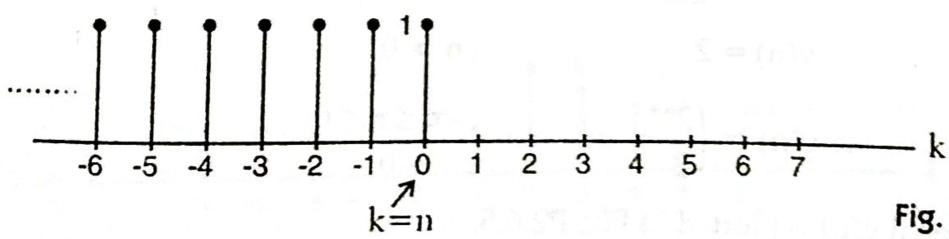


Fig. P2.6.4

When $-\infty \le n$ and $n \le 0$ [i.e., $-\infty \le n \le 0$]

$$y(n) = \sum_{k=-\infty}^{n} x(k)\, h(n-k)$$

$$= \sum_{k=-\infty}^{n} 2^k$$

Put $l = -k$ then,

$$\therefore y(n) = \sum_{l=\infty}^{-n} 2^{l}$$

$$= \sum_{l=-n}^{\infty} 2^{-l}$$

Put $m = l + n$ then,

$$y(n) = \sum_{m=0}^{\infty} 2^{-(m-n)}$$

$$= 2^{n} \sum_{m=0}^{\infty} (½)^{m}$$

$$= 2^{n} \frac{1}{1 - ½}$$

$$y(n) = 2^{n+1} \qquad ; -\infty \le n \le 0$$

When $n > 0$ then,

$$y(n) = \sum_{k=-\infty}^{0} x(k)\, h(n-k)$$

$$= \sum_{k=-\infty}^{0} 2^{k}$$

$$= \sum_{k=0}^{\infty} 2^{-k}$$

$$y(n) = 2 \qquad ; n > 0$$

$$\therefore y(n) = \begin{cases} 2^{n+1} & ; -\infty \le n \le 0 \\ 2 & ; n > 0 \end{cases}$$

The signal y(n) is plotted in Fig. P2.6.5.

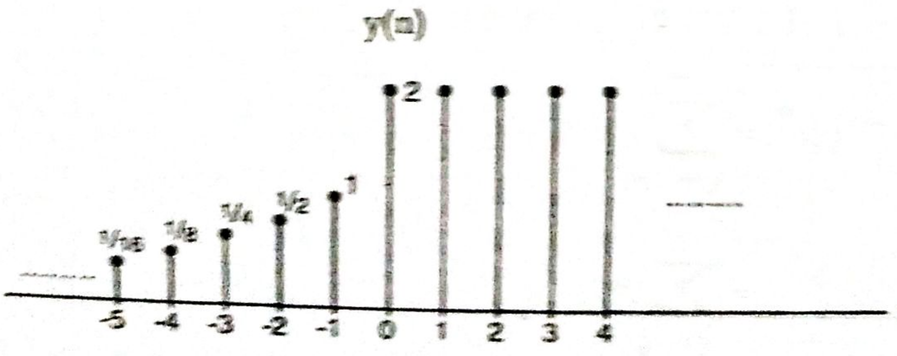


Fig. P2.6.5

Example 2.7 Evaluate the discrete-time convolution sum given below.

$$y(n) = (½)^n u(n-2) * u(n)$$

Solution : Consider $x_1(n) = (½)^n u(n-2)$

$$x_2(n) = u(n)$$

$$y(n) = (½)^n u(n-2) * u(n)$$

$$y(n) = x_1(n) * x_2(n)$$

i.e., $$y(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

The signals $x_1(n)$ and $x_2(n)$ are plotted in Fig. P2.7.1 and Fig. P2.7.2 respectively.

$x_1(n)=(½)^n u(n-2)$

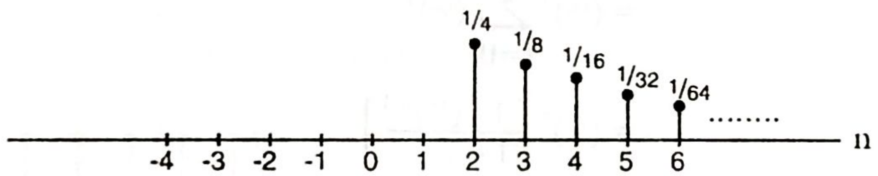


Fig. P2.7.1

$x_2(n)=u(n)$

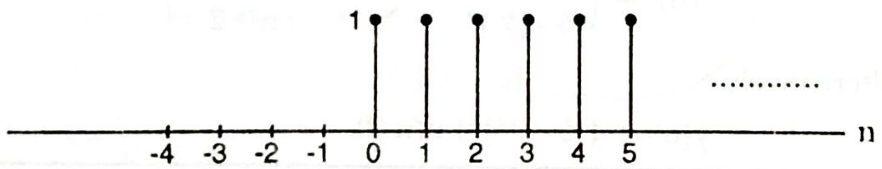


Fig. P2.7.2

$x_1(k)=(½)^k u(k-2)$

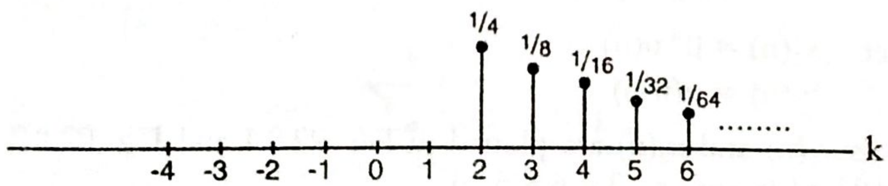


Fig. P2.7.3

$x_2(n-k)$

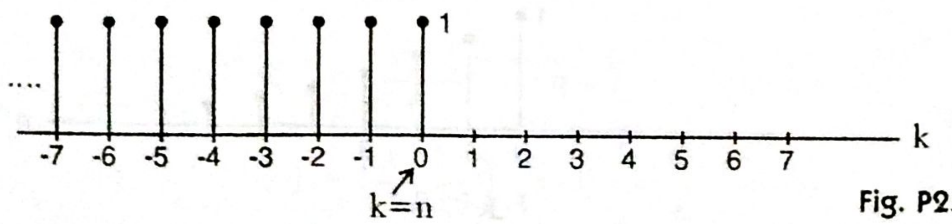


Fig. P2.7.4

When $n < 2$, $x_1(k)\, x_2(n-k) = 0$

$$\therefore \quad y(n) = 0 \quad ; n < 2$$

When $n \geq 2$, $y(n) = \sum_{k=2}^{n} x_1(k)\, x_2(n-k)$

$$= \sum_{k=2}^{n} (½)^k$$

Put l = k-2, then

$$\therefore y(n) = \sum_{l=0}^{n-2} (½)^{l+2}$$

$$= (½)^2 \sum_{l=0}^{n-2} (½)^l$$

$$= (½)^2 \left[\frac{1 - (½)^{n-1}}{1 - ½}\right]$$

$$y(n) = [½ - (½)^n] \quad ; n \geq 2$$

$$\therefore y(n) = \begin{cases} 0 & ; n < 2 \\ ½ - (½)^n & ; n \geq 2 \end{cases}$$

Alternatively,

$$y(n) = \{½ - (½)^n\}\, u(n-2)$$

Example 2.8 Evaluate the discrete-time convolution sum given below.

$$y(n) = \beta^n\, u(n) * u(n-3) \quad ; |\beta| < 1$$

Solution : Let $x_1(n) = \beta^n\, u(n)$

$x_2(n) = u(n-3)$

The signals $x_1(n)$ and $x_2(n)$ are plotted in Fig. P2.8.1 and Fig. P2.8.2 respectively. Given : $|\beta| < 1$ (assuming β is positive).

$x_1(n) = \beta^n u(n)$

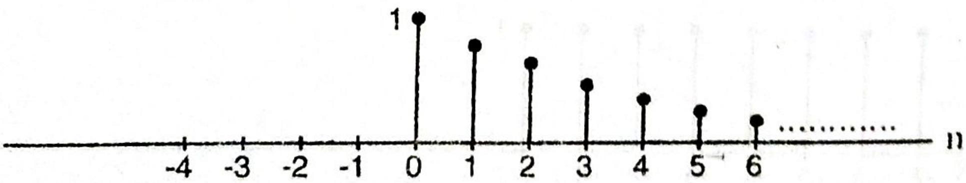


Fig. P2.8.1

$x_2(n) = u(n-3)$

Fig. P2.8.2

$x_1(k) = \beta^k u(k)$

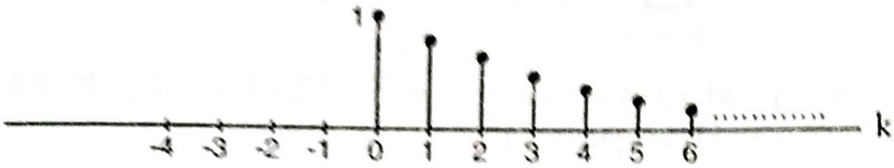


Fig. P2.8.3

$x_2(n-k)$

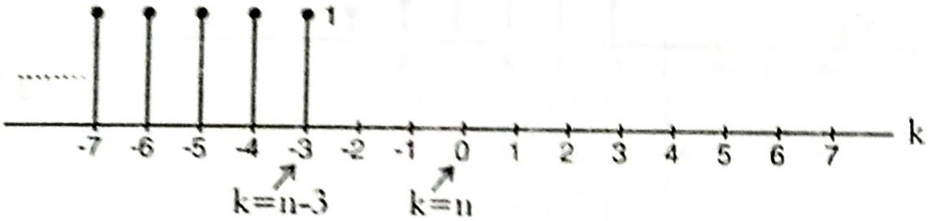


Fig. P2.8.4

When $n-3<0$ [i.e., $n<3$]

$$x_1(k) \cdot x_2(n-k) = 0$$

$$\therefore\ y(n) = 0 \qquad ; n<3$$

When $n-3 \geq 0$ [i.e., $n \geq 3$]

$$y(n) = \sum_{k=0}^{n-3} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=0}^{n-3} \beta^k$$

$$y(n) = \frac{1-\beta^{n-2}}{1-\beta} \qquad ; n \geq 3$$

$$\therefore\ y(n) = \left(\frac{1-\beta^{n-2}}{1-\beta}\right) u(n-3)$$

Example 2.9 Find the discrete-time convolution sum given below.

$$y(n) = \beta^n u(n) * \alpha^n u(n) \qquad ; |\beta|<1 \quad ; \quad |\alpha|<1$$

Solution : Let $x_1(n) = \beta^n u(n)$

$$x_2(n) = \alpha^n u(n)$$

$$y(n) = \beta^n u(n) * \alpha^n u(n) \qquad ; |\beta|<1 \quad ; \quad |\alpha|<1$$

$$= x_1(n) * x_2(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

The signals $x_1(n)$ and $x_2(n)$ are plotted in Fig. P2.9.1 and Fig. P2.9.2 respectively. (assuming both α and β are positive).

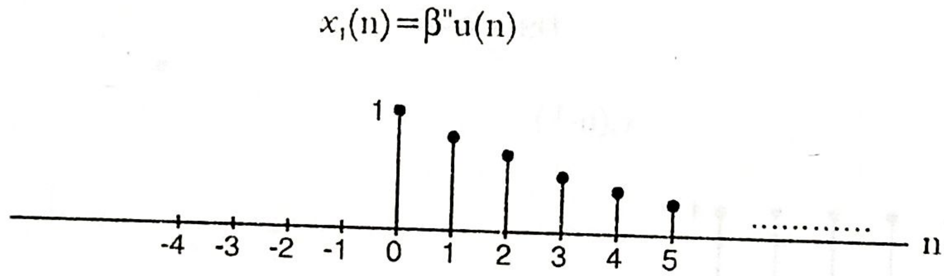


Fig. P2.9.1

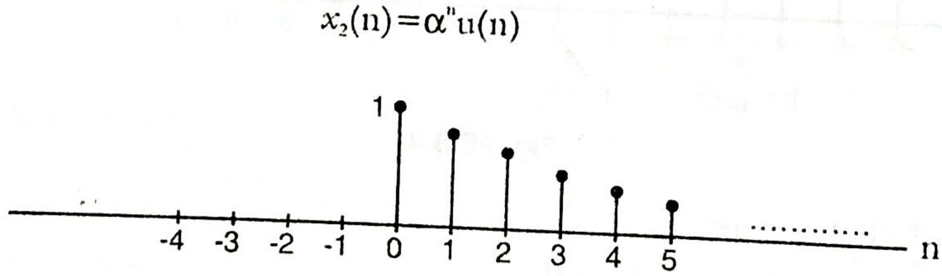


Fig. P2.9.2

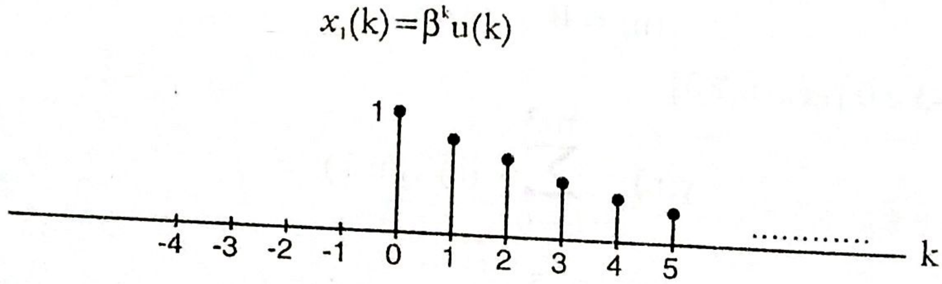


Fig. P2.9.3

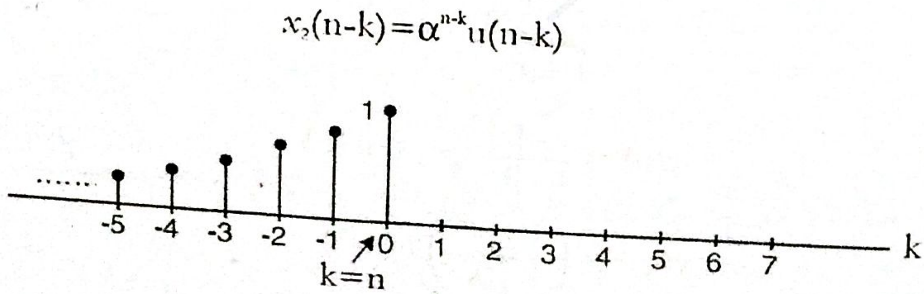


Fig. P2.9.4

When $n < 0$; $x_1(k)\, x_2(n-k) = 0$

$\therefore\ y(n) = 0$; $n<0$

When $n \geq 0$,

$$y(n) = \sum_{k=0}^{n} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=0}^{n} \beta^k \alpha^{n-k}$$

$$= \alpha^n \sum_{k=0}^{n} \left(\frac{\beta}{\alpha}\right)^k$$

$$y(n) = \alpha^n \cdot \frac{1-\left(\frac{\beta}{\alpha}\right)^{n+1}}{1-\left(\frac{\beta}{\alpha}\right)} \qquad ;\ \alpha \neq \beta$$

$$= \alpha^n(n+1) \qquad ;\ \alpha = \beta \text{ [using L-Hospital's rule]}$$

$$\therefore \quad y(n) = \left[\alpha^n \cdot \frac{1-\left(\frac{\beta}{\alpha}\right)^{n+1}}{1-\left(\frac{\beta}{\alpha}\right)}\right] u(n) \qquad ;\ \alpha \neq \beta$$

$$= [\, \alpha^n(n+1)\,]\, u(n) \qquad ;\ \alpha=\beta$$

Example 2.10 Evaluate the discrete-time convolution sum given below.

$$y(n) = [u(n+10) - 2u(n+5) + u(n-6)] * \beta^n u(n) \qquad ;\ |\beta|<1$$

Solution : Let $x_1(n) = [u(n+10) - 2u(n+5) + u(n-6)]$ and
$x_2(n) = \beta^n u(n)$

The signals $x_1(n)$ and $x_2(n)$ are plotted in Fig. P2.10.1 and Fig. P2.10.2 respectively.

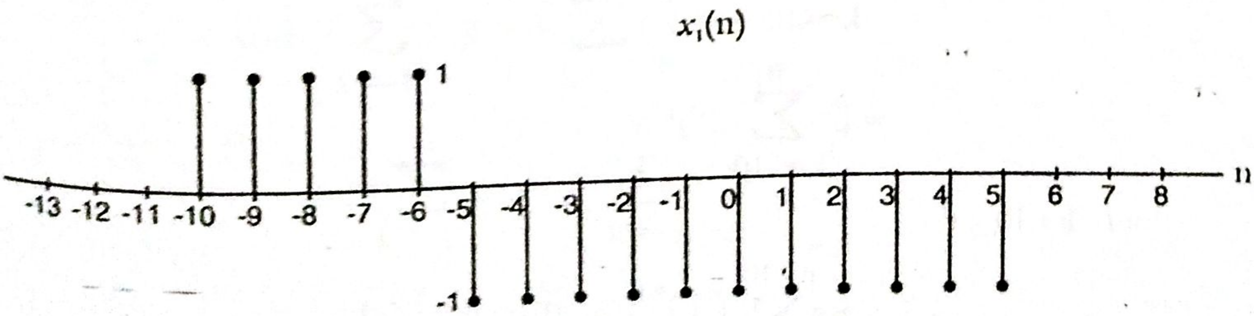


Fig. P2.10.1

$x_2(n) = \beta^n u(n)$

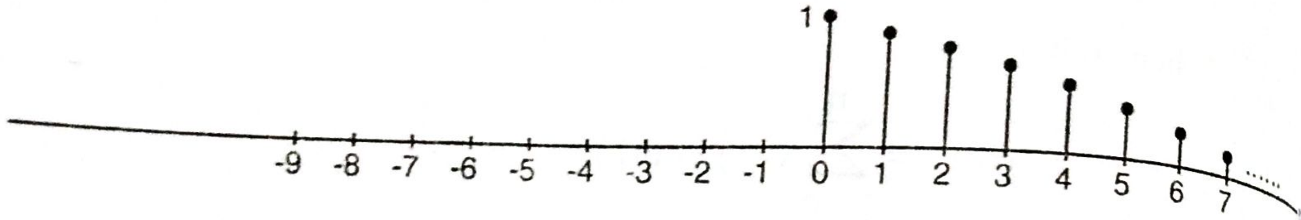


Fig. P2.10.2

$x_1(k)$

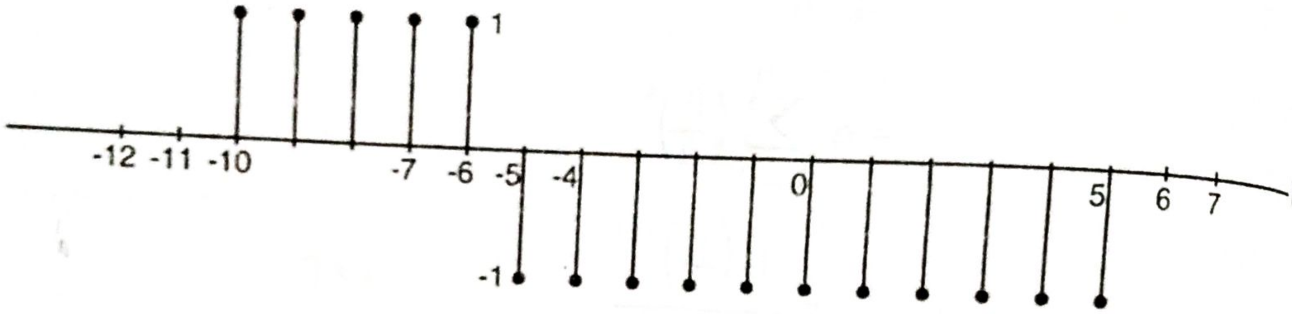


Fig. P2.10.3

$x_2(n-k) = \beta^{n-k} u(n-k)$

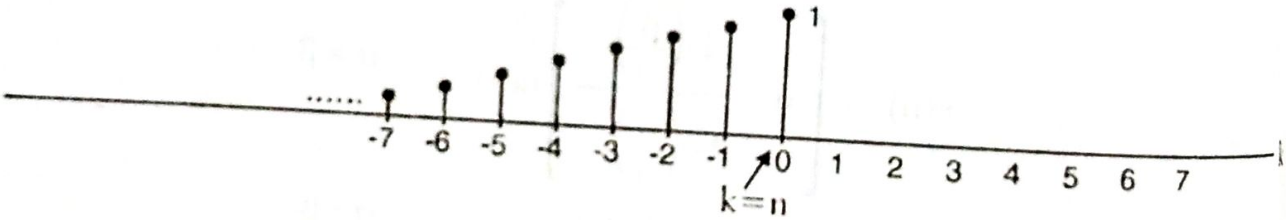


Fig. P2.10.4

When $n < -10$

$$x_1(k)\, x_2(n-k) = 0$$

$$\therefore y(n) = 0 \qquad ; n < -10$$

When $n \geq -10$ and $n \leq -6$ (i.e., $-10 \leq n \leq -6$)

$$y(n) = \sum_{k=-10}^{n} 1.\beta^{n-k}$$

$$= \beta^n \sum_{k=-10}^{n} (\beta^{-1})^k$$

Put $l = k + 10$; then

$$y(n) = \beta^n \sum_{l=0}^{n+10} \left(\frac{1}{\beta}\right)^{l-10}$$

$$= \beta^n \left(\frac{1}{\beta}\right)^{-10} \sum_{l=0}^{n+10} \left(\frac{1}{\beta}\right)^l$$

$$= \beta^{n+10} \frac{1 - \left(\frac{1}{\beta}\right)^{n+11}}{1 - \left(\frac{1}{\beta}\right)}$$

$$y(n) = \frac{1 - \beta^{n+11}}{1 - \beta} \qquad ; -10 \leq n \leq -6$$

When $n > -6$ and $n \leq 5$, (i.e., $-6 < n \leq 5$)

$$y(n) = \sum_{k=-10}^{-6} (1)\beta^{n-k} + \sum_{k=-5}^{n} (-1)\beta^{n-k}$$

$$= \beta^n \sum_{k=-10}^{-6} \left(\frac{1}{\beta}\right)^k - \beta^n \sum_{k=-5}^{n} \left(\frac{1}{\beta}\right)^k$$

$$= \beta^n \left[\sum_{l=0}^{4} \left(\frac{1}{\beta}\right)^{l-10} - \sum_{m=0}^{n+5} \left(\frac{1}{\beta}\right)^{m-5}\right]$$

$$= \beta^n \left[\beta^{10} \sum_{l=0}^{4} \left(\frac{1}{\beta}\right)^l - \beta^5 \sum_{m=0}^{n+5} \left(\frac{1}{\beta}\right)^m\right]$$

$$= \beta^n \left[\beta^{10} \frac{1 - (^1/_\beta)^5}{1 - (^1/_\beta)} - \beta^5 \frac{1 - (^1/_\beta)^{n+6}}{1 - (^1/_\beta)}\right]$$

$$y(n) = \frac{\beta^n}{1 - \beta} \left[2\beta^6 - \beta^{11} - \beta^{-n}\right] \qquad ; -6 < n \leq 5$$

When $n > 5$

$$y(n) = \sum_{k=-10}^{-6} 1 \, . \, \beta^{n-k} + \sum_{k=-5}^{5} (-1)\beta^{n-k}$$

$$= \beta^n \sum_{k=-10}^{-6} (\beta^{-1})^k - \beta^n \sum_{k=-5}^{5} (\beta^{-1})^k$$

$$y(n) = \frac{\beta^n}{1 - \beta} \left[2\beta^6 - \beta^{11} - \beta^{-5}\right] \qquad ; n > 5$$

$$\therefore y(n) = \begin{cases} 0 & ; n < -10 \\ \dfrac{1-\beta^{n+11}}{1-\beta} & ; -10 \le n \le -6 \\ \dfrac{\beta^{n}}{1-\beta}\,[2\beta^{6} - \beta^{11} - \beta^{-n}] & ; -6 < n \le 5 \\ \dfrac{\beta^{n}}{1-\beta}\,[2\beta^{6} - \beta^{11} - \beta^{-5}] & ; n > 5 \end{cases}$$

2.2.2 The representation for continuous-time LTI systems in terms of Impulse Response

Any arbitrary continuous-time signal $x(t)$ can be expressed as the weighted superposition of time-shifted impulses as below.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \qquad \text{........ (2.7)}$$

Now consider a continuous-time LTI system as shown in Fig. 2.7.

$$x(t) \longrightarrow \boxed{T\{\cdot\}} \longrightarrow y(t)$$

Fig. 2.7

where $x(t)$ is the input to the system.
$y(t)$ is the output of the system.
$T\{\cdot\}$ is system operator.

$\therefore$ We have $y(t) = T\{x(t)\}$ .... (2.8)

Substituting eqn. 2.7 in eqn. 2.8 we get,

$$y(t) = T\left\{ \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \right\}$$

Using linearity property we get,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,T\{\delta(t-\tau)\}\,d\tau \qquad \text{........ (2.9)}$$

In eqn. 2.9, $T\{\delta(t-\tau)\}$ corresponds to the operation of the system performed on time-shifted impulse $\delta(t-\tau)$.

$$\therefore T\{\delta(t-\tau)\} = h(t-\tau) \qquad \text{........ (2.10)}$$

where $h(t)$ is the *impulse response* of the system.

Substituting eqn. 2.10 in eqn. 2.9, we get,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \qquad \text{........ (2.11)}$$

$$\therefore \quad \boxed{y(t) = x(t) * h(t)} \qquad \text{........ (2.12)}$$

From eqn. 2.11, we can say that the output $y(t)$ is obtained as a weighted superposition of impulse responses time-shifted by τ. The weights are $x(\tau)\, d\tau$. The eqn. 2.11 is known as *convolution integral*.

Examples

Example 2.11 Consider a continuous-time LTI system with unit impulse response,

$$h(t) = u(t) \quad \text{and} \quad \text{input } x(t) = e^{-at}\, u(t) \qquad ; a>0$$

Find the output $y(t)$ of the system.

Solution : We have,

$$y(t) = x(t) * h(t)$$

i.e.,
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

The input signal $x(t)$ and the impulse response $h(t)$ are plotted in Fig. P2.11.1 and Fig. P2.11.2 respectively.

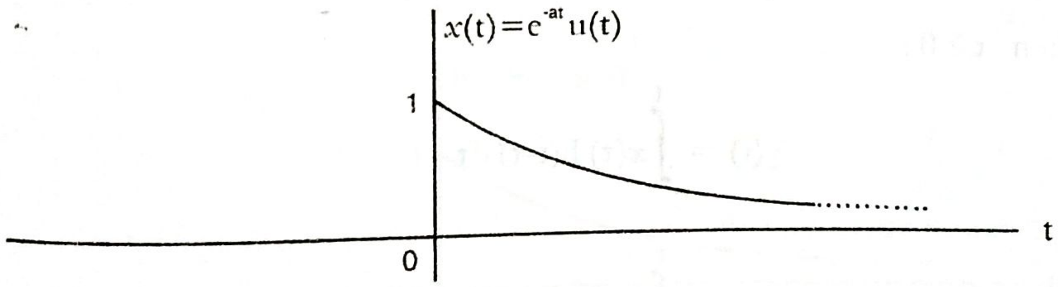


Fig. P2.11.1

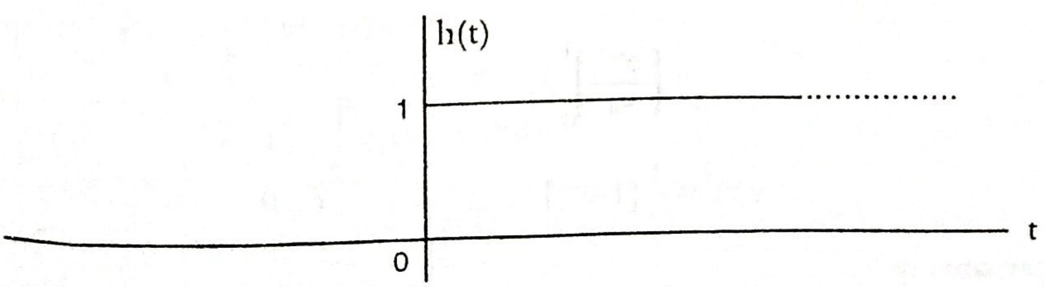


Fig. P2.11.2

Let us plot $x(\tau)$ and $h(t-\tau)$ (with $t=0$) as shown below in Fig. P2.11.3 and P2.11.4 respectively.

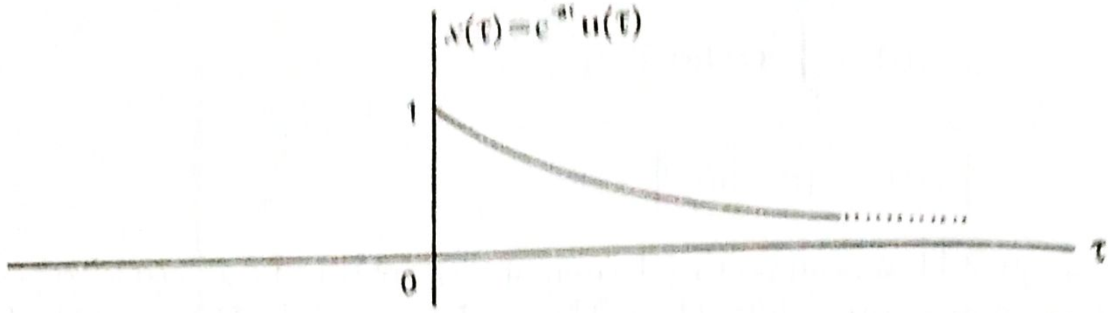


Fig. P2.11.3

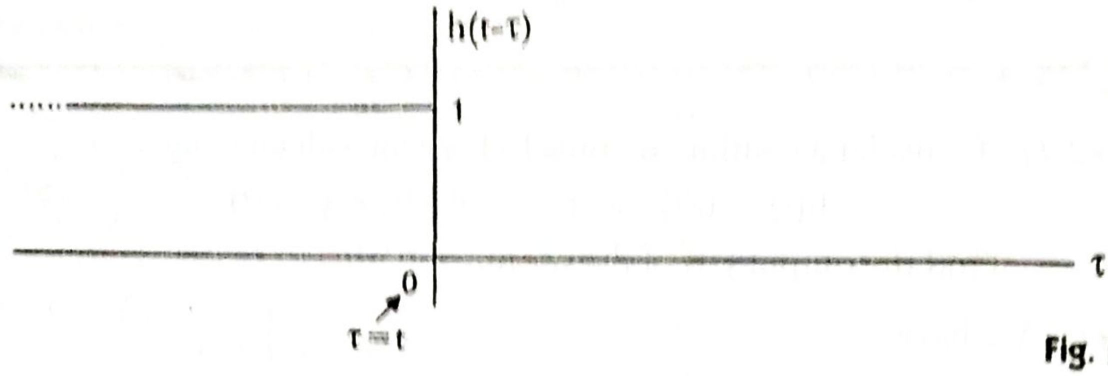


Fig. P2.11.4

Now, if 't' is positive, it means forward shift of $h(t-\tau)$ and negative means backward shift of $h(t-\tau)$.

When $t < 0$;

$$x(\tau)\, h(t-\tau) = 0$$

$$\therefore\ y(t) = 0 \qquad ; t<0$$

When $t \geq 0$;

$$y(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_0^t e^{-a\tau} . 1 . d\tau$$

$$= \left[\frac{e^{-a\tau}}{-a} \right]_0^t$$

$$y(t) = \frac{1}{a} [1 - e^{-at}] \qquad ; t \geq 0$$

Alternatively,

$$\therefore\ y(t) = \frac{1}{a} [1 - e^{-at}]\, u(t)$$

The signal y(t) is shown in Fig. P2.11.5.

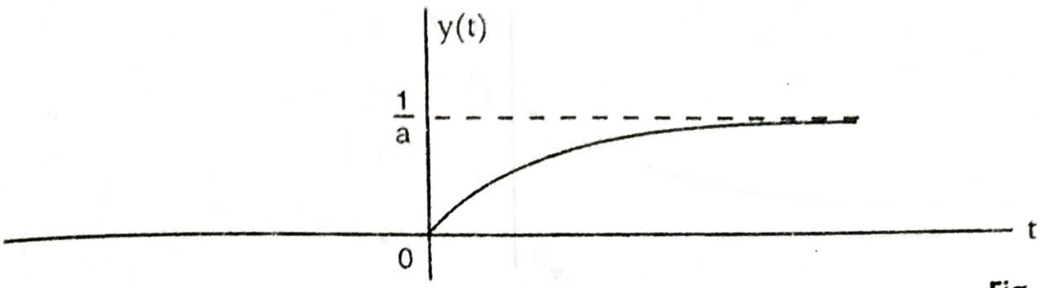


Fig. P2.11.5

Example 2.12 Consider a LTI system with unit impulse response,

$$h(t) = e^{-t}\,u(t)$$

If the input applied to this system is,

$$x(t) = e^{-3t}\,\{u(t) - u(t-2)\}$$

find the output y(t) of the system.

Solution : The input signal $x(t)$ and the impulse response h(t) are plotted in Fig. P2.12.1 and Fig. P2.12.2 respectively.

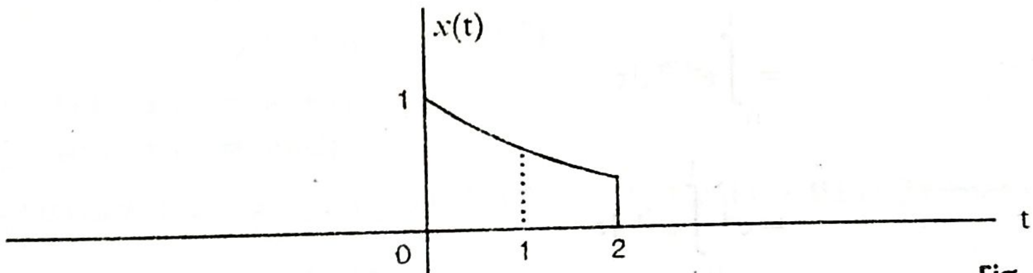


Fig. P2.12.1

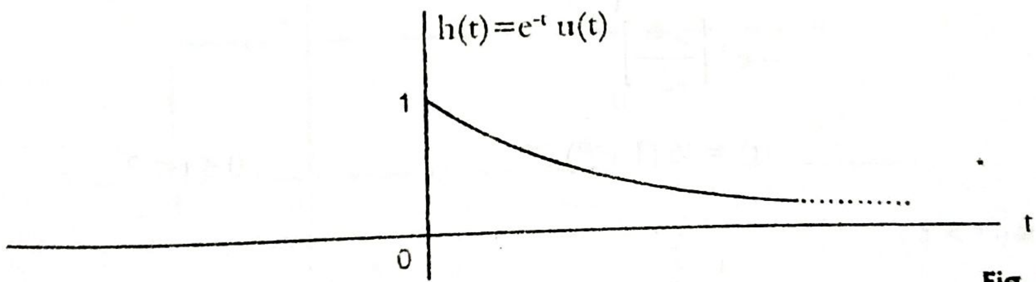


Fig. P2.12.2

We know that

$$y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

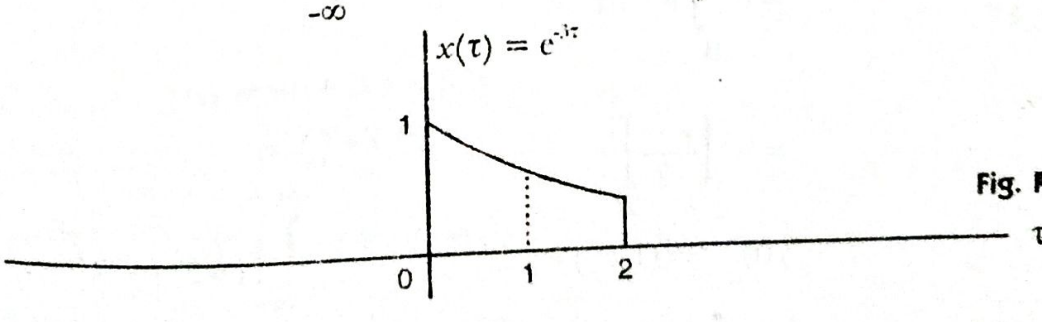


Fig. P2.12.3

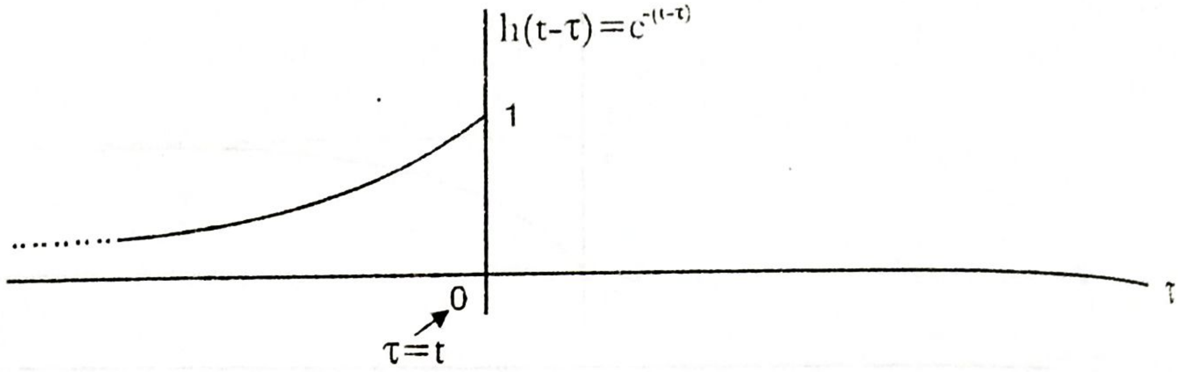


Fig. P2.12.4

When t<0

$$x(\tau)\, h(t-\tau) = 0$$

$$\therefore\ y(t) = 0 \qquad ; t<0$$

When $t \geq 0$ and $t < 2$, (i.e., $0 \leq t < 2$), we get,

$$= \int_0^t e^{-3\tau}\, e^{-(t-\tau)}\, d\tau$$

$$= \int_0^t e^{-t-2\tau}\, d\tau$$

$$= e^{-t} \int_0^t e^{-2\tau}\, d\tau$$

$$= e^{-t} \left[\frac{e^{-2\tau}}{-2}\right]_0^t$$

$$y(t) = ½\,(1-e^{-2t})\, e^{-t} \qquad ; 0 \leq t < 2$$

When $t > 2$

$$y(t) = \int_0^2 e^{-3\tau}\, e^{-(t-\tau)}\, d\tau$$

$$= e^{-t} \int_0^2 e^{-2\tau}\, d\tau$$

$$= e^{-t} \left[\frac{e^{-2\tau}}{-2}\right]_0^2$$

$$y(t) = ½\,(1-e^{-4})\, e^{-t} \qquad ; t > 2$$

$$\therefore\ y(t) = \begin{cases} 0 & ; t < 0 \\ \tfrac{1}{2}(1-e^{-2t})e^{-t} & ; 0 \le t < 2 \\ \tfrac{1}{2}(1-e^{-4})e^{-t} & ; t > 2 \end{cases}$$

The signal y(t) is plotted in Fig. P2.12.5.

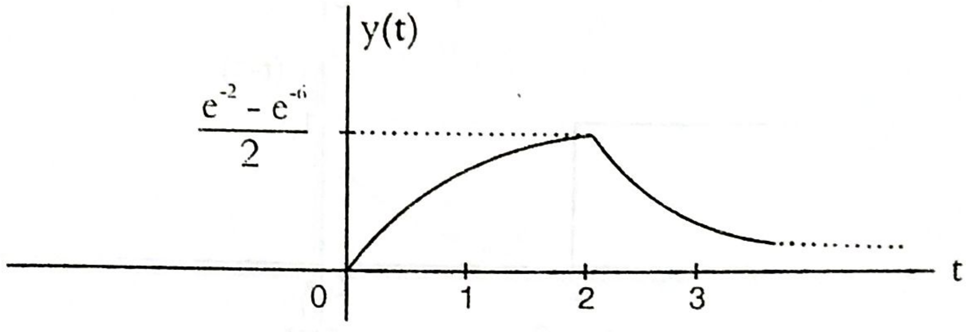


Fig. P2.12.5

Example 2.13 Evaluate the following continuous-time convolution integral.

$$y(t) = u(t+1) * u(t-2)$$

Solution : Let $x_1(t) = u(t+1)$
and $x_2(t) = u(t-2)$

The signals $x_1(t)$ and $x_2(t)$ are plotted in Fig. P2.13.1 and Fig. P2.13.2 respectively.

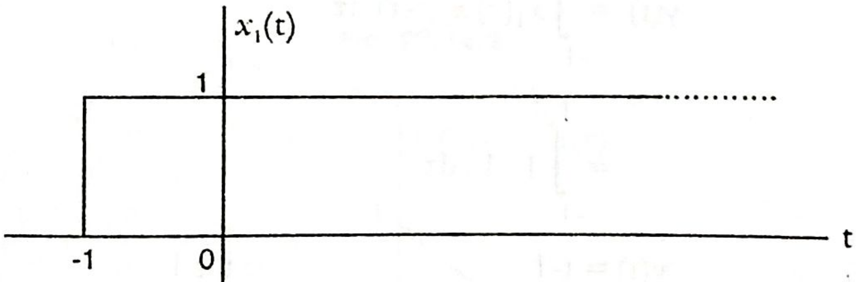


Fig. P2.13.1

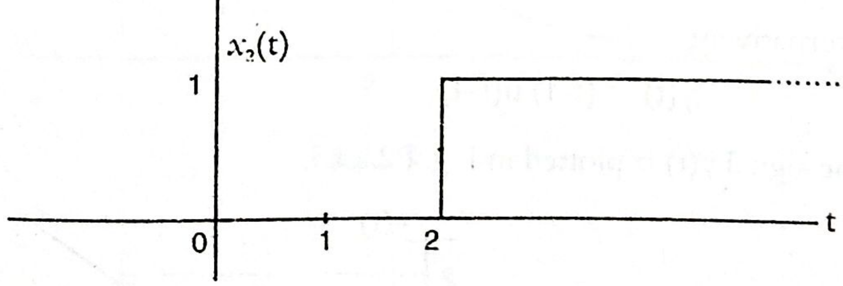


Fig. P2.13.2

$$y(t) = u(t+1) * u(t-2)$$
$$= x_1(t) * x_2(t)$$

i.e.,
$$y(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

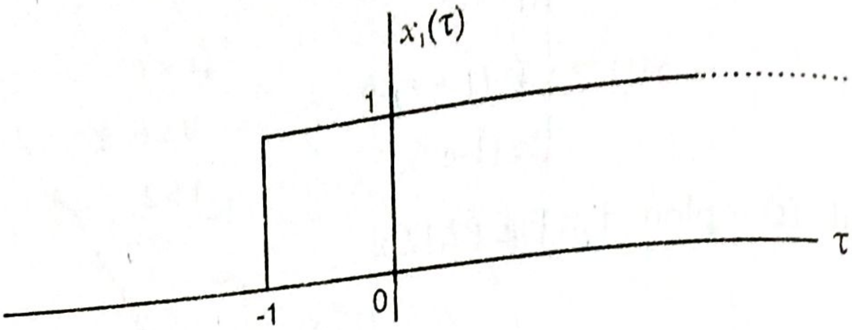


Fig. P2.1

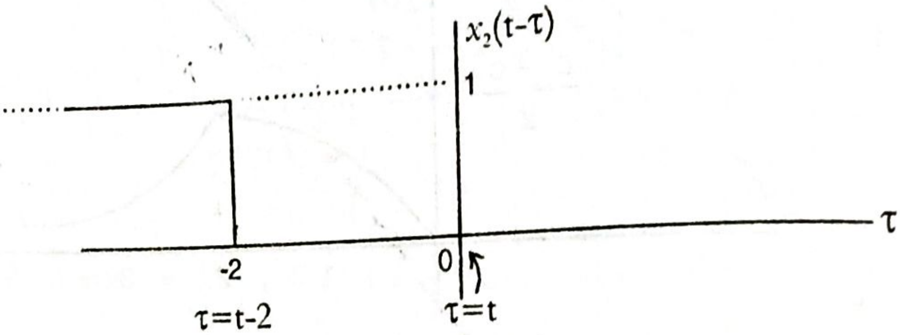


Fig. P2.1

When t-2 < -1 [i.e., t < 1]

$$x_1(\tau)\, x_2(t-\tau) = 0$$

$$\therefore y(t) = 0 \qquad ; t<1$$

When t-2 ≥ -1 [i.e., t ≥ 1]

$$y(t) = \int_{-1}^{t-2} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= \int_{-1}^{t-2} 1\,.\,1\,.\,d\tau$$

$$y(t) = t-1 \qquad ; t \geq 1$$

Alternatively,

$$y(t) = (t-1)\, u(t-1)$$

The signal y(t) is plotted in Fig. P2.13.5.

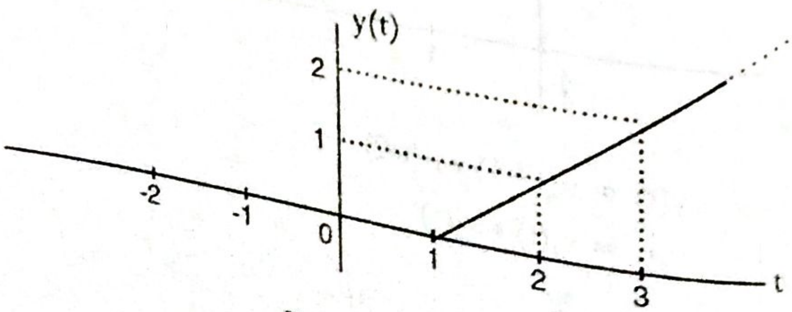


Fig. P2.13.5

Example 2.14 Evaluate the continuous-time convolution integral given below.

$$y(t) = e^{-2t}\,u(t) * u(t+2)$$

Solution : Let $x_1(t) = e^{-2t}u(t)$

$x_2(t) = u(t+2)$

The signals $x_1(t)$ and $x_2(t)$ are plotted in Fig. P2.14.1 and Fig. P2.14.2 respectively.

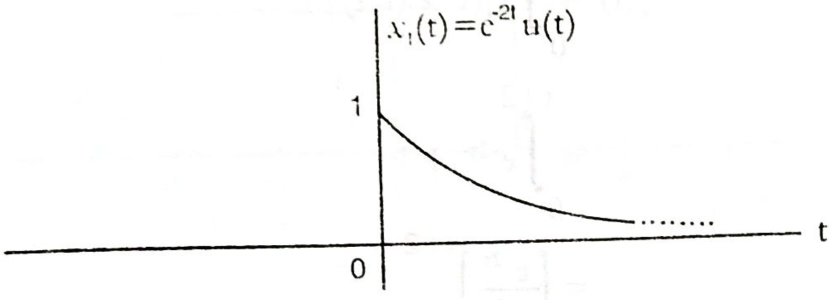


Fig. P2.14.1

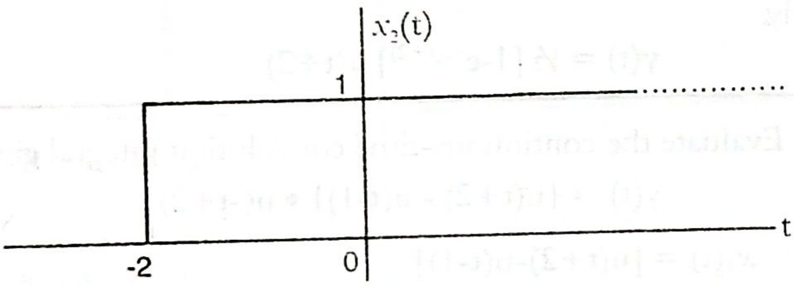


Fig. P2.14.2

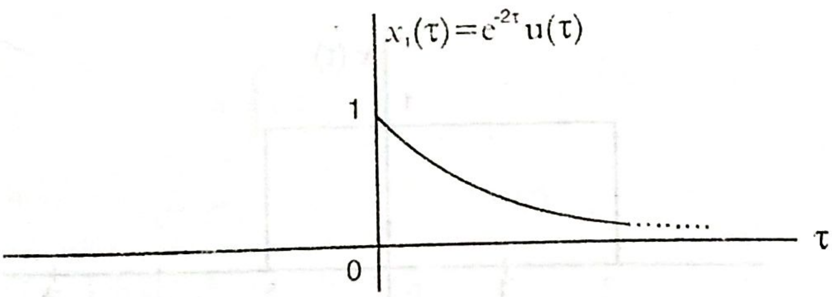


Fig. P2.14.3

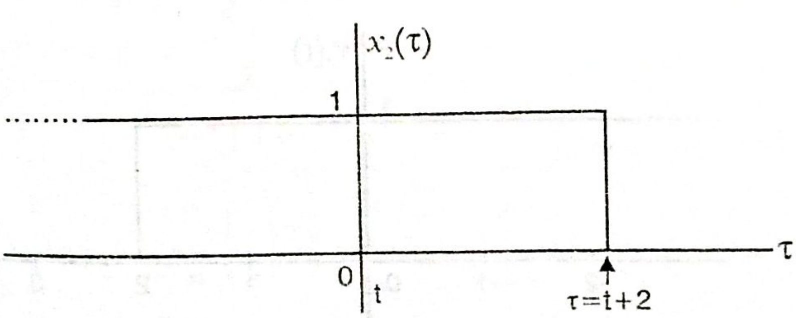


Fig. P2.14.4

When $t+2 < 0$, [i.e., $t < -2$]

$$x_1(\tau)\, x_2(t-\tau) = 0 \qquad ; t < -2$$

$$\therefore\ y(t) = 0$$

When $t+2 \geq 0$ [i.e., $t \geq -2$]

$$y(t) = \int_0^{t+2} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= \int_0^{t+2} e^{-2\tau} .\, 1\, d\tau$$

$$= \left[\frac{e^{-2\tau}}{-2}\right]_0^{t+2}$$

$$y(t) = \tfrac{1}{2}\left[1-e^{-2(t+2)}\right] \qquad ; t \geq -2$$

Alternatively,

$$y(t) = \tfrac{1}{2}\left[1-e^{-2(t+2)}\right] u(t+2)$$

Example 2.15 Evaluate the continuous-time convolution integral given below.

$$y(t) = \{u(t+2) - u(t-1)\} * u(-t+2)$$

Solution : Let $x_1(t) = [u(t+2)-u(t-1)]$

and $x_2(t) = u(-t+2)$

The signals $x_1(t)$ and $x_2(t)$ are plotted in Fig. P2.15.1 and P2.15.2 respectively.

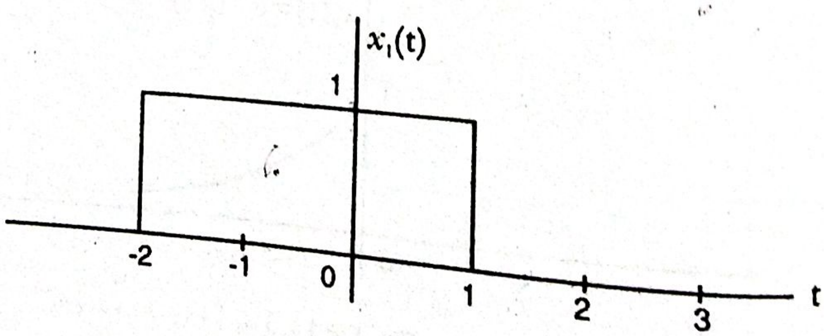


Fig. P2.1

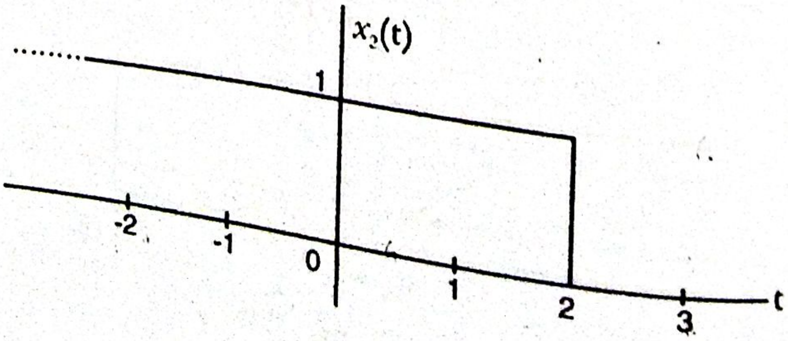


Fig. P2.1

$$y(t) = x_1(t) * x_2(t)$$

$$y(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

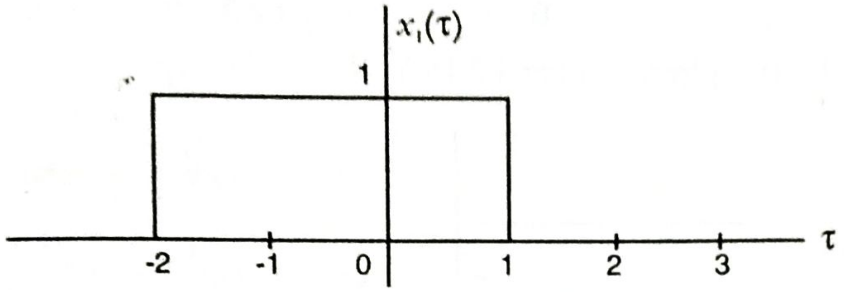


Fig. P2.15.3

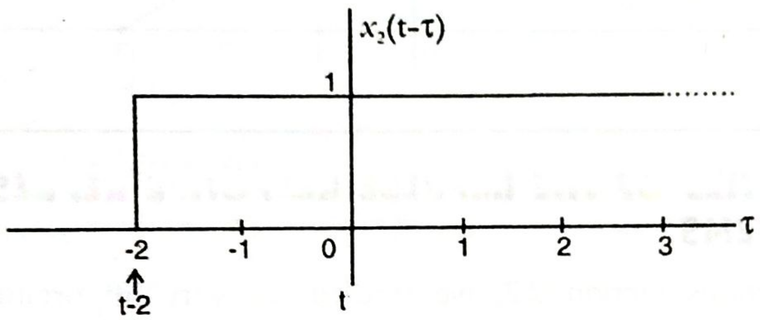


Fig. P2.15.4

When t-2 < -2 [i.e., t < 0]

$$y(t) = \int_{-2}^{1} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= \int_{-2}^{1} d\tau$$

$$y(t) = 3 \qquad ; t<0$$

When t-2 ≥ -2 and t-2 < 1 [i.e., 0 ≤ t<3]

$$y(t) = \int_{t-2}^{1} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= \int_{t-2}^{1} 1\, d\tau$$

$$= 1 - t+2$$

$$y(t) = 3-t \qquad ; 0 \le t<3$$

When t-2 ≥ 1 [i.e., t ≥ 3]

$$x_1(\tau)\, x_2(t-\tau) = 0$$

$$\therefore\ y(t) = 0$$

$$\therefore\ y(t) = 3 \quad ;\ t < 0$$
$$= 3 - t \quad ;\ 0 \le t < 3$$
$$= 0 \quad ;\ t \ge 3$$

The signal y(t) is plotted in Fig. P2.15.5

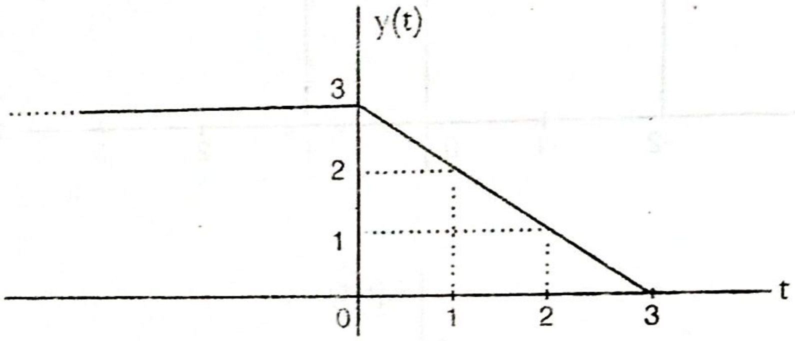


Fig. P2.15.5

2.3 PROPERTIES OF THE IMPULSE RESPONSE REPRESENTATION FOR LTI SYSTEMS

In the previous section 2.2, we studied the very important representations of continuous-time and discrete-time LTI systems in terms of their unit impulse responses. That means the characteristics of an LTI system are completely determined by its impulse response. Hence the properties of an LTI system such as memory, causality, stability etc. are related to its impulse response. Also the impulse response of an interconnection of LTI systems is related to the impulse response of each subsystem.

2.3.1 The Commutative Property

The convolution in both continuous-time and discrete-time are *commutative*.

i.e., $x(n) * h(n) = h(n) * x(n)$

& $x(t) * h(t) = h(t) * x(t)$

Consider a discrete-time LTI system having impulse response h(n) and with input $x(n)$. Then the output is given by,

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

Put n-k=m then,

$$y(n) = \sum_{m=-\infty}^{\infty} x(n-m)\, h(m)$$

$$y(n) = h(n) * x(n)$$

$$\therefore\ y(n) = x(n) * h(n) = h(n) * x(n) \qquad \ldots\ldots (2.13)$$

Similarly, consider a continuous-time LTI system having impulse response h(t) and with input x(t). Then the output is given by,

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Put $t-\tau = m$, $d\tau = -dm$, then

$$y(t) = -\int_{\infty}^{-\infty} x(t-m).\, h(m)\, dm$$

$$= \int_{-\infty}^{\infty} h(m)\, x(t-m)\, dm$$

$$y(t) = h(t) * x(t)$$

$$\therefore\ y(t) = x(t) * h(t) = h(t) * x(t) \qquad \ldots\ldots (2.14)$$

∴ Convolution operation is commutative.

2.3.2 The Distributive Property

Another basic property of convolution is *distributive* property.

i.e., In discrete-time,

$$x(n) * \{h_1(n) + h_2(n)\} = x(n) * h_1(n) + x(n) * h_2(n)$$

and in continuous-time,

$$x(t) * \{h_1(t) + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$$

Consider an interconnection of discrete-time LTI system as shown in Fig. 2.8

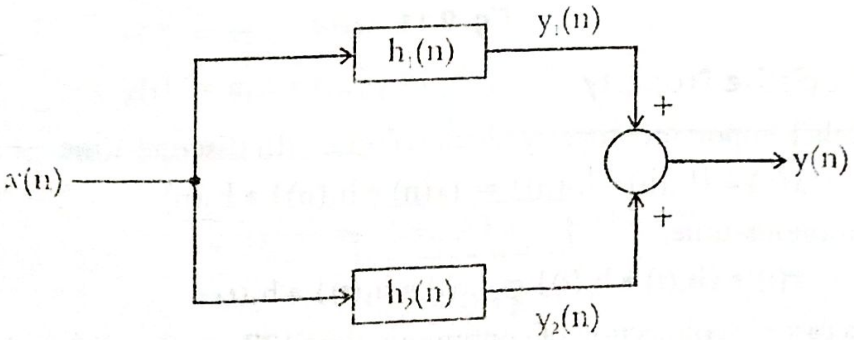


Fig. 2.8

From Fig. 2.8 we have,

$$y_1(n) = x(n) * h_1(n)$$
$$y_2(n) = x(n) * h_2(n)$$
$$y(n) = y_1(n) + y_2(n) = [x(n) * h_1(n)] + [x(n) * h_2(n)]$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h_1(n-k) + \sum_{k=-\infty}^{\infty} x(k)\, h_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, \{h_1(n-k) + h_2(n-k)\}$$

$$\therefore\ y(n) = x(n) * \{h_1(n) + h_2(n)\} \qquad \text{------ (2.15)}$$

From eqn. 2.15, we can write the system shown in Fig. 2.8 as shown below in Fig. 2.9.

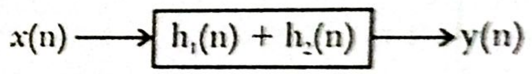


Fig. 2.9

Similarly, in continuous-time,

$$x(t) * \{h_1(t) + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$$

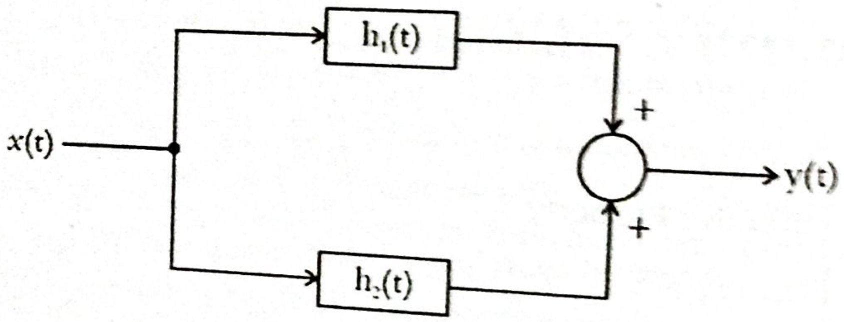


Fig. 2.10

The system shown in Fig. 2.10 can be written as shown Fig. 2.11.

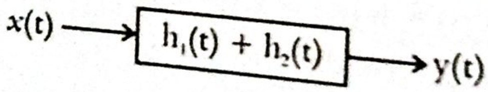


Fig. 2.11

2.3.3 The Associative Property

This is another important property of convolution. In discrete-time,

$$x(n) * \{h_1(n) * h_2(n)\} = \{x(n) * h_1(n)\} * h_2(n)$$

and in continuous-time,

$$x(t) * \{h_1(t) * h_2(t)\} = \{x(t) * h_1(t)\} * h_2(t)$$

Consider a cascade connection of continuous-time LTI system as shown in Fig. 2.12

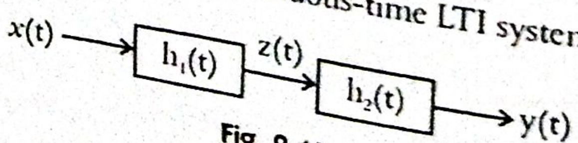


Fig. 2.12

From Fig. 2.12 we have,

$$y(t) = z(t) * h_2(t)$$

$$= \int_{-\infty}^{\infty} z(\tau)\, h_2(t-\tau)\, d\tau \qquad \text{.......} \quad (2.16)$$

where z(t) is the output of the first system

∴ We have,

$$z(\tau) = x(\tau) * h_1(\tau)$$

$$= \int_{-\infty}^{\infty} x(\eta)\, h_1(\tau-\eta)\, d\eta \qquad \text{.......} \quad (2.17)$$

Substituting eqn. 2.17 in 2.16 we get,

$$y(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\eta)\, h_1(\tau-\eta)\, h_2(t-\tau)\, d\eta\, d\tau$$

Substituting $m=\tau-\eta$ and interchanging the order of integration we get,

$$= \int_{-\infty}^{\infty} x(\eta) \left[\int_{-\infty}^{\infty} h_1(m)\, h_2(t-\eta-m)dm \right] d\eta$$

$$= \int_{-\infty}^{\infty} x(\eta)\, [h(t-\eta)]\, d\eta$$

where $\quad h(t-\eta) = h_1(t-\eta) * h_2(t-\eta)$

$$\therefore\ h(t) = h_1(t) * h_2(t) \qquad \text{.......} \quad (2.18)$$

$$\therefore\ y(t) = x(t) * h(t) \qquad \text{.......} \quad (2.19)$$

$$\therefore\ y(t) = x(t) * [h_1(t) * h_2(t)]$$

Observing eqn. 2.18 and 2.19, we can write the system shown in Fig. 2.12 as shown in Fig. 2.13 below.

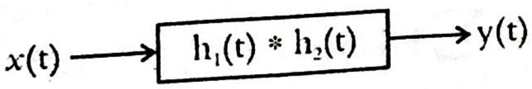


Fig. 2.13

Hence, the overall impulse response of two LTI system connected in cascade is the convolution of the individual impulse responses.

Also we know that convolution is commutative. Therefore we can write the system in Fig. 2.13 as below.

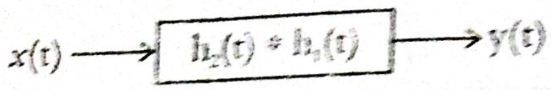


Fig. 2.14

Fig. 2.15

From the above discussion we conclude that the output of a cascade combination of LTI systems is independent of the order in which the systems are connected.

Similarly, for discrete-time we have,

$$x(n) * \{h_1(n) * h_2(n)\} = x(n) * \{h_2(n) * h_1(n)\}$$

2.3.4 Memoryless System

A system is *memoryless* if its output at any time depends only on the value of the input at the same time. This is true for a discrete-time LTI system only if $h(n) = 0$ for $n \neq 0$ and for continuous-time LTI system only if $h(t) = 0$ for $t \neq 0$.

2.3.5 Causal System

We know that the output of a *causal* system depends only on the present and/or past values of the input to the system. i.e., for a discrete-time LTI system to be causal, its $h(n) = 0$ for $n<0$. Similarly, for a continuous-time LTI system to be causal, its $h(t)=0$ for $t<0$.

2.3.6 Stable System

A system is *stable* if every bounded input produces a bounded output.

Consider a discrete-time LTI system with an input $x(n)$ is bounded.

i.e., $|x(n)| < M$ for all 'n'

If the impulse response of the system is $h(n)$, then the magnitude of the output is given by,

$$\begin{aligned} |y(n)| &= |x(n) * h(n)| \\ &= |h(n) * x(n)| \\ &= \left|\sum_{k=-\infty}^{\infty} h(k)\, x(n-k)\right| \end{aligned}$$

Since the magnitude of the sum of a set of numbers is smaller than the sum of the magnitudes of the numbers, we get,

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)\, x(n-k)|$$

$$|y(n)| \le \sum_{k=-\infty}^{\infty} |h(k)|\ |x(n-k)|$$

Since $|x(n)| < M$, then $|x(n-k)| < M$, for all values of 'k' and 'n'.

$$\therefore\ |y(n)| \le M \sum_{k=-\infty}^{\infty} |h(k)| \quad \text{for all 'n'}$$

i.e., the magnitude of y(n) is bounded only if,

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

$\therefore$ A discrete-time LTI system is stable only if its impulse response is *absolutely summable.*

Similarly consider a continuous-time LTI system with impulse response h(t).

If the input is bounded,

i.e., $|x(t)| < M$

then,

$$|y(t)| = |h(t) * x(t)|$$

$$= \left| \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \right|$$

$$\le \int_{-\infty}^{\infty} |h(\tau)\, x(t-\tau)\, d\tau|$$

$$\le \int_{-\infty}^{\infty} |h(\tau)|\ |x(t-\tau)|\, d\tau$$

$$\le M \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$$

i.e., y(t) is bounded only if

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

$\therefore$ A continuous-time LTI system is stable only if its impulse response is *absolutely integrable.*

2.3.7 The Unit-step Response of an LTI System

The output y(n) of a discrete-time LTI system characterized by an impulse response h(n) with input $x(n)$ is,

$$y(n) = h(n) * x(n)$$

If the input is unit step i.e., $x(n) = u(n)$, then the step response is given by,

$$s(n) = h(n) * u(n)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, u(n-k) \qquad \ldots\ldots. \quad (2.20)$$

We know that $u(n-k)=1 \quad ; n-k \geq 0$ or $k \leq n$

$= 0 \quad ; n-k < 0$ or $k > n$

∴ From eqn. 2.20 we get,

$$s(n) = \sum_{k=-\infty}^{n} h(k) \qquad \ldots\ldots. \quad (2.21)$$

Therefore the step response of a discrete-time LTI system is the *running sum* of the impulse response.

Similarly, for continuous-time LTI system we have,

$$y(t) = h(t) * x(t)$$

If the input is unit step i.e., $x(t) = u(t)$, then the step response is given by,

$$\therefore s(t) = h(t) * u(t)$$

$$= \int_{-\infty}^{\infty} h(\tau)\, u(t-\tau)\, d\tau \qquad \ldots\ldots. \quad (2.22)$$

We know that $u(t-\tau)=1 \quad ; t-\tau \geq 0$ or $\tau \leq t$

$= 0 \quad ; t-\tau < 0$ or $\tau > t$

∴ From eqn. 2.22 we get,

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau \qquad \ldots\ldots. \quad (2.23)$$

Therefore the step response of a continuous-time LTI system is the *running integral* of the impulse response.

Examples

Example 2.16 Consider the interconnection of LTI systems shown in Fig.P2.16. Express the overall impulse response h(n) in terms of $h_1(n)$, $h_2(n)$, $h_3(n)$, $h_4(n)$ and $h_5(n)$.

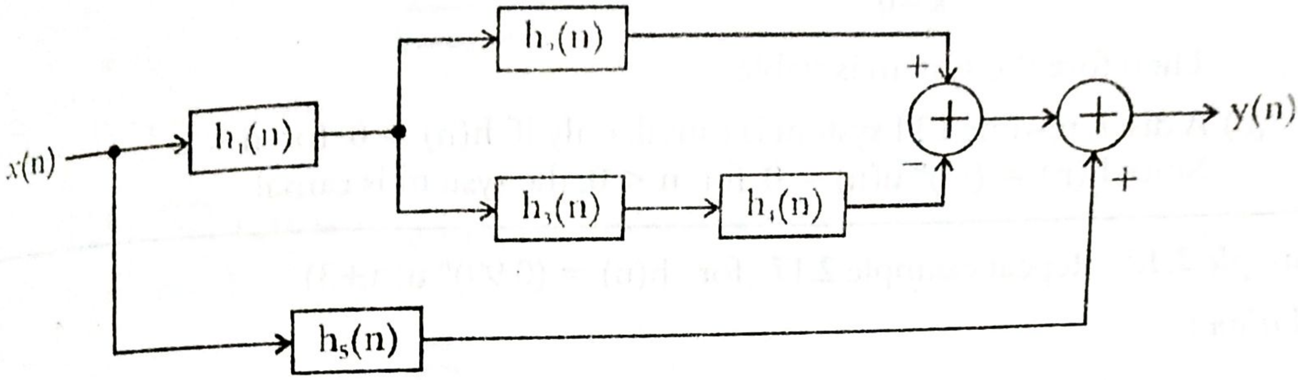


Fig. P2.16

Solution : Fig. P2.16 can be written as shown in Fig. P2.16.1(a) & (b) below.

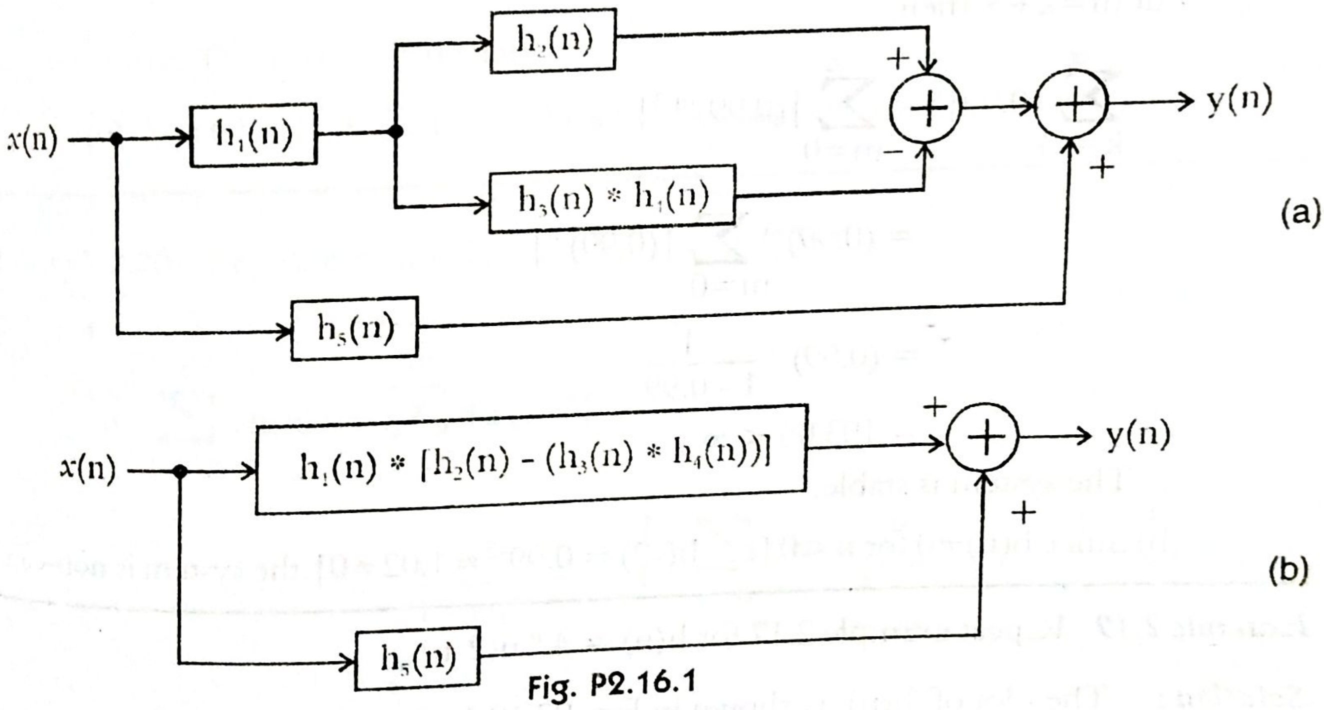


Fig. P2.16.1

Therefore the overall impulse response is,

$$\therefore\ h(n) = h_5(n) + \{\, h_1(n) * [h_2(n) - (h_3(n) * h_4(n))]\}$$

Example 2.17 Determine a discrete-time LTI system characterized by impulse response $h(n) = (½)^n\, u(n)$ is (a) stable and (b) causal.

Solution : Given : $h(n) = (½)^n\, u(n)$

(a) We know that, a discrete-time LTI system stable, only if its impulse response is absolutely summable.

i.e., $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$

$$\therefore \quad \sum_{k=0}^{\infty} |(½)^k| = \frac{1}{1-½} = 2 < \infty$$

Therefore the system is stable.

(b) A discrete-time LTI system is causal, only if $h(n) = 0$ for $n < 0$.
Since $h(n) = (½)^n u(n) = 0$ for $n < 0$, the system is causal.

Example 2.18 Repeat example 2.17 for $h(n) = (0.99)^n u(n+3)$

Solution :

$$\text{(a)} \quad \sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=-3}^{\infty} |(0.99)^k|$$

Put $m=k+3$ then,

$$\sum_{k=-3}^{\infty} |0.99|^k = \sum_{m=0}^{\infty} |(0.99)^{m-3}|$$

$$= (0.99)^{-3} \sum_{m=0}^{\infty} |(0.99)^m|$$

$$= (0.99)^{-3} \frac{1}{1-0.99}$$

$$= 103.06 < \infty$$

$\therefore$ The system is stable.

(b) Since $h(n) \neq 0$ for $n < 0$ [eg., $h(-2) = 0.99^{-2} = 1.02 \neq 0$], the system is non-causal.

Example 2.19 Repeat example 2.17 for $h(n) = 4^{-n} u(2-n)$

Solution : The plot of $h(n)$ is shown in Fig. P2.19.1.

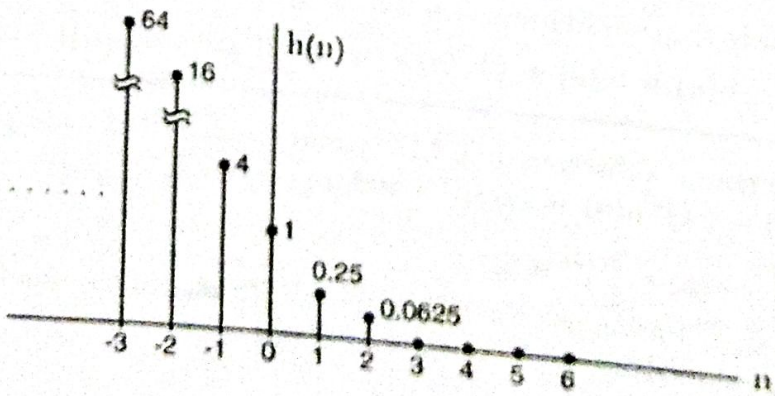


Fig. P2.19.1

$$\text{i.e.,} \quad \sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

$$\therefore \quad \sum_{k=0}^{\infty} |(½)^k| = \frac{1}{1 - ½} = 2 < \infty$$

Therefore the system is stable.

(b) A discrete-time LTI system is causal, only if $h(n) = 0$ for $n < 0$.
Since $h(n) = (½)^n\, u(n) = 0$ for $n < 0$, the system is causal.

Example 2.18 Repeat example 2.17 for $h(n) = (0.99)^n\, u(n+3)$

Solution :

$$\text{(a)} \quad \sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=-3}^{\infty} |(0.99)^k|$$

Put $m=k+3$ then,

$$\sum_{k=-3}^{\infty} |0.99|^k = \sum_{m=0}^{\infty} |(0.99)^{m-3}|$$

$$= (0.99)^{-3} \sum_{m=0}^{\infty} |(0.99)^m|$$

$$= (0.99)^{-3} \frac{1}{1 - 0.99}$$

$$= 103.06 < \infty$$

$\therefore$ The system is stable.

(b) Since $h(n) \neq 0$ for $n < 0$ [eg., $h(-2) = 0.99^{-2} = 1.02 \neq 0$], the system is non-causal.

Example 2.19 Repeat example 2.17 for $h(n) = 4^{-n}\, u(2-n)$

Solution : The plot of $h(n)$ is shown in Fig. P2.19.1.

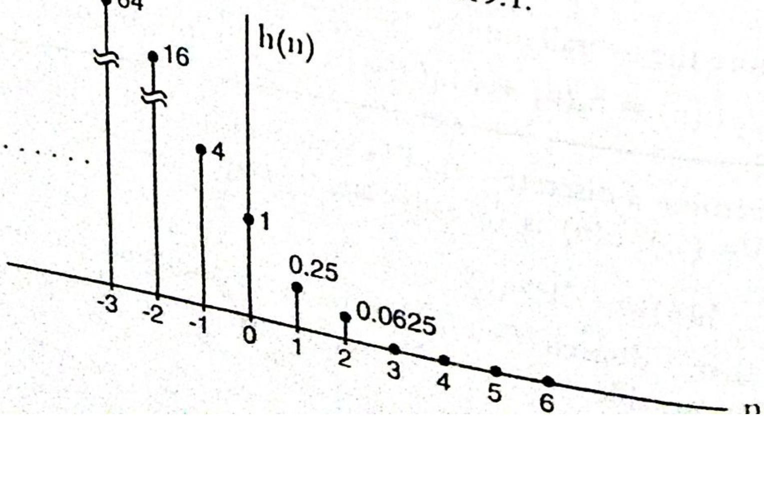


Fig. P2.19.1

(a) $$\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=-\infty}^{2} |(4)^{-k}|$$

$$= \sum_{k=-2}^{\infty} |4^k|$$

Put $l=k+2$ then,

$$\sum_{k=-2}^{\infty} |4^k| = \sum_{l=0}^{\infty} |4^{l-2}|$$

$$= \frac{1}{16} \sum_{l=0}^{\infty} (4)^l$$

$$= \infty$$

∴ The system is unstable.

(b) Since $h(n) \neq 0$ for $n < 0$, the system is non-causal.

Example 2.20 Repeat example 2.17 for $h(n) = n(½)^n u(n)$.

Solution :

(a) $$\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=0}^{\infty} |k(½)^k|$$

$$= \frac{½}{(1-½)^2} \qquad \left[\because \sum_{n=0}^{\infty} n\alpha^n = \frac{\alpha}{(1-\alpha)^2} \quad ; |\alpha|<1\right]$$

$$= 2 < \infty$$

∴ System is stable.

(b) Since $h(n) = 0$ for $n<0$, the system is causal.

Example 2.21 A continuous-time LTI system is represented by the impulse response,

$$h(t) = e^{-3t} u(t-1)$$

Determine whether it is (a) stable (b) causal

Solution : (a) We know that a continuous-time LTI system is stable only if its impulse i.e., $$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

$$\therefore \int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \int_{1}^{\infty} e^{-3\tau}\, d\tau$$

$$= \left.\frac{e^{-3\tau}}{-3}\right|_{1}^{\infty}$$

$$= e^{-3}/3 < \infty.$$

$\therefore$ The system is stable.

(b) A continuous-time LTI system is causal if $h(t) = 0$ for $t < 0$.
Since the given $h(t) = 0$ for $t<0$, the system is causal.

Example 2.22 Repeat example 2.21 for,

$$h(t) = e^{-t}\, u(t+100)$$

Solution :

(a) $$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \int_{-100}^{\infty} |e^{-\tau}|\, d\tau$$

$$= \left.\frac{e^{-\tau}}{-1}\right|_{-100}^{\infty}$$

$$= e^{100}$$

$$= 2.69 \times 10^{43} < \infty$$

$\therefore$ The system is stable.

(b) Since $h(t) \neq 0$ for $t<0$, the system is non-causal.

Example 2.23 Repeat example 2.21 for,

$$h(t) = e^{t}\, u(-1-t)$$

Solution : The plot of $h(t)$ is shown in Fig. P2.23.1.

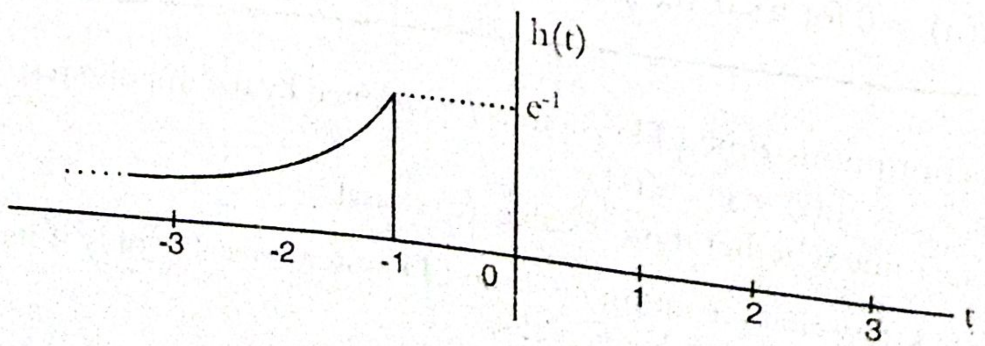


Fig. P2.23.1

(a) $$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \int_{-\infty}^{-1} |e^{\tau}|\, d\tau$$

$$= e^{\tau} \Big|_{-\infty}^{-1}$$

$$= e^{-1} < \infty$$

$\therefore$ the system is stable.

(b) Since $h(t) \neq 0$ for $t<0$, the system is non-causal.

Example 2.24 Repeat example 2.21 for,

$$h(t) = e^{-4|t|}$$

Solution : The plot of $h(t)$ is shown in Fig. P2.24.1.

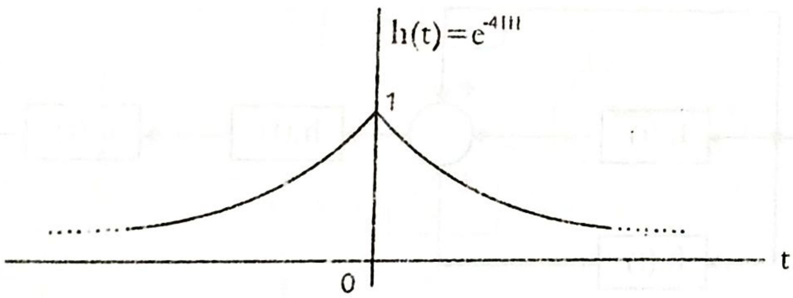


Fig. P2.24.1

(a) $$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \int_{-\infty}^{0} |e^{4\tau}|\, d\tau + \int_{0}^{\infty} |e^{-4\tau}| d\tau$$

$$= \frac{e^{4\tau}}{4} \Big|_{-\infty}^{0} + \frac{e^{-4\tau}}{-4} \Big|_{0}^{\infty}$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2} < \infty$$

$\therefore$ the system is stable.

(b) Since $h(t) \neq 0$ for $t<0$, the system is non-causal.

Example 2.25 Find the overall impulse response h(t) in terms of the impulse response of each subsystem for the system shown in Fig. P2.25.

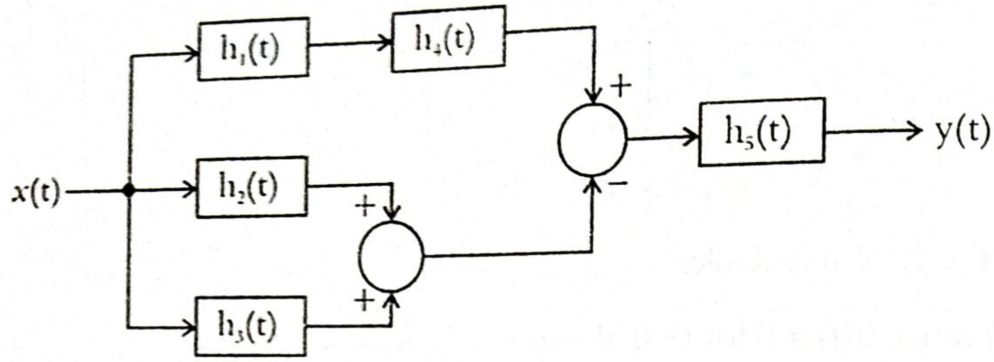


Fig. P2.25

Solution :

$$h(t) = \{[h_1(t) * h_4(t)] - [h_2(t) + h_3(t)]\} * h_5(t)$$

Example 2.26 Repeat example 2.25 for the system shown below in Fig. P2.26.

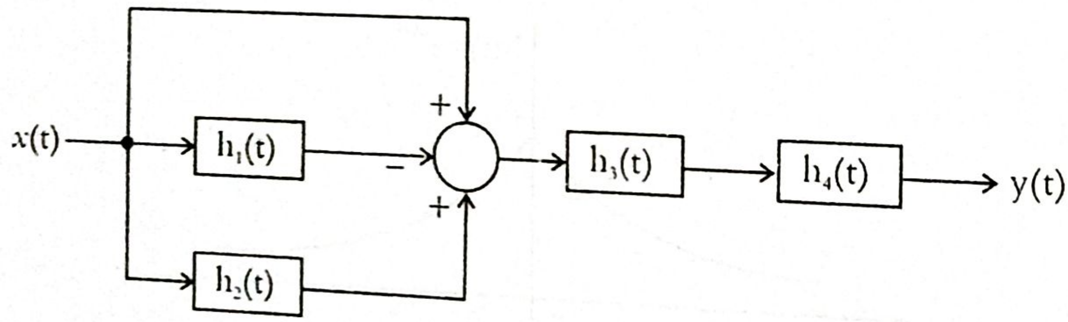


Fig. P2.26

Solution :

$$h(t) = [\delta(t) - h_1(t) + h_2(t)] * h_3(t) * h_4(t)$$

Example 2.27 Let $h_1(t)$, $h_2(t)$, $h_3(t)$ and $h_4(t)$ be the impulse responses of LTI system. Draw the interconnection of systems required to obtain the overall impulse response,

Solution :

$$h(t) = h_1(t) + \{h_2(t) + h_3(t)\} * h_4(t)$$

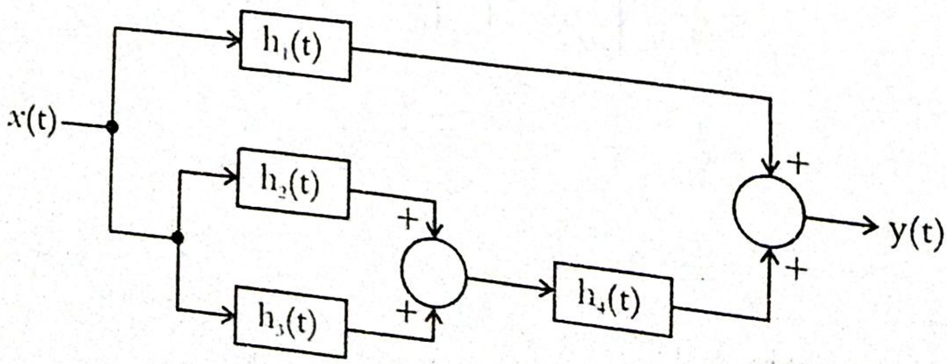


Fig. P2.27.1

Example 2.25 Find the overall impulse response h(t) in terms of the impulse response of each subsystem for the system shown in Fig. P2.25.

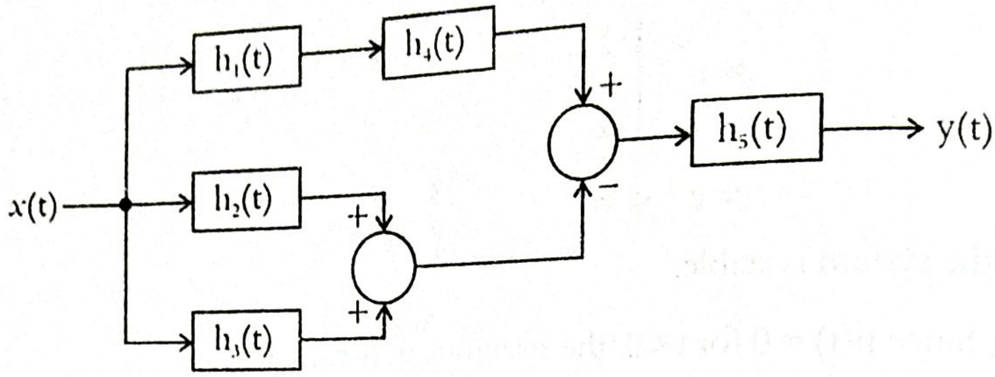


Fig. P2.25

Solution :

$$h(t) = \{[h_1(t) * h_4(t)] - [h_2(t) + h_3(t)]\} * h_5(t)$$

Example 2.26 Repeat example 2.25 for the system shown below in Fig. P2.26.

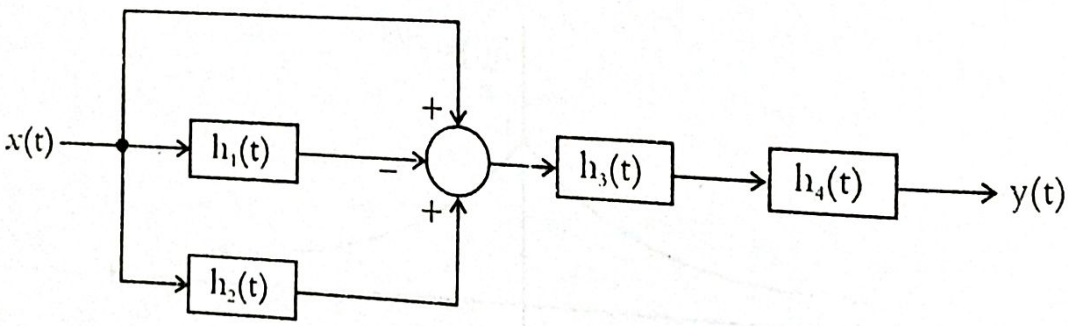


Fig. P2.26

Solution :

$$h(t) = [\delta(t) - h_1(t) + h_2(t)] * h_3(t) * h_4(t)$$

Example 2.27 Let $h_1(t)$, $h_2(t)$, $h_3(t)$ and $h_4(t)$ be the impulse responses of LTI system. Draw the interconnection of systems required to obtain the overall impulse response,

$$h(t) = h_1(t) + \{h_2(t) + h_3(t)\} * h_4(t)$$

Solution :

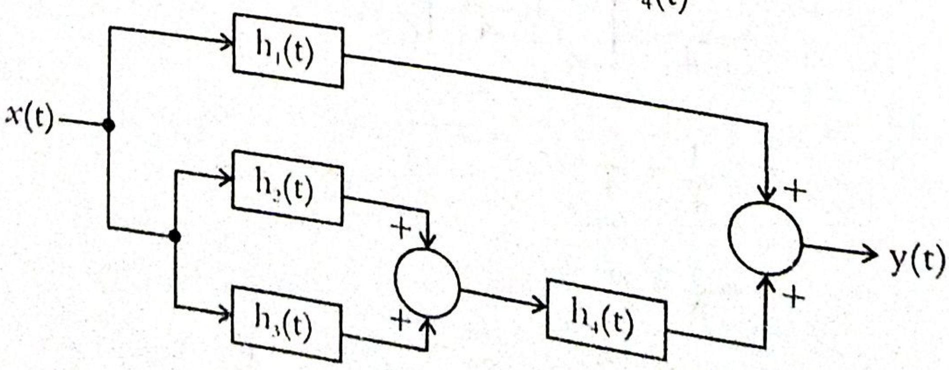


Fig. P2.27.1

Example 2.28 Repeat example 2.27 to obtain the overall impulse response,

$$h(t) = h_1(t) * \{h_2(t) + h_3(t) + h_4(t)\}$$

Solution :

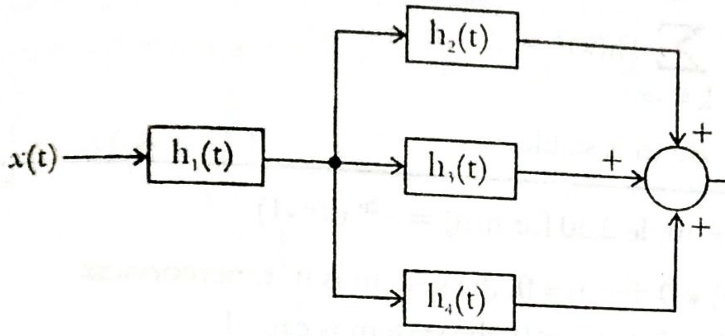


Fig. P2.28.1

Example 2.29 Repeat example 2.27 to obtain the overall impulse response,

$$h(t) = h_1(t) * h_2(t) + h_3(t) * h_4(t)$$

Solution :

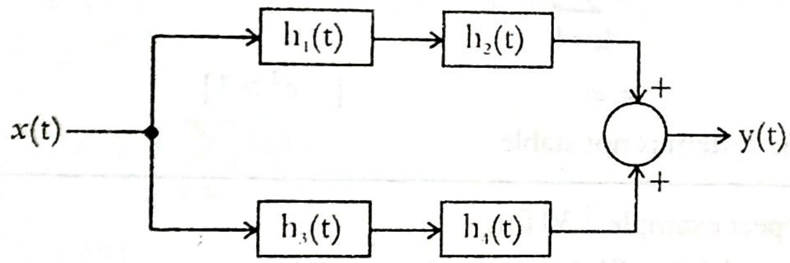


Fig. P2.29.1

Example 2.30 For the impulse response given below determine whether the corresponding system is (i) memoryless (ii) causal (iii) stable.

$$h(n) = 2u(n) - 2u(n-1)$$

Solution : (i) A discrete-time LTI system is memoryless if $h(n) = 0$ for $n \neq 0$.

The plot of $h(n)$ is shown in Fig. P2.30.1.

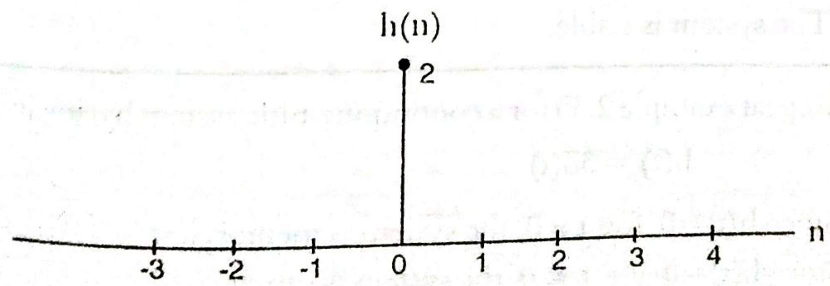


Fig. P2.30.1

Since $h(n) = 0$ for $n \neq 0$, the system is memoryless.

(ii) Since $h(n) = 0$ for $n < 0$, the system is causal.

(iii) We have $h(n) = 2u(n) - 2u(n-1) = 2\delta(n)$

$$\therefore \sum_{k=-\infty}^{\infty} |h(k)| = 2 < \infty$$

$\therefore$ The system is stable.

Example 2.31 Repeat example 2.30 for $h(n) = e^{2n}\, u(n-1)$

Solution : (i) Since $h(n) \neq 0$ for $n \neq 0$, the system is not memoryless.

(ii) Since $h(n) = 0$ for $n < 0$, the system is causal.

(iii) $$\therefore \sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=1}^{\infty} e^{2k}$$

$$= \sum_{k=1}^{\infty} (e^2)^k$$

$$= \infty \qquad [\because e^2 > 1]$$

$\therefore$ The system is not stable.

Example 2.32 Repeat example 2.30 for,

$$h(n) = \delta(n) + \sin(n\pi)$$

Solution : We have $h(n) = \delta(n) + \sin(n\pi)$

$\therefore h(n) = \delta(n)$. $\qquad [\because \sin(n\pi)=0]$

(i) Since $h(n) = 0$ for $n \neq 0$, the system is memoryless.

(ii) Since $h(n) = 0$ for $n < 0$, the system is causal.

(iii) $$\therefore \sum_{k=-\infty}^{\infty} |h(k)| = 1 < \infty$$

$\therefore$ The system is stable.

Example 2.33 Repeat example 2.30 for a continuous-time system having impulse response

$$h(t) = 3\delta(t)$$

Solution : (i) Since $h(t)=0$ for $t \neq 0$, the system is memoryless.

(ii) Since $h(t)=0$ for $t < 0$, the system is causal.

(iii) $$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = 3.$$ $\therefore$ The system is stable.

Example 2.34 Repeat example 2.33 for,

$$h(t) = e^{2t}\, u(t-1)$$

Solution : (i) Since $h(t) \neq 0$ for $t \neq 0$; the system is not memoryless.

(ii) Since $h(t) = 0$ for $t < 0$; the system is causal.

(iii) $$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \int_{1}^{\infty} |e^{2\tau}|\, d\tau$$

$$= \frac{e^{2\tau}}{2}\Bigg|_{1}^{\infty}$$

$$= \infty$$

$\therefore$ The system is unstable.

Example 2.35 Find the step response for the LTI system represented by the impulse response $h(n) = (½)^n\, u(n)$

Solution : For a discrete-time LTI system with impulse response $h(n)$, the step response $s(n)$ is given by,

$$s(n) = \sum_{k=-\infty}^{n} h(k)$$

For $n<0$; $s(n) = 0$

For $n\geq 0$ $\therefore$ $$s(n) = \sum_{k=0}^{n} (½)^k$$

$$s(n) = \frac{1-(½)^{n+1}}{1-½}$$

$$= 2[1-(½)^{n+1}] \qquad ; n \geq 0$$

$$\therefore\ s(n) = 2[1-(½)^{n+1}]\, u(n)$$

Example 2.36 Show that,

(a) $x(n) * \delta(n) = x(n)$

(b) $x(n) * \delta(n-n_o) = x(n-n_o)$

(c) $$x(n) * u(n) = \sum_{k=-\infty}^{n} x(k)$$

(d) $$x(n) * u(n-n_o) = \sum_{k=-\infty}^{n-n_o} x(k)$$

Solution :

We have $$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

(a) $$\therefore x(n) * \delta(n) = \sum_{k=-\infty}^{\infty} x(k)\, \delta(n-k)$$

$$= x(k)\Big|_{k=n} \qquad \text{[using sifting property]}$$

$$= x(n)$$

(b) $$x(n) * \delta(n-n_o) = \sum_{k=-\infty}^{\infty} x(k)\, \delta(n-n_o-k)$$

$$= x(k)\Big|_{k=n-n_o} \qquad \text{[using sifting property]}$$

$$= x(n-n_o)$$

(c) $$x(n) * u(n) = \sum_{k=-\infty}^{\infty} x(k)\, u(n-k) \qquad \text{..... (P2.36.1)}$$

We have $u(n-k) = 1$; when $n-k \geq 0$ i.e., $k \leq n$

$= 0$; when $n-k < 0$ i.e., $k > n$ ..... (P2.36.2)

Substituting eqn. P2.36.2 in eqn. P2.36.1 we get,

$$\therefore x(n) * u(n) = \sum_{k=-\infty}^{n} x(k)$$

(d) $$x(n) * u(n-n_o) = \sum_{k=-\infty}^{\infty} x(k)\, u(n-n_o-k) \qquad \text{..... (P2.36.3)}$$

We have $u(n-n_o-k) = 1$; when $n-n_o-k \geq 0$ i.e., $k \leq n-n_o$

$= 0$; when $n-n_o-k < 0$ i.e., $k > n-n_o$ ..... (P2.36.4)

Substituting eqn. P2.36.4 in eqn. P2.36.3 we get,

$$x(n) * u(n-n_o) = \sum_{k=-\infty}^{n-n_o} x(k)$$

Example 2.37 Show that,

(a) $x(t) * \delta(t) = x(t)$

(b) $x(t) * \delta(t-t_o) = x(t-t_o)$

(c) $x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

(d) $x(t) * u(t-t_o) = \int_{-\infty}^{t-t_o} x(\tau) d\tau$

Solution : We have,

$$x(t) * (h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

(a) $$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau) d\tau$$

$$= x(\tau)\Big|_{\tau=t} \qquad \text{[using sifting property]}$$

$$= x(t)$$

(b) $$x(t) * \delta(t-t_o) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-t_o-\tau) d\tau$$

$$= x(\tau)\Big|_{\tau=t-t_o} \qquad \text{[using sifting property]}$$

$$= x(t-t_o)$$

(c) $$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau) d\tau \qquad \text{..... (P2.37.1)}$$

We know that $u(t-\tau) = 1$; when $t-\tau \geq 0$ i.e., $\tau \leq t$

$= 0$; when $t-\tau < 0$ i.e., $\tau > t$ ..... (P2.37.2)

Substituting eqn. P2.37.2 in eqn. P2.37.1 we get,

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)$$

(d) $$x(t) * u(t-t_o) = \int_{-\infty}^{\infty} x(\tau)\, u(t-t_o-\tau) d\tau \qquad \text{..... (P2.37.3)}$$

We know that $u(t-t_o-\tau) = 1$; when $t-t_o-\tau \geq 0$ i.e., $\tau \leq t-t_o$

$= 0$; when $t-t_o-\tau < 0$ i.e., $\tau > t-t_o$ ..... (P2.37.4)

Substituting eqn. P2.37.4 in eqn. P2.37.3 we get,

$$x(t) * u(t-t_o) = \int_{-\infty}^{t-t_0} x(\tau)\, d\tau$$

Example 2.38 Find and sketch the step response for the LTI system characterized by the impulse response $h(n) = \delta(n) - \delta(n-1)$

Solution :

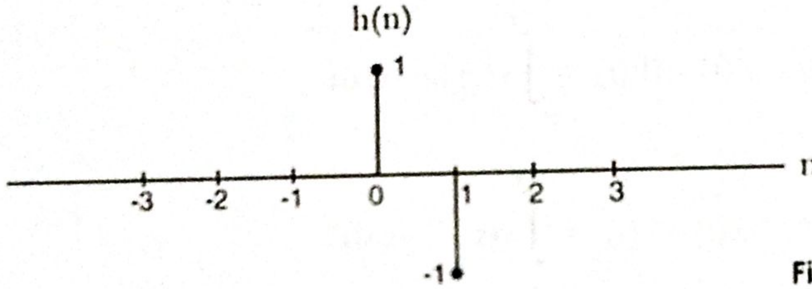


Fig. P2.38.1

The step response s(n) is given by the running sum of the impulse response. The step response is plotted in Fig. P2.38.2 below.

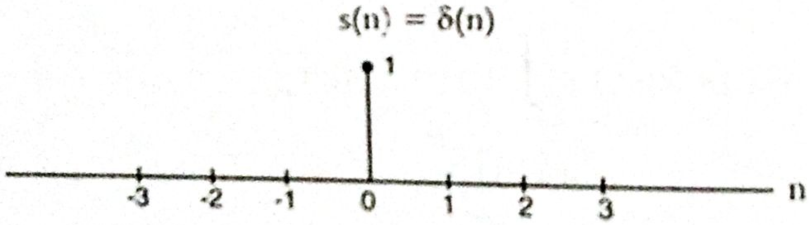


Fig. P2.38.2

$\therefore s(n) = \delta(n)$.

Example 2.39 Repeat example 2.38 for,

$$h(n) = u(n)$$

Solution : The plot of h(n) and s(n) are shown in Fig. P2.39.1 and Fig. P2.39.2 respectively.

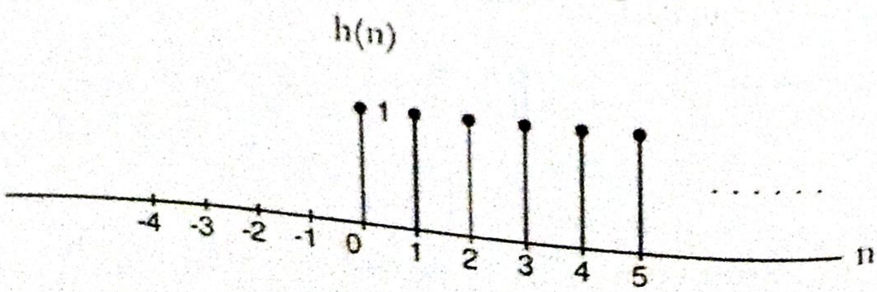


Fig. P2.39.1

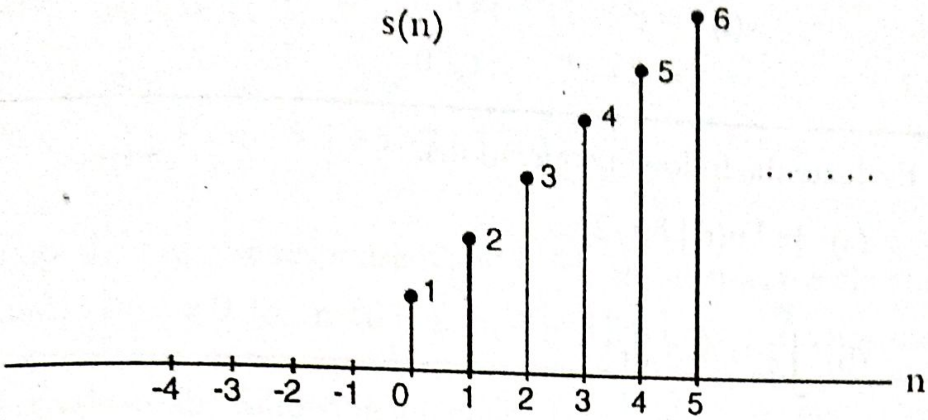


Fig. P2.39.2

$\therefore s(n) = (n+1)\,u(n).$

Example 2.40 Evaluate the step response for the LTI system represented by the impulse response

$$h(t) = t\,u(t)$$

Solution : We know that the step response is given by,

$$s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$$

$$= \int_{0}^{t} \tau\,d\tau$$

$$= \frac{\tau^2}{2}\Bigg|_{0}^{t}$$

$$= \frac{t^2}{2} \qquad ; t \geq 0$$

$\therefore$ The step response $s(t) = \dfrac{t^2}{2}u(t)$

Example 2.41 Repeat example 2.40 for,

$$h(t) = e^{-|t|}$$

Solution : For $t < 0$; $s(t) = \displaystyle\int_{-\infty}^{t} e^{\tau}\,d\tau = e^{t}$

For $t \geq 0$; $s(t) = \displaystyle\int_{-\infty}^{0} e^{\tau}\,d\tau + \int_{0}^{t} e^{-\tau}\,d\tau$

$$= 2 - e^{-t}$$

$$\therefore \quad s(t) = e^{t} \qquad ; t < 0$$
$$= 2 - e^{-t} \qquad ; t \geq 0$$

Example 2.42 Evaluate the following operations.

(a) $[e^{-t}u(t)]\,\delta(t-2)$

(b) $\int_{-\infty}^{\infty} e^{-t}\delta(t-2)dt$

(c) $e^{-t}u(t) * \delta(t-2)$

Solution :

(a) From sampling property we have,

$$x(t)\,\delta(t-t_o) = x(t_o)\,\delta(t-t_o)$$
$$\therefore [e^{-t}u(t)]\,\delta(t-2) = e^{-2}u(2)\,\delta(t-2)$$
$$= e^{-2}.\,\delta(t-2)$$

(b) From sifting property, we have,

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_o)dt = x(t_o)$$

$$\therefore \int_{-\infty}^{\infty} e^{-t}\,\delta(t-2)dt = e^{-2}$$

(c) We have $\quad x(t) * \delta(t-t_o) = x(t-t_o)$

$$\therefore \quad e^{-t}u(t) * \delta(t-2) = e^{-(t-2)}u(t-2)$$

Example 2.43 For a discrete time LTI system, the input and output are related by the equation

$$y(n) = x(n+1)+5x(n) - 7x(n-1)+4x(n-2)$$

(a) Find the impulse response of the system.

(b) Also comment on the stability and causality of the system.

Solution : Given : $y(n) = x(n+1)+5x(n) - 7x(n-1)+4x(n-2)$ ..... (P2.43.1)

Eqn. P2.43.1 can be written as,

$$y(n) = x(n) * [\delta(n+1)+5\delta(n) - 7\delta(n-1)+4\delta(n-2)] \quad ..... \text{(P2.43.2)}$$

Comparing eqn. P2.43.2 with $y(n) = x(n) * h(n)$ we get,

$$h(n) = \delta(n+1)+5\delta(n) - 7\delta(n-1)+4\delta(n-2)$$

$$\therefore\ h(n) = \{1, \underset{\uparrow}{5}, -7, 4\}$$

$$\sum_{k=-\infty}^{\infty} |h(k)| = 1+5+7+4 = 17 < \infty$$

Since the impulse response is absolutely summable, the system is stable.

Since $h(n) \neq 0$ for $n<0$ [eg., $h(-1)=1$], the system is non-causal.

Example 2.44 Evaluate the following integrals.

(a) $$\int_{-\infty}^{\infty} (t^2+\cos\pi t)\ \delta(t-1)dt$$

(b) $$\int_{-\infty}^{\infty} e^{-t}\ \delta(t)dt$$

(c) $$\int_{-\infty}^{\infty} e^{-t}\ \delta(2t-2)dt$$

Solution : (a) From sifting property we have,

$$\int_{-\infty}^{\infty} x(t)\ \delta(t-t_o)dt = x(t_o)$$

$$\therefore\ \int_{-\infty}^{\infty} (t^2+\cos\pi t)\ \delta(t-1)dt = (t^2+\cos\pi t)\big|_{t=1} = 0$$

(b) Also we have,

$$\int_{-\infty}^{\infty} x(t)\ \delta(t)dt = x(0)$$

$$\therefore\ \int_{-\infty}^{\infty} e^{-t}\ \delta(t)dt = e^{-t}\big|_{t=0} = 1$$

(c) Given $$\int_{-\infty}^{\infty} e^{-t}\ \delta(2t-2)dt$$

$$= \int_{-\infty}^{\infty} e^{-t}\ \delta(2(t-1))dt \qquad \text{..... (P2.44.1)}$$

Put $t-1=m$; $dt=dm$

$\therefore$ From eqn. P2.44.1 we get,

$$= \int_{-\infty}^{\infty} e^{-(m+1)}\,\delta(2m)dm$$

$$= \frac{e^{-1}}{2}\int_{-\infty}^{\infty} e^{-m}\,\delta(m)dm \qquad \left[\because \delta(at) = \frac{1}{a}\delta(t) \quad ; a>0\right]$$

$$= \frac{e^{-1}}{2}\left[e^{-m}\Big|_{m=0}\right] \qquad \left[\because \int_{-\infty}^{\infty} x(t)\,\delta(t)dt = x(0)\right]$$

$$= \frac{e^{-1}}{2}$$

Note : Convolution property of two delayed unit impulse sequences is given by,

$$\delta(n-l) * \delta(n-m) = \delta(n-l-m)$$

Example 2.45 Determine the convolution of two given sequences

$$x(n) = \{1, \underset{\uparrow}{2}, 3, 4\} \text{ and } h(n) = \{\underset{\uparrow}{1}, 1, 3, 2\}$$

Solution : Given that

$$x(n) = \{1, 2, 3, 4\} = \delta(n+1) + 2\delta(n) + 3\delta(n-1) + 4\delta(n-2)$$
$$h(n) = \{1, 1, 3, 2\} = \delta(n) + \delta(n-1) + 3\delta(n-2) + 2\delta(n-3)$$
$$\therefore x(n)*h(n) = [\delta(n+1) + 2\delta(n) + 3\delta(n-1) + 4\delta(n-2)] * [\delta(n) + \delta(n-1) + 3\delta(n-2) + 2\delta(n-3)]$$

Using convolution property of two delayed unit impulse sequences we get,

$$x(n)*h(n) = \delta(n+1) + \delta(n) + 3\delta(n-1) + 2\delta(n-2) + 2\delta(n) + 2\delta(n-1) + 6\delta(n-2) + 4\delta(n-3) + 3\delta(n-1) + 3\delta(n-2) + 9\delta(n-3) + 6\delta(n-4) + 4\delta(n-2) + 4\delta(n-3) + 12\delta(n-4) + 8\delta(n-5)$$

$$= \delta(n+1) + 3\delta(n) + 8\delta(n-1) + 15\delta(n-2) + 17\delta(n-3) + 18\delta(n-4) + 8\delta(n-5)$$

$$\therefore x(n)*h(n) = \{1, \underset{\uparrow}{3}, 8, 15, 17, 18, 8\}$$

2.4 DIFFERENTIAL / DIFFERENCE EQUATION REPRESENTATION FOR LTI SYSTEMS

This is another type of time-domain representation for LTI systems. It gives the relationship between the input and the output of LTI systems. *Differential equation* is used to represent continuous-time system whereas *difference equation* is used to represent discrete-time system.

2.4.1 Differential equation representation for continuous-time LTI systems

The general form of a linear constant-coefficient differential equation is,

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t) \qquad ..... \quad (2.24)$$

where $x(t)$ is the input to the system and $y(t)$ is the output of the system. The integer 'N' in eqn. 2.24 is known as the *order* of the differential equation. The expression for the output $y(t)$ (i.e., the solution) described by a differential equation has 2 components.

(i) ***The natural response*** : This is the output due to the initial conditions and is denoted as $y^{(n)}(t)$. It is also known as *zero-input* response.

(ii) ***The forced response*** : This is due to only the input and denoted as $y^{(f)}(t)$. It is also known as *zero-state* response.

The natural response is the system output with no input whereas the forced response is the system output for zero initial conditions.

(i) The Natural Response : $y^{(n)}(t)$ [Zero-Input Response]

This is the output of the system with zero input. Therefore, the general form of a linear constant-coefficient differential equation with $x(t) = 0$ becomes,

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y^{(n)}(t) = 0$$

The differential equation with input equal to zero is known as *homogeneous equation.* The natural response for a continuous time system is of the form,

$$y^{(n)}(t) = \sum_{i=1}^{N} C_i e^{r_i t} \qquad ...... \quad (2.25)$$

where r_i are the N roots of the characteristic equation given by,

$$\sum_{k=0}^{N} a_k r^k = 0$$

Note : (i) If any root r_i repeats 'm' times, then we include 'm' distinct terms in the natural response as below.

$$e^{r_i t},\ te^{r_i t},\ t^2 e^{r_i t},\ ......,\ t^{m-1} e^{r_i t}.$$

(ii) The nature of each term in the natural response depends on the roots of the characteristic equation [i.e., 'r_i']. If r_i are real, the natural response consists of exponential terms. Imaginary roots results in sinusoidal terms and complex roots leads to exponentially damped sinusoids.

Example 2.46 Find the natural response for the system described by the differential equation,

$$5\frac{dy(t)}{dt} + 10\,y(t) = 2\,x(t) \qquad ; y(0)=3$$

Solution : Given :

$$5\frac{dy(t)}{dt} + 10\,y(t) = 2\,x(t) \qquad ; y(0)=3$$

To find the natural response, we have to consider input $x(t)=0$ (i.e., get homogeneous equation)

$$\therefore \quad 5\frac{dy(t)}{dt} + 10\,y(t) = 0 \qquad \text{..... P2.46.1}$$

To obtain the characteristic equation replace $\frac{d^k y(t)}{dt^k} = r^k$

∴ From eqn. P2.46.1, we get the characteristic equation,

$$5r + 10 = 0$$

$$\therefore r = -2.$$

∴ The natural response (using eqn. 2.25)

$$y^{(n)}(t) = Ce^{-2t} \qquad \text{..... P2.46.2}$$

Substituting $y(0)=3$ in eqn. P2.46.2 we get,

$$y(0)=3 = C.e^{0}$$

$$\therefore C=3.$$

∴ The natural response $y^{n}(t) = 3e^{-2t}$

Example 2.47 Find the zero input response of the system described by the differential equation.

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t) + \frac{dx(t)}{dt}$$

$$y(0) = 0 \qquad ; \qquad \left.\frac{dy(t)}{dt}\right|_{t=0} = 1$$

Solution : Natural response is the solution of homogeneous equation [i.e., with $x(t)=0$]

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 0$$

∴ Characteristic equation is,

$$r^2 + 3r + 2 = 0$$

∴ The roots of characteristic equation are,

$$r_1 = -1 \quad , \quad r_2 = -2$$

Since both roots are real, the natural response is of the form,

$$y^{(n)}(t) = C_1e^{r_1t} + C_2e^{r_2t}$$

$$\therefore\ y^{n}(t) = C_1e^{-t} + C_2e^{-2t} \qquad \text{..... P2.47.1}$$

$$\therefore\ \frac{dy^{(n)}t}{dt} = -C_1e^{-t} - 2C_2e^{-2t} \qquad \text{..... P2.47.2}$$

Put $y(0) = 0$ in eqn. P2.47.1 we get,

$$0 = C_1+C_2 \qquad \text{..... P2.47.3}$$

Put $\left.\frac{dy(t)}{dt}\right|_{t=0} = y'(0)=1$ in eqn. P2.47.2 we get,

$$1= -C_1 - 2C_2 \qquad \text{..... P2.47.4}$$

Solving eqn. P2.47.3, and P2.47.4 we get,

$$C_2 = -1$$

$$\therefore\ C_1 = 1$$

$\therefore$ The natural response is,

$$y^{(n)}(t) = e^{-t} - e^{-2t}$$

Example 2.48 Solve the following homogeneous differential equation with the specified initial conditions.

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 0$$

with $\quad y(0)=0 \quad ; y'(0)=0$

Solution : The solution of a homogeneous differential equation is nothing but the natural response. (i.e., response due to initial conditions). Since all the initial conditions are zero, the natural response is zero.

i.e., $\quad y^{(n)}(t)=0$ (i.e., system is at rest).

Example 2.49 Solve the following homogeneous differential equation with the initial conditions specified.

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = 0$$

with $y(0)=1 \quad ; y'(0)=1$

Solution : Given :

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = 0$$

$\therefore$ Characteristic equation.

$$r^2+2r+1=0$$

$$(r+1)^2 = 0$$

$r_1 = -1$ and $r_2 = -1$ i.e., $r_1 = r_2 = -1$

Since the roots are repeated and both the roots are real, the natural response is of the form,

$$y^{(n)}(t) = C_1 e^{r_1 t} + C_2\, t e^{r_1 t}$$

$$y^{(n)}(t) = C_1 e^{-t} + C_2 t e^{-t} \quad \text{..... P2.49.1}$$

$$\therefore \quad \frac{dy^{(n)}t}{dt^n} = -C_1 e^{-t} + C_2[-te^{-t} + e^{-t}] \quad \text{..... P2.49.2}$$

Put $y(0) = 1$ in eqn. P2.49.1 we get,

$$1 = C_1 \quad \text{..... P2.49.3}$$

Put $y'(0) = 1$ in eqn. P2.49.2 we get,

$$1 = -C_1 + C_2 \quad \text{..... P2.49.4}$$

Solving eqn. P2.49.3 and P2.49.4 we get,

$$C_2 = 2$$

$\therefore$ The natural response is,

$$y^{(n)}(t) = e^{-t} + 2te^{-t}$$

Example 2.50 Determine the natural response for the system described by the following difference equation,

$$\frac{d^2y(t)}{dt^2} + 4\,y(t) = 3\frac{dx(t)}{dt}$$

$$\text{with } y(0) = -1, \quad \left.\frac{dy(t)}{dt}\right|_{t=0} = 1$$

Solution : Homogeneous equation,

$$\frac{d^2y(t)}{dt^2} + 4\,y(t) = 0$$

Characteristic equation,

$$r^2 + 4 = 0$$

$$\therefore r = \pm j2$$

$$\therefore r_1 = j2 \text{ and } r_2 = -j2$$

Since the roots are non-repeated and purely imaginary, the natural response is of the form,

$$y^{(n)}(t) = C_1 \cos 2t + C_2 \sin 2t \quad \text{..... P2.50.1}$$

$$\therefore \frac{dy^{(n)}(t)}{dt} = -2C_1 \sin 2t + 2C_2 \cos 2t \quad \text{..... P2.50.2}$$

Put $y(0) = -1$ in eqn. P2.50.1 we get,

$$-1 = C_1 \quad \text{..... P2.50.3}$$

Put $\left.\frac{dy(t)}{dt}\right|_{t=0} = y'(0) = 1$ in eqn. P2.50.2 we get,

$$1 = 2C_2$$

$$\therefore C_2 = ½$$

$$\therefore C_1 = -1 \text{ and } C_2 = ½$$

∴ The natural response is,

$$y^{(n)}(t) = -\cos 2t + \frac{1}{2}\sin 2t$$

Example 2.51 Find the natural response for the system described by differential equation,

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$$

with $\quad y(0) = 1, \quad \left.\frac{dy(t)}{dt}\right|_{t=0} = 0$

Solution : Homogeneous equation,

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = 0$$

∴ Characteristic equation is,

$$r^2 + 2r + 2 = 0$$

$$\therefore r = -1 \pm j1$$

$$\therefore r_1 = -1+j1 \;\&\; r_2 = -1-j1$$

Since the roots are non-repeated and complex, the natural response is of the form,

$$y^{(n)}(t) = e^{-t}\{C_1\cos(1.t) + C_2\sin(1.t)\} \quad \text{...... P2.51.1}$$

$$\therefore \frac{dy^{n}(t)}{dt} = e^{-t}\{-C_1\sin t + C_2\cos t\} - e^{-t}\{C_1\cos t + C_2\sin t\} \quad \text{...... P2.51.2}$$

Put $y(0)=1$ in eqn. P2.51.1 we get,

$$1 = C_1$$

Put $\left.\frac{dy(t)}{dt}\right|_{t=0} = y'(0)=0$ in eqn. P2.51.2 we get,

$$0 = C_2 - C_1$$

$$\therefore C_2 = 1$$

∴ The natural response is,

$$y^{(n)}(t) = e^{-t}\{\cos t + \sin t\}$$

(ii) The Forced Response : $y^{(f)}(t)$ [Zero-State Response]

The forced response $y^{(f)}(t)$ is the solution of the differential equation for the given input with initial conditions are zero. It has two components.

(i) a term resembling the natural response $y^{(n)}(t)$.

(ii) particular solution $y^{(p)}(t)$.

The particular solution for the given input is obtained by assuming the system output has the same form as the input.

Eg., If the input is $x(t) = Ae^{-at}$, then we assume that the particular solution is of the form $y^{(p)}(t) = Ke^{-at}$. Table 2.1 gives the form of a particular solution corresponding to several common inputs. The constants 'K' are determined such that $y^{(p)}(t)$ satisfies the differential equation of the system.

Input : $x(t)$	Particular solution : $y^{(p)}(t)$
A (constant)	K
Ae^{-at}	Ke^{-at}
$A\cos(\omega t+\phi)$	$K_1 \cos\omega t + K_2 \sin\omega t$

Table 2.1

Note : When the input has the form of one of the components in the natural response, then we must write a particular solution that is independent of all terms in the natural response. For example, if the natural response consists the term e^{-at} and te^{-at}, then for the input $x(t) = e^{-at}$ the particular solution is of the form $y^{(p)}(t) = Kt^2 e^{-at}$

Examples

Example 2.52 Determine the forced response for the system given by,

$$5\frac{dy(t)}{dt} + 10y(t) = 2x(t)$$

with input $x(t) = 2\,u(t)$

Solution : Given : $5\dfrac{dy(t)}{dt} + 10y(t) = 2x(t)$ P2.52.1

The forced response is due to input only. It has 2 terms : (i) a term resembling the natural response (ii) particular solution.

$$y^{(f)}(t) = y^{(n)}(t) + y^{(p)}(t)$$

Characteristic equation,

$$5r+10=0$$

$$r = -2$$

$$\therefore y^{(n)}(t) = Ce^{-2t}$$

Particular solution is of the form of input $x(t)$. P2.52.2

The given $x(t)=2u(t)$ is constant. Therefore the particular solution of the form.

$$y^{(p)}(t) = K$$ P2.52.3

Substituting eqn. P2.52.3 in eqn. P2.52.1 we get,

$$5.\frac{d(K)}{dt} + 10K = 2x(t) = (2).2\,u(t)$$

$$0 + 10K = 4u(t)$$

$$10K = 4(1)$$

$$K = 2/5$$

$\therefore$ Forced Response $\quad y^{(f)}(t) = Ce^{-2t} + 2/5$ P2.52.4

For finding forced response we assume initial conditions are zero. i.e., $y(0)=0$

$\therefore$ From eqn. P2.52.4 we get,

$$0 = C.e^{-0} + 2/5$$

$$C = -2/5$$

$$\therefore y^{(f)}(t) = -2/5\, e^{-2t} + 2/5$$

Forced Response : $y^{(f)}(t) = 2/5\,(1-e^{-2t}) \quad ; t \geq 0$

Example 2.53 Find the zero-state response for the system given by,

$$5\frac{dy(t)}{dt} + 10y(t) = 2x(t)$$

with input $x(t) = e^{-t}\,u(t)$

Solution : Given : $5\frac{dy(t)}{dt} + 10y(t) = 2x(t)$ P2.53.1

$$y^{(f)}(t) = y^{(n)}(t) + y^{(p)}(t)$$

Characteristic equation,

$$5r + 10 = 0$$

$$r = -2$$

$\therefore y^{(n)}(t) = Ce^{-2t}$ P2.53.2

Since $x(t) = e^{-t}\,u(t)$, particular solution is of the form,

$y^{(p)}(t) = Ke^{-t}$ P2.53.3

Substituting eqn. P2.53.3 in eqn. P2.53.1 we get,

$$-5K\,e^{-t} + 10Ke^{-t} = 2\,.\,e^{-t}$$

$$-5K + 10K = 2$$

$$K = 2/5$$

$\therefore y^{(f)}(t) = Ce^{-2t} + 2/5\,e^{-t}$ P2.53.4

Assume initial condition $y(0)=0$. From eqn. P2.53.4 we get,

$$0 = C + 2/5$$

$$\therefore \quad C = -2/5$$

$\therefore$ Forced Response : $\quad y^{(f)}(t) = 2/5\,(e^{-t} - e^{-2t})$

Example 2.54 Solve $\dfrac{d^2y(t)}{dt^2} + 9y(t) = x(t)$

where $x(t) = \cos 4t$

Solution : Given : $\dfrac{d^2y(t)}{dt^2} + 9y(t) = x(t)$ P2.54.1

$$y^{(f)}(t) = y^{(n)}(t) + y^{(p)}(t)$$

Characteristic equation $r^2+9=0$

$$\therefore r = \pm j3 \text{ (imaginary roots)}$$

$$\therefore y^{(n)}(t) = C_1 \cos 3t + C_2 \sin 3t \qquad \text{....... P2.54.2}$$

Since $x(t) = \cos 4t$, the particular solution is of the form

$$y^{(p)}(t) = K_1 \cos 4t + K_2 \sin 4t \qquad \text{....... P2.54.3}$$

Substituting eqn. P2.54.3 in eqn. P2.54.1 we get,

$$\frac{d^2}{dt^2}[K_1 \cos 4t + K_2 \sin 4t] + 9[K_1 \cos 4t + K_2 \sin 4t] = \cos 4t$$

$$-16K_1 \cos 4t - 16K_2 \sin 4t + 9K_1 \cos 4t + 9K_2 \sin 4t = \cos 4t$$

$$-7K_1 \cos 4t - 7K_2 \sin 4t = \cos 4t$$

Comparing LHS and RHS we get,

$-7K_1=1 \qquad \therefore K_1 = -1/7$

$-7K_2=0 \qquad \therefore K_2 = 0$

$$\therefore y^{(f)}(t) = C_1 \cos 3t + C_2 \sin 3t - 1/7 \cos 4t \qquad \text{....... P2.54.4}$$

Put initial condition $y(0) = 0$ and $\left.\dfrac{dy(t)}{dt}\right|_{t=0} = 0$

From eqn. P2.54.4 we get,

$0 = C_1 - 1/7 \qquad \therefore C_1 = 1/7$

$0 = 3C_2 \qquad \therefore C_2=0$

$\therefore$ The forced response is,

$$\therefore y^{(f)}(t) = 1/7 \cos 3t - 1/7 \cos 4t$$

$$y^{(f)}(t) = 1/7 [\cos 3t - \cos 4t]$$

Example 2.55 Find the forced response for the system described by,

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = 2x(t) + \frac{dx(t)}{dt}$$

with input $x(t) = 2e^{-t}u(t)$

Solution : Given : $\dfrac{d^2y(t)}{dt^2} + 5\dfrac{dy(t)}{dt} + 6y(t) = 2x(t) + \dfrac{dx(t)}{dt}$ P2.55.1

$$y^{(f)}(t) = y^{(n)}(t)+y^{(p)}(t)$$

Characteristic equation $r^2+5r+6=0$

$$\therefore r_1 = -3 \text{ and } r_2 = -2$$

$$y^{(n)}(t) = C_1e^{-3t} + C_2e^{-2t} \qquad \text{....... P2.55.2}$$

Since $x(t) = 2e^{-t}\,u(t)$, the particular solution is of the form,

$$y^{(p)}(t) = K\,e^{-t} \qquad \text{...... P2.55.3}$$

Substituting eqn. P2.55.3 in eqn. P2.55.1 we get,

$$\frac{d^2}{dt^2}(Ke^{-t}) + 5\frac{d}{dt}(Ke^{-t}) + 6.Ke^{-t} = 4e^{-t} - 2e^{-t}$$

$$Ke^{-t} - 5Ke^{-t} + 6Ke^{-t} = 2e^{-t}$$

$$2Ke^{-t} = 2e^{-t}$$

$$K = 1$$

$$\therefore y^{(f)}(t) = C_1e^{-3t} + C_2e^{-2t} + e^{-t} \qquad \text{....... P2.55.4}$$

Assume initial conditions are zero i.e., $y(0)=0$; $\left.\frac{dy(t)}{dt}\right|_{t=0} = 0$

From eqn. P2.55.4 we get,

$$\therefore 0 = C_1+C_2+1$$

$$0 = -3C_1-2C_2-1$$

Solving, we get $C_1=1$ and $C_2= -2$

The forced response is,

$$\therefore y^{(f)}(t) = e^{-3t} - 2e^{-2t} + e^{-t}$$

Example 2.56 Find the forced response for the system given by the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t) + \frac{dx(t)}{dt}$$

with input $x(t) = 5\,u(t)$

Solution : Given : $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t) + \frac{dx(t)}{dt}$ P2.56.1

$$y^{(f)}(t) = y^{(n)}(t)+y^{(p)}(t)$$

Characteristic equation $r^2+3r+2=0$

$$\therefore r_1 = -1 \text{ and } r_2 = -2$$

$$\therefore y^{(n)}(t) = C_1e^{-t} + C_2e^{-2t} \qquad \text{....... P2.56.2}$$

Since $x(t) = 5\,u(t)$, the particular solution is of the form,

$$y^{(p)}(t) = K \qquad \text{....... P2.56.3}$$

Substituting eqn. P2.56.3 in eqn. P2.56.1 we get,

$$\frac{d^2}{dt^2}[K] + 3\frac{d}{dt}[K] + 2K = 5+0$$

$$0 + 0 + 2K = 5$$

$$K = 5/2$$

$$\therefore y^{(f)}(t) = C_1e^{-t} + C_2e^{-2t} + 5/2 \qquad \text{....... P2.56.4}$$

Assuming initial conditions are zero [i.e., y(0)=0 & y'(0) = 0]

From eqn. P2.56.4 we get,

$$\therefore 0 = C_1 + C_2 + 5/2$$

$$0 = -C_1 - 2C_2$$

$\therefore$ Solving, we get $C_1 = -5$ and $C_2 = 5/2$

$$\therefore y^{(f)}(t) = -5e^{-t} + 5/2\, e^{-2t} + 5/2 = 5/2\,[1 + e^{-2t} - 2e^{-t}]$$

Example 2.57 Find the forced response for the system given by,

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt} \text{ with input } x(t) = 2e^{-t}u(t)$$

Solution : Given : $\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt}$ P2.57.1

$$y^{(f)}(t) = y^{(n)}(t) + y^{(P)}(t)$$

Characteristic equation $r^2 + 2r + 1 = 0$

$$\therefore (r+1)^2 = 0$$

$$\therefore r_1 = r_2 = -1 \text{ (Repeated roots)}$$

$$\therefore y^{(n)}(t) = C_1e^{-t} + C_2te^{-t} \qquad \text{....... P2.57.2}$$

The input $x(t) = 2e^{-t}u(t)$, resembles the term in the equation corresponding to $y^{(n)}(t)$

$$\therefore y^{(P)}(t) = Kt^2e^{-t} \qquad \text{....... P2.57.3}$$

Substituting eqn. P2.57.3 in eqn. P2.57.1, we get,

$$\frac{d^2}{dt^2}(Kt^2e^{-t}) + 2\frac{d}{dt}(Kt^2e^{-t}) + Kt^2e^{-t} = -2e^{-t}$$

Solving, we get $K = -1$

$$\therefore y^{(f)}(t) = C_1e^{-t} + C_2te^{-t} - t^2e^{-t} \qquad \text{....... P2.57.4}$$

Assuming zero initial conditions, [i.e., y(0)=0 & y'(0) = 0)]

From eqn. P2.57.4 we get,

$$\therefore 0 = C_1$$

$$0 = -C_1 + C_2 + 0 \qquad \therefore C_2 = 0$$

$\therefore$ The forced response is,

$$y^{(f)}(t) = -t^2e^{-t}$$

(iii) The Complete Response : y(t)

The *complete response* or the *total response* of the system is the sum of the natural response and the forced response i.e., to determine the complete response we must take both the initial conditions and the input into consideration.

Examples

Example 2.58 Find the output of the system given by the differential equation,

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt}$$

$$y(0) = 0 \quad ; \quad \left.\frac{dy(t)}{dt}\right|_{t=0} = 1 \text{ and } x(t) = e^{-2t}u(t)$$

Solution : Given : $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt}$ P2.58.1

$$y(t) = y^{(n)}(t) + y^{(P)}(t)$$

Characteristic equation $r^2 + 5r + 4 = 0$

$$\therefore r_1 = -4 \text{ and } r_2 = -1$$

$\therefore y^{(n)}(t) = C_1e^{-4t} + C_2e^{-t}$ P2.58.2

Since $x(t) = 2e^{-2t}u(t)$, the particular solution is of the form,

$y^{(P)}(t) = Ke^{-2t}$ P2.58.3

Substituting eqn. P2.58.3 in eqn. P2.58.1 we get,

$$\frac{d^2}{dt^2}[Ke^{-2t}] + 5\frac{d}{dt}[Ke^{-2t}] + 4Ke^{-2t} = -2e^{-2t}$$

$$4Ke^{-2t} - 10Ke^{-2t} + 4Ke^{-2t} = -2e^{-2t}$$

$$\therefore K = 1$$

$\therefore y(t) = C_1e^{-4t} + C_2e^{-t} + e^{-2t}$ P2.58.4

Initial conditions, $y(0)=0 \quad ; \left.\frac{dy(t)}{dt}\right|_{t=0} = 1$

From eqn. P2.58.4 we get,

$$\therefore 0 = C_1 + C_2 + 1$$

$$1 = -4C_1 - C_2 - 2$$

Solving we get,

$$C_1 = -2/3 \text{ and } C_2 = -1/3$$

$\therefore$ The complete response is,

$$y(t) = -2/3\, e^{-4t} - 1/3\, e^{-t} + e^{-2t}$$

$$\therefore y(t) = e^{-2t} - 1/3\,(2e^{-4t} + e^{-t})$$

Example 2.59 Find the total response of the system given by,

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 2x(t)$$

$$\text{with } y(0) = -1 \; ; \left.\frac{dy(t)}{dt}\right|_{t=0} = 1 \text{ and } x(t) = \cos t\, u(t)$$

Solution : Given : $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 2x(t)$ P2.59.1

$$y(t) = y^{(n)}(t) + y^{(P)}(t)$$

Characteristic equation : $r^2 + 3r + 2 = 0$

$$\therefore r_1 = -2 \text{ and } r_2 = -1$$

$$\therefore y^{(n)}(t) = C_1e^{-2t} + C_2e^{-t}$$ P2.59.2

Since $x(t) = \cos t\, u(t)$, the particular solution is of the form,

$$y^{(P)}(t) = K_1 \cos t + K_2 \sin t$$ P2.59.3

Substituting eqn. P2.59.3 in eqn. P2.59.1 we get,

$$\frac{d^2}{dt^2}[K_1\cos t+K_2\sin t] + 3\frac{d}{dt}[K_1\cos t+K_2\sin t] + 2[K_1\cos t+K_2\sin t] = 2\cos t$$

$$-K_1\cos t - K_2\sin t - 3K_1\sin t + 3K_2\cos t + 2K_1\cos t + 2K_2\sin t = 2\cos t$$

$$(K_1+3K_2)\cos t + (K_2-3K_1)\sin t = 2\cos t$$

Comparing LHS and RHS we get,

$$K_1+3K_2 = 2$$

$$\& \quad K_2 - 3K_1 = 0$$

Solving we get, $K_1 = 1/5$ and $K_2 = 3/5$

$$y(t) = C_1e^{-2t} + C_2e^{-t} + 1/5 \cos t + 3/5 \sin t$$ P2.59.4

Initial conditions, $y(0)=-1 \; ; \left.\frac{dy(t)}{dt}\right|_{t=0} = 1$

From eqn. P2.59.4 we get,

$$-1 = C_1 + C_2 + 1/5$$

$$1 = -2C_1 - C_2 + 3/5$$

Solving we get,

$$C_1 = 4/5 \text{ and } C_2 = -2$$

$\therefore$ The response is,

$$y(t) = 4/5\, e^{-2t} - 2e^{-t} + 1/5 \cos t + 3/5 \sin t$$

$$y(t) = 1/5[\cos t + 3\sin t + 4e^{-2t} - 10e^{-t}]$$

Example 2.60 Obtain the response of the system given by,

$$\frac{d^2y(t)}{dt^2} + y(t) = 3\frac{dx(t)}{dt}$$

$$\text{with } y(0) = -1 \quad ; \left.\frac{dy(t)}{dt}\right|_{t=0} = y'(0) = 1 \text{ and } x(t) = 2e^{-t}u(t)$$

Solution : Given : $\frac{d^2y(t)}{dt^2} + y(t) = 3\frac{dx(t)}{dt}$ P2.60.1

$$y(t) = y^{(n)}(t) + y^{(p)}(t)$$

Characteristic equation : $r^2 + 1 = 0$

$$r = \pm j1$$

$\therefore y^{(n)}(t) = C_1\cos t + C_2\sin t$ P2.60.2

Since $x(t) = 2e^{-t}u(t)$, the particular solution is of the form,

$y^{(p)}(t) = Ke^{-t}$ P2.60.3

Substituting eqn. P2.60.3 in eqn. P2.60.1 we get,

$$\frac{d^2}{dt^2}[Ke^{-t}] + Ke^{-t} = -6e^{-t}$$

$$2Ke^{-t} = -6e^{-t}$$

$$\therefore K = -3$$

$\therefore y(t) = C_1\cos t + C_2\sin t - 3e^{-t}$ P2.60.4

Initial conditions are, $y(0) = -1 \quad ; \left.\frac{dy(t)}{dt}\right|_{t=0} = y'(0) = 1$

From eqn. P2.60.4 we get,

$$-1 = C_1 - 3 \qquad \therefore C_1 = 2$$

$$1 = C_2 + 3 \qquad \therefore C_2 = -2$$

$\therefore$ The response,

$$y(t) = 2\cos t - 2\sin t - 3e^{-t}$$

i.e., $y(t) = 2(\cos t - \sin t) - 3e^{-t}$

2.4.2 Difference equation representation for discrete-time LTI systems

The general form of a linear constant-coefficient difference equation is,

$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k)$ .... (2.26)

where $x(n)$ is the input to the system and $y(n)$ is the output of the system. The integer N in eqn. 2.26 is known as the *order* of the difference equation. The expression for the output y(n) (i.e., the solution of the difference equation) of a system described by a difference equation has 2 components.

(i) ***The natural response*** : This is the output associated with the initial conditions and denoted as $y^{(n)}(n)$. It is also known as *zero-input* response.

(ii) ***The forced response*** : This is output due to only the input and denoted as $y^{(f)}(n)$. It is also known as *zero-state* response.

The natural response is the system output with no input whereas the forced response is the system output for zero initial conditions.

(i) The Natural Response : $y^{(n)}(n)$ [Zero-Input Response]

This is the output of the system with zero input. Therefore the general form of a linear constant-coefficient difference equation with $x(n) = 0$ becomes,

$$\sum_{k=0}^{N} a_k y(n-k) = 0$$

The difference equation with input equal to zero is known as *homogeneous equation*.

The natural response for a discrete-time system is of the form,

$$y^{(n)}(n) = \sum_{i=1}^{N} C_i r_i^n \qquad \ldots\ldots \quad (2.27)$$

where r_i's are the N roots of the characteristic equation given by,

$$\sum_{k=0}^{N} a_k r^{-k} = 0$$

Note : If any root r_i repeats 'p' times, then we include 'p' distinct terms in the natural response as below.

$$r_i^n,\ n r_i^n,\ n^2 r_i^n,\ \ldots,\ n^{p-1} r_i^n$$

Examples

Example 2.61 Find the natural response for the system described by the following difference equation.

$$y(n) - \frac{9}{16} y(n-2) = x(n-1)$$

with $y(-1) = 1$ and $y(-2) = -1$

the output y(n) (i.e., the solution of the difference equation) of a system described by a difference equation has 2 components.

(i) ***The natural response*** : This is the output associated with the initial conditions and denoted as $y^{(n)}(n)$. It is also known as *zero-input* response.

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The natural response is the system output with no input whereas the forced response is the system output for zero initial conditions.

(i) The Natural Response : $y^{(n)}(n)$ [Zero-Input Response]

This is the output of the system with zero input. Therefore the general form of a linear constant-coefficient difference equation with $x(n) = 0$ becomes,

$$\sum_{k=0}^{N} a_k y(n-k) = 0$$

The difference equation with input equal to zero is known as *homogeneous equation.*

The natural response for a discrete-time system is of the form.

$$y^{(n)}(n) = \sum_{i=1}^{N} C_i r_i^{\,n} \qquad \text{......} \quad (2.27)$$

where r_i's are the N roots of the characteristic equation given by,

$$\sum_{k=0}^{N} a_k r^{-k} = 0$$

Note : If any root r_i repeats 'p' times, then we include 'p' distinct terms in the natural response as below.

$$r_i^{\,n},\ n\, r_i^{\,n},\ n^2 r_i^{\,n},\ \ldots,\ n^{p-1} r_i^{\,n}$$

Examples

Example 2.61 Find the natural response for the system described by the following difference equation.

$$y(n) - \frac{9}{16} y(n-2) = x(n-1)$$

with $y(-1) = 1$ and $y(-2) = -1$

Solution : To solve a N^{th} order difference equation, we need 'N' initial conditions. The given system is of order of 2 and 2 initial conditions y(-1) and y(-2) are given. To find the natural response we have to consider input $x(n) = 0$ (get homo-geneous equation).

$$\therefore y(n) - \frac{9}{16}y(n-2) = 0 \qquad \text{...... P2.61.1}$$

To obtain the characteristic equation replace $y(n-k) = r^{-k}$

$\therefore$ From eqn. P2.61.1, we get the characteristic equation,

$$1 - \frac{9}{16}r^{-2} = 0$$

$$\therefore r = \pm 3/4$$

i.e., $r_1 = 3/4$ and $r_2 = -3/4$

$\therefore$ The natural response is of the form (using eqn. 2.27)

$$y^{(n)}(n) = C_1 (r_1)^n + C_2(r_2)^n$$

$$y^{(n)}(n) = C_1(3/4)^n + C_2(-3/4)^n \qquad \text{....... P2.61.2}$$

From eqn. P2.61.1,

$$y(n) = \frac{9}{16}y(n-2) \qquad \text{....... P2.61.3}$$

Using eqn. P2.61.2, eqn. 2.61.3 and the initial conditions we get,

$$y(0) = \frac{9}{16}y(-2) = \frac{9}{16}(-1) = C_1 + C_2$$

$$y(1) = \frac{9}{16}y(-1) = \frac{9}{16}(1) = \frac{3}{4}C_1 - \frac{3}{4}C_2$$

Solving we get, $C_1 = \frac{3}{32}$ and $C_2 = \frac{-21}{32}$

$\therefore$ The natural response is,

$$y^{(n)}(n) = \frac{3}{32}\left(\frac{3}{4}\right)^n - \frac{21}{32}\left(-\frac{3}{4}\right)^n \quad ; n \geq 0$$

Example 2.62 Find the natural response of the system described by difference equation,

$$y(n) - 1/4\, y(n-1) - 1/8\, y(n-2) = x(n) + x(n-1)$$

with y(-1) = 0 and y(-2) = 1

Solution : Homogeneous equation is,

$$y(n) - 1/4\, y(n-1) - 1/8\, y(n-2) = 0 \qquad \text{....... P2.62.1}$$

$\therefore$ Characteristic equation is,

$$1 - 1/4\, r^{-1} - 1/8\, r^{-2} = 0$$

$$r^2 - {}^1\!/\!_4 r - {}^1\!/\!_8 = 0$$

$$r_1 = {}^1\!/\!_2 \text{ and } r_2 = -{}^1\!/\!_4$$

The roots are non-repeated and real.

$\therefore$ The natural response is of the form,

$$y^{(n)}(n) = C_1\,({}^1\!/\!_2)^n + C_2\,(-{}^1\!/\!_4)^n \qquad \text{....... P2.62.2}$$

From eqn. P2.62.1 we get,

$$y(n) = {}^1\!/\!_4\, y(n-1) + {}^1\!/\!_8\, y(n-2) \qquad \text{....... P2.62.3}$$

Using eqn. P2.62.2, eqn. P2.62.3 and the initial conditions we get,

$$y(0) = {}^1\!/\!_4\, y(-1) + {}^1\!/\!_8\, y(-2) = {}^1\!/\!_4\,(0) + {}^1\!/\!_8\,(1) = {}^1\!/\!_8 = C_1 + C_2$$

$$y(1) = {}^1\!/\!_4\, y(0) + {}^1\!/\!_8\, y(-1) = {}^1\!/\!_4\,({}^1\!/\!_8) + {}^1\!/\!_8\,(0) = {}^1\!/\!_{32} = {}^1\!/\!_2\, C_1 - {}^1\!/\!_4\, C_2$$

Solving we get, $C_1 = {}^1\!/\!_{12}$ and $C_2 = {}^1\!/\!_{24}$

$\therefore$ The natural response is,

$$y^{(n)}(n) = {}^1\!/\!_{12}\,({}^1\!/\!_2)^n + {}^1\!/\!_{24}\,(-{}^1\!/\!_4)^n \quad ; n \geq 0$$

Example 2.63 Find the zero-input response of the system described by the homogeneous difference equation,

$$y(n) - 3y(n-1) - 4y(n-2) = 0$$

with initial conditions $y(-1) = 5$ and $y(-2) = 0$

Solution : Homogeneous equation is,

$$y(n) - 3y(n-1) - 4y(n-2) = 0 \qquad \text{....... P2.63.1}$$

$\therefore$ Characteristic equation is,

$$1 - 3r^{-1} - 4r^{-2} = 0$$

$$r^2 - 3r - 4 = 0$$

$$\therefore r_1 = -1 \text{ and } r_2 = 4$$

The roots are non-repeated and real. $\therefore$ The natural response is of the form,

$$y^{(n)}(n) = C_1(-1)^n + C_2\,(4)^n \qquad \text{....... P2.63.2}$$

From eqn. P2.63.1 we get,

$$y(n) = 3y(n-1) + 4y(n-2) \qquad \text{....... P2.63.3}$$

Using eqn. P2.63.2, eqn. 2.63.3 and the initial conditions we get,

$$y(0) = 3y(-1) + 4y(-2) = 3(5) + 4(0) = 15 = C_1 + C_2$$

$$y(1) = 3y(0) + 4y(-1) = 3(15) + 4(5) = 65 = -C_1 + 4C_2$$

Solving we get $C_1 = -1$ and $C_2 = 16$

$\therefore$ The natural response is,

$$y^{(n)}(n) = (-1)\,(-1)^n + 16(4)^n$$

$$\therefore y^{(n)}(n) = (-1)^{n+1} + (4)^{n+2} \quad ; n \geq 0$$

Example 2.64 Find the zero-input response for the system described by the difference equation,

$$y(n) + \frac{9}{16}y(n-2) = x(n-1)$$

with initial conditions $y(-1) = 1$ and $y(-2) = -1$

Solution : Homogeneous equation is,

$$y(n) + \frac{9}{16}y(n-2) = 0 \qquad \text{......} \quad \text{P2.64.1}$$

$\therefore$ Characteristic equation is,

$$1 + \frac{9}{16}r^{-2} = 0$$

$$r^2 + \frac{9}{16} = 0$$

$$r = \pm j\,3/4$$

$$\therefore\ r_1 = j\,3/4 \text{ and } r_2 = -j\,3/4$$

The roots are non-repeated and imaginary. $\therefore$ The natural response is of the form,

$$y^{(n)}(n) = C_1(j\,3/4)^n + C_2(-j\,3/4)^n \qquad \text{......} \quad \text{P2.64.2}$$

From eqn. P2.64.1 we get,

$$y(n) = -\frac{9}{16}\,y(n-2) \qquad \text{......} \quad \text{P2.64.3}$$

Using eqn. P2.64.2, eqn. 2.64.3 and the initial conditions we get,

$$y(0) = -\frac{9}{16}\,y(-2) = \frac{-9}{10}(-1) = \frac{9}{16} = C_1 + C_2$$

$$y(1) = \frac{-9}{16}\,y(-1) = \frac{-9}{16}(1) = \frac{-9}{16} = j\frac{3}{4}C_1 - j\frac{3}{4}C_2$$

Solving, $C_1 = \frac{9}{32} + j\frac{3}{8}$ and $C_2 = \frac{9}{32} - j\frac{3}{8}$

$\therefore$ The natural response is,

$$y^{(n)}(n) = \left(\frac{9}{32} + j\frac{3}{8}\right)\left(j\frac{3}{4}\right)^n + \left(\frac{9}{32} - j\frac{3}{8}\right)\left(-j\frac{3}{4}\right)^n$$

Example 2.65 Solve the homogeneous difference equation,

$$y(n) + y(n-1) + \tfrac{1}{2}\,y(n-2) = 0$$

with $y(-1) = -1$ and $y(-2) = 1$

Solution : Homogeneous equation is,

$$y(n) + y(n-1) + \tfrac{1}{2}\,y(n-2) = 0 \qquad \text{......} \quad \text{P2.65.1}$$

$\therefore$ Characteristic equation is,

$$1 + r^{-1} + ½ r^{-2} = 0$$

$$r^2 + r + ½ = 0$$

$$r = \frac{-1 \pm \sqrt{1-2}}{2} = \frac{-1 \pm j}{2}$$

$$\therefore r_1 = \frac{-1+j}{2} \text{ and } r_2 = \frac{-1-j}{2}$$

The roots are non-repeated and complex. $\therefore$ The natural response is of the form,

$$y^{(n)}(n) = C_1\left(\frac{-1+j}{2}\right)^n + C_2\left(\frac{-1-j}{2}\right)^n \qquad \text{....... P2.65.2}$$

From eqn. P2.65.1 we get,

$$y(n) = -y(n-1) - ½\, y(n-2) \qquad \text{....... P2.65.3}$$

Using eqn. P2.65.2, eqn. 2.65.3 and the initial conditions we get,

$$y(0) = -y(-1) - ½\, y(-2) = 1 - ½ = ½ = C_1 + C_2$$

$$y(1) = -y(0) - ½\, y(-1) = -½ + ½ = 0 = C_1\left(\frac{-1+j}{2}\right) + C_2\left(\frac{-1-j}{2}\right)$$

Solving we get, $C_1 = ¼\,(1-j)$ and $C_2 = ¼\,(1+j)$

$\therefore$ The natural response is,

$$y^{(n)}(n) = ¼\,(1-j)\left(\frac{-1+j}{2}\right)^n + ¼\,(1+j)\left(\frac{-1-j}{2}\right)^n$$

(ii) The Forced Response : $y^{(f)}(n)$ [Zero-State Response]

The forced response $y^{(f)}(n)$ is the solution to the difference equation for the given input with initial conditions are zero. It has two components.

(i) a term resembling the natural response $y^{(n)}(n)$.

(ii) particular solution $y^{(p)}(n)$.

The particular solution for the given input is obtained by assuming the system output has the same form as the input.

Eg., If the input is $x(n) = A\alpha^n$, then we assume the particular solution is of the form $y^{(p)}(n) = K\alpha^n$. Table 2.2 gives the form of a particular solution corresponding to several common inputs. The constants 'K' are determined such that $y^{(p)}(n)$ satisfies the difference equation of the system.

Input : $x(n)$	Particular solution : $y^{(p)}(n)$
A (constant)	K
$A\alpha^n$	$K\alpha^n$
$A\cos(\Omega n+\phi)$	$K_1\cos(\Omega n) + K_2\sin(\Omega n)$

Table 2.2

Note : When the input has the form of one of the components in the natural response, then we must write a particular solution that is independent of all terms in the natural response. For eg., if the natural response consists the term 'α^n' and '$n\alpha^n$' then for the input $x(n) = \alpha^n$ the particular solution is of the form $y^{(p)}(n) = Kn^2\alpha^n$

Examples

Example 2.66 Find the forced response for the system given by the difference equation,

$$y(n) - \tfrac{1}{4}y(n-1) - \tfrac{1}{8}y(n-2) = x(n)+x(n-1)$$

with input $x(n) = (\tfrac{1}{8})^n u(n)$

Solution : Given :

$$y(n) - \tfrac{1}{4}y(n-1) - \tfrac{1}{8}y(n-2) = x(n)+x(n-1) \quad \text{......} \quad P2.66.1$$

i.e., $$y(n) = \tfrac{1}{4}y(n-1) + \tfrac{1}{8}y(n-2) + x(n)+x(n-1) \quad \text{......} \quad P2.66.2$$

$$y^{(f)}(n) = y^{(n)}(n) + y^{(p)}(n)$$

Characteristic equation,

$$1 - \tfrac{1}{4}r^{-1} - \tfrac{1}{8}r^{-2} = 0$$

$$r^2 - \tfrac{1}{4}r - \tfrac{1}{8} = 0$$

$$r_1 = -\tfrac{1}{4} \text{ and } r_2 = \tfrac{1}{2}$$

$$\therefore y^{(n)}(n) = C_1(-\tfrac{1}{4})^n + C_2(\tfrac{1}{2})^n \quad \text{......} \quad P2.62.3$$

Since $x(n) = (\tfrac{1}{8})^n u(n)$, the particular solution is of the form,

$$y^{(p)}(n) = K(\tfrac{1}{8})^n u(n) \quad \text{......} \quad P2.66.4$$

Substituting eqn. P2.66.4 in eqn. P2.66.1, we get,

$$K(\tfrac{1}{8})^n u(n) - \tfrac{1}{4}K(\tfrac{1}{8})^{n-1}u(n-1) - \tfrac{1}{8}K(\tfrac{1}{8})^{n-2}u(n-2) = (\tfrac{1}{8})^n u(n) + (\tfrac{1}{8})^{n-1}u(n-1)$$

While determining the value of K, none of the term should vanish. That is possible only for $n \geq 2$.

Multiplying both the sides by $(\tfrac{1}{8})^{-n}$ we get,

$$K - \tfrac{1}{4}K(\tfrac{1}{8})^{-1} - \tfrac{1}{8}K(\tfrac{1}{8})^{-2} = 1 + (\tfrac{1}{8})^{-1}$$

$$K[1 - 2 - 8] = 9$$

$$K = -1$$

$$\therefore y^{(f)}(n) = C_1(-\tfrac{1}{4})^n + C_2(\tfrac{1}{2})^n - (\tfrac{1}{8})^n \quad \text{......} \quad P2.66.5$$

Assume initial conditions are zero. [i.e., $y(-1) = 0$; $y(-2) = 0$)],

Using eqn. P2.66.2 and eqn. P2.66.5 we get,

$$y(0) = \tfrac{1}{4}y(-1) + \tfrac{1}{8}y(-2) + 1 + 0 = 1 = C_1 + C_2 - 1$$

$$y(1) = \tfrac{1}{4}y(0) + \tfrac{1}{8}y(-1) + \tfrac{1}{8} + 1 = \tfrac{11}{8} = -\tfrac{1}{4}C_1 + \tfrac{1}{2}C_2 - \tfrac{1}{8}$$

Solving we get, $C_1 = -\tfrac{2}{3}$ and $C_2 = \tfrac{8}{3}$

$\therefore$ The forced response is,

$$y^{(f)}(n) = -\tfrac{2}{3}(-\tfrac{1}{4})^n + \tfrac{8}{3}(\tfrac{1}{2})^n - (\tfrac{1}{8})^n \qquad ; n \geq 0$$

Example 2.67 Determine the zero-state response for the system described by the difference equation,

$$y(n) - \tfrac{5}{6}\,y(n-1) + \tfrac{1}{6}\,y(n-2) = x(n)$$

where the forcing function $x(n) = 2^n$; $n \geq 0$ and zero elsewhere.

Solution : Given : $y(n) - \tfrac{5}{6}\,y(n-1) + \tfrac{1}{6}\,y(n-2) = x(n)$ P2.67.1

$$y(n) = \tfrac{5}{6}\,y(n-1) - \tfrac{1}{6}\,y(n-2) + x(n) \qquad \text{...... P2.67.2}$$

$$y^{(f)}(n) = y^{(h)}(n) + y^{(p)}(n)$$

Characteristic equation,

$$1 - \tfrac{5}{6}r^{-1} + \tfrac{1}{6}r^{-2} = 0$$

$$r^2 - \tfrac{5}{6}r + \tfrac{1}{6} = 0$$

$$\therefore r_1 = \tfrac{1}{3} \text{ and } r_2 = \tfrac{1}{2}$$

$$\therefore y^{(h)}(n) = C_1(\tfrac{1}{3})^n + C_2(\tfrac{1}{2})^n \qquad \text{...... P2.67.3}$$

Since $x(n) = 2^n\,u(n)$, the particular solution is of the form,

$$y^{(p)}(n) = K\,2^n\,u(n) \qquad \text{...... P2.67.4}$$

Substituting eqn. P2.67.4 in eqn. P2.67.1 we get,

$$K\,2^n - \tfrac{5}{6}K\,2^{n-1} + \tfrac{1}{6}K\,2^{n-2} = 2^n \quad ; n \geq 2$$

$$K - \tfrac{5}{6}K(2)^{-1} + \tfrac{1}{6}K(2)^{-2} = 1$$

$$K\,[1 - \tfrac{5}{12} + \tfrac{1}{24}] = 1$$

$$K = \tfrac{8}{5}$$

$$\therefore y^{(f)}(n) = C_1(\tfrac{1}{3})^n + C_2(\tfrac{1}{2})^n + \tfrac{8}{5}\,.\,2^n \qquad \text{...... P2.67.5}$$

Assume initial conditions are zero.

Using eqn. P2.67.2 and eqn. P2.67.5 we get,

$$y(0) = \tfrac{5}{6}\,y(-1) - \tfrac{1}{6}\,y(-2) + 1 = 1 = C_1 + C_2 + \tfrac{8}{5}$$

$$y(1) = \tfrac{5}{6}\,y(0) - \tfrac{1}{6}\,y(-1) + 2 = \tfrac{17}{6} = \tfrac{1}{3}C_1 + \tfrac{1}{2}C_2 + \tfrac{16}{5}$$

Solving we get, $C_1 = \tfrac{2}{5}$ and $C_2 = -1$

$\therefore$ The forced response is,

$$y^{(f)}(n) = (\tfrac{2}{5})\,(\tfrac{1}{3})^n - 1\,(\tfrac{1}{2})^n + \tfrac{8}{5}\,.\,2^n \qquad ; n \geq 0$$

Example 2.68 Find the forced response for the system described by the difference equation

$$y(n) + y(n-1) + \tfrac{1}{2}\,y(n-2) = x(n) + 2x(n-1)$$

with input $x(n) = u(n)$.

Solution : Given : $y(n) + y(n-1) + {}^1/_2\, y(n-2) = x(n) + 2x(n-1)$

i.e., $y(n) = -y(n-1) - {}^1/_2\, y(n-2) + x(n) + 2x(n-1)$ P2.68.1

$$y^{(f)}(n) = y^{(n)}(n) + y^{(p)}(n) \quad \text{..... P2.68.2}$$

Characteristic equation is,

$$1 + r^{-1} + {}^1/_2\, r^{-2} = 0$$

$$r^2 + r + {}^1/_2 = 0$$

$$r = \frac{-1 \pm j}{2}$$

$$\therefore\ y^{(n)}(n) = C_1\left(\frac{-1+j}{2}\right)^n + C_2\left(\frac{-1-j}{2}\right)^n \quad \text{...... P2.68.3}$$

Since $x(n) = u(n)$, the particular solution is of the form,

$$y^{(P)}(n) = K\,u(n) \quad \text{...... P2.68.4}$$

Substituting eqn. P2.68.4 in eqn. P2.68.1 we get,

$$K + K + {}^1/_2\, K = 1+2 \qquad ; n \geq 2$$

$$2.5K = 3$$

$$\therefore\ K = \frac{6}{5}$$

$$\therefore\ y^{(f)}(n) = C_1\left(\frac{-1+j}{2}\right)^n + C_2\left(\frac{-1-j}{2}\right)^n + \frac{6}{5} \quad \text{...... P2.68.5}$$

Assume initials conditions are zero.

Using eqn. P2.68.2 and eqn. P2.68.5 we get,

$$y(0) = -y(-1) - {}^1/_2\, y(-2) + 1 + 0 = 1 = C_1 + C_2 + {}^6/_5$$

$$y(1) = -y(0) - {}^1/_2\, y(-1) + 1 + 2 = 2 = C_1\left(\frac{-1+j}{2}\right) + C_2\left(\frac{-1-j}{2}\right) + \frac{6}{5}$$

Solving we get, $C_1 = -{}^1/_{10} - j\,{}^7/_{10}$ and $C_2 = -{}^1/_{10} + j\,{}^7/_{10}$

$\therefore$ The forced response is,

$$y^{(f)}(n) = (-{}^1/_{10} - j\,{}^7/_{10})\left(\frac{-1+j}{2}\right)^n + (-{}^1/_{10} + j\,{}^7/_{10})\left(\frac{-1-j}{2}\right)^n + \frac{6}{5} \qquad ; n \geq 0$$

(iii) The Complete Response : y(n)

The *complete response* or the *total response* of the system is the sum of the natural response and the forced response, i.e., to determine the complete response we must take both the initial conditions and the input into consideration.

Examples

Example 2.69 Find the response of the system described by the difference equation,

$$y(n) - {}^1/_9\, y(n-2) = x(n-1)$$

with $y(-1)=1$, $y(-2)=0$ and $x(n) = u(n)$

Solution : Given : $y(n) - {}^1\!/_9\, y(n-2) = x(n-1)$ P2.69.1

$$y(n) = {}^1\!/_9\, y(n-2) + x(n-1) \qquad \text{..... P2.69.2}$$

$$y(n) = y^{(h)}(n) + y^{(P)}(n)$$

Characteristic equation,

$$1 - {}^1\!/_9\, r^{-2} = 0$$

$$r^2 = {}^1\!/_9$$

$$r = \pm\, {}^1\!/_3$$

$$\therefore\ r_1 = {}^1\!/_3 \text{ and } r_2 = -{}^1\!/_3$$

$$\therefore\ y^{(h)}(n) = C_1({}^1\!/_3)^n + C_2(-{}^1\!/_3)^n \qquad \text{..... P2.69.3}$$

Since $x(n) = u(n)$, the particular solution is of the form,

$$y^{(P)}(n) = K\, u(n) \qquad \text{..... P2.69.4}$$

Substituting eqn. P2.69.4 in eqn. P2.69.1 we get,

$$K - {}^1\!/_9\, K = 1$$

$$\therefore\ K = {}^9\!/_8$$

$$\therefore\ y(n) = C_1({}^1\!/_3)^n + C_2(-{}^1\!/_3)^n + \frac{9}{8} \qquad \text{...... P2.69.5}$$

We have initial conditions $y(-1) = 1$ and $y(-2) = 0$

Using eqn. P2.69.2 and eqn. P2.69.5 we get,

$$y(0) = {}^1\!/_9\, y(-2) + x(-1) = 0 = C_1 + C_2 + \frac{9}{8}$$

$$y(1) = {}^1\!/_9\, y(-1) + x(0) = {}^{10}\!/_9 = {}^1\!/_3\, C_1 - {}^1\!/_3\, C_2 + \frac{9}{8}$$

Solving we get $C_1 = \frac{-7}{12}$ and $C_2 = \frac{-13}{24}$

$\therefore$ The response is,

$$y(n) = \frac{-7}{12}\left(\frac{1}{3}\right)^n - \frac{13}{24}\left(\frac{-1}{3}\right)^n + \frac{9}{8} \qquad ;\ n \geq 0$$

2.5 BLOCK DIAGRAM REPRESENTATIONS

The impulse response and differential/difference equation representations we studied so far would provide only the input-output behaviour of the system. But it doesnot provide any descriptions about the different internal operations performed by the system. In this section, we will discuss about another time-domain representation for system known as *block diagram representation*.

Block diagram representation is a pictorial representation which describes the different set of internal computations used to determine the output from the input. Block diagram representation has a great significance because it helps in the implementation of the system using computer. For example, the block diagram representation for a continuous-time system is the basis for analog computer simulation of systems and it can be directly translated into a program for the simulation of such a system on a digital computer. Also the representations suggests a simple and efficient ways to implement the systems using digital computers.

2.5.1 For discrete-time system

Let us discuss the block diagram representation of a discrete-time LTI system described by difference equation. The 3 elementary operations on signals are,

(i) ***Scalar multiplication***

$$x(n) \xrightarrow{a} y(n) = ax(n)$$

(ii) ***Addition***

$$x(n) \rightarrow \boxed{\Sigma} \rightarrow z(n) = x(n) + y(n)$$

$$\uparrow$$

$$y(n)$$

'Σ' can also be written as '$\oplus$'

(iii) ***Time-shift***

$$x(n) \rightarrow \boxed{S} \rightarrow y(n)=x(n-1)$$

'$\boxed{S}$' can also be written as '$\boxed{D}$'

'S' corresponds to shift and 'D' corresponds to delay.

Consider a block diagram representation for a discrete-time system as shown in Fig. 2.16

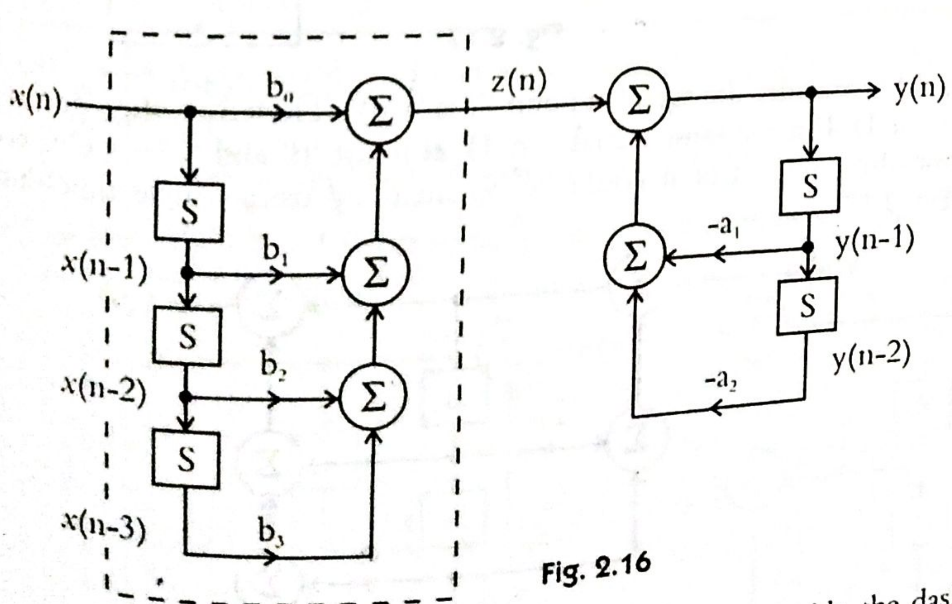

Fig. 2.16

Writing the equation for the portion of the system shown inside the dashed line we get,

$$z(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + b_3 x(n-3) \qquad \cdots\cdots (2.28)$$

Similarly, for the remaining part we get,

$$y(n) = z(n) - a_1 y(n-1) - a_2 y(n-2) \qquad \cdots\cdots (2.29)$$

Substituting eqn. 2.28 in eqn. 2.29 we get,

$$y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + b_3 x(n-3) - a_1 y_1(n-1) - a_2 y(n-2)$$

$$\therefore y(n) + a_1 y_1(n-1) + a_2 y(n-2) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + b_3 x(n-3) \quad \cdots\cdots (2.30)$$

The eqn. 2.30 is the difference equation corresponding to the system shown in Fig. 2.16.

The block diagram representation is not unique. Since convolution is associative, i.e., consider two systems having impulse response $h_1(n)$ and $h_2(n)$ are connected in cascade. We may interchange their order without changing the input-output relation of the system. Therefore the system shown in Fig. 2.16 can be written as shown in Fig. 2.17.

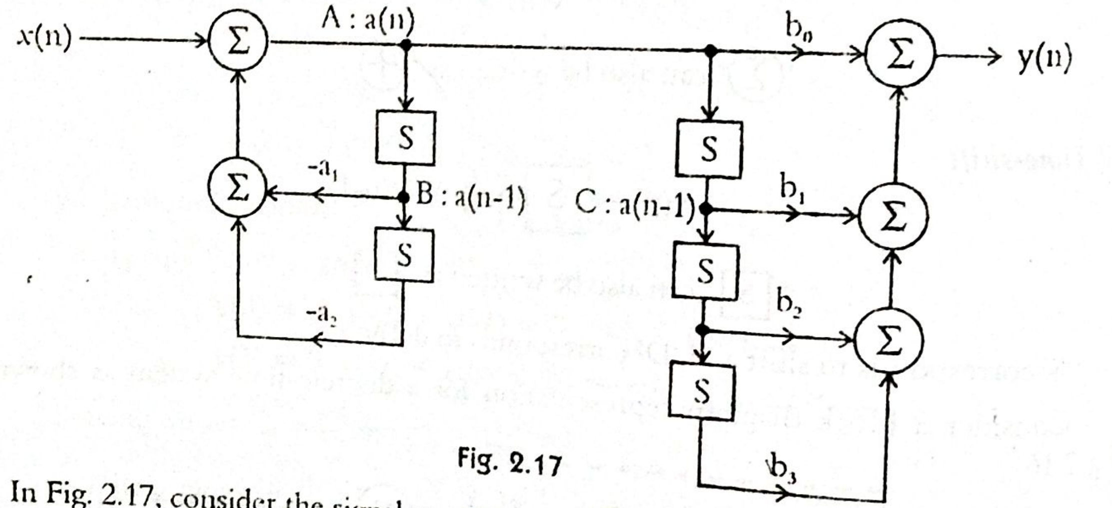

Fig. 2.17

In Fig. 2.17, consider the signal at point 'A' is 'a(n)'. Therefore, the signal at point 'B' and 'C' are a(n-1). But the same signal 'a(n-1)' at point 'B' and 'C' are obtained by two separate time-shift units. But it could be obtained by using single time-shift unit as shown in Fig. 2.18.

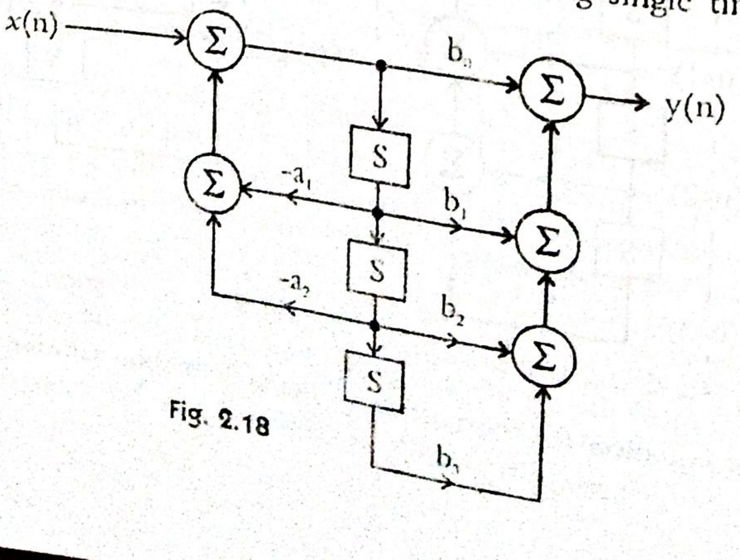

Fig. 2.18

The block diagram representation shown in Fig. 2.16 is known as *direct form I implementation* whereas that shown in Fig. 2.18 is known as *direct form II implementation*. The direct form II implementation uses less time-shift unit or memory elements.

Examples

Example 2.70 Find the difference equation corresponding to the block diagram shown in Fig. P2.70.1

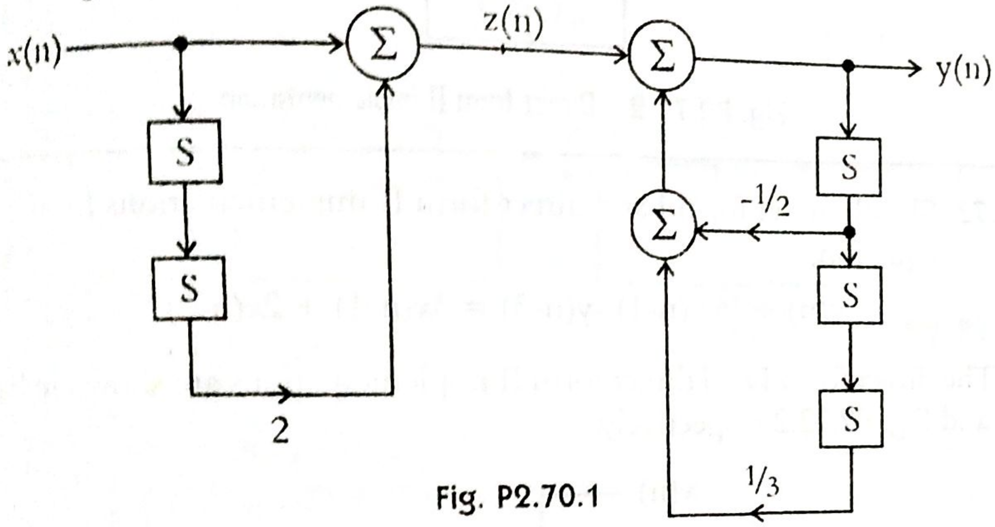

Fig. P2.70.1

Solution : From Fig. P2.70.1 we get,

$$z(n) = x(n) + 2x(n-2) \qquad \ldots\ldots \quad \text{P2.70.1}$$

and $\quad y(n) = z(n) - \frac{1}{2}\, y(n-1) + \frac{1}{3}\, y(n-3) \qquad \ldots\ldots \quad \text{P2.70.2}$

Put eqn. P2.70.1 in eqn. P2.70.2 we get,

$$y(n) = x(n) + 2x(n-2) - \frac{1}{2}\, y(n-1) + \frac{1}{3}\, y(n-3)$$

$$\therefore\ y(n) + \frac{1}{2}\, y(n-1) - \frac{1}{3}\, y(n-3) = x(n) + 2x(n-2)$$

Example 2.71 Draw the direct form I and direct form II implementation for the system described by,

$$y(n) + \frac{1}{4}\, y(n-1) - \frac{1}{8}\, y(n-2) = x(n) + x(n-1)$$

Solution : The direct form I and direct form II implementations are shown in Fig. P2.71.1 and Fig. P2.71.2 respectively.

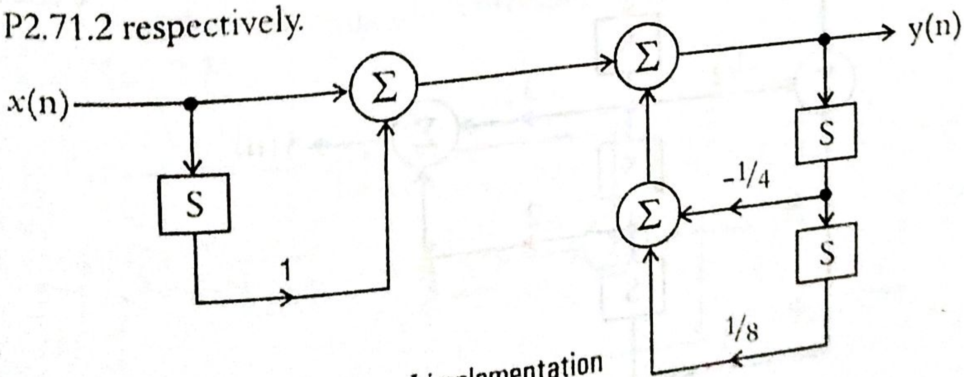

Fig. P2.71.1 Direct form I implementation

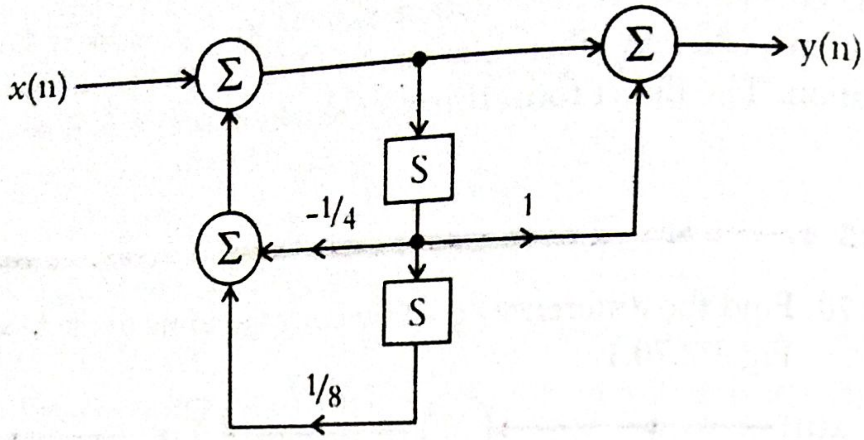

Fig. P2.71.2 Direct form II implementation

Example 2.72 Sketch direct form I and direct form II implementations for the difference equation,

$$y(n) + \tfrac{1}{2}\, y(n-1) - y(n-3) = 3x(n-1) + 2x(n-2)$$

Solution : The direct form I and direct form II implementations are shown in Fig. P2.72.1 and Fig. P2.72.2 respectively.

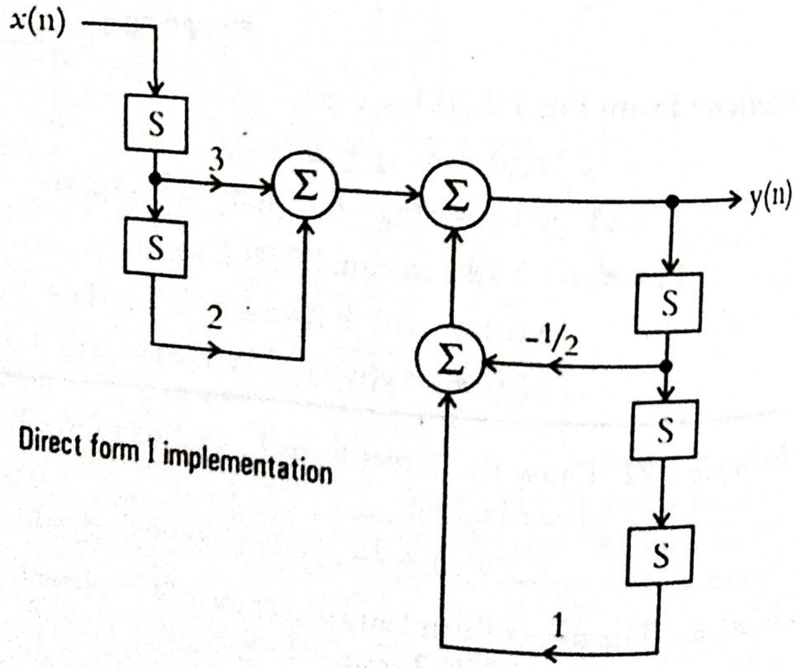

Fig. P2.72.1 Direct form I implementation

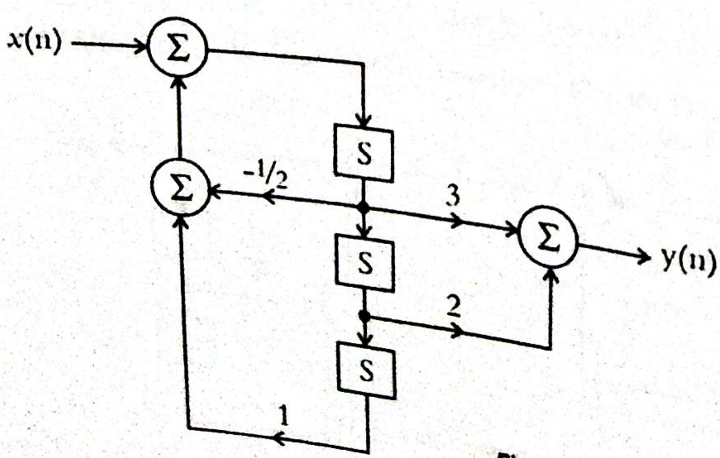

Fig. P2.72.2 Direct form II implementation

Example 2.73 Find the difference-equation representation corresponding to the block diagram shown in Fig. P2.73.

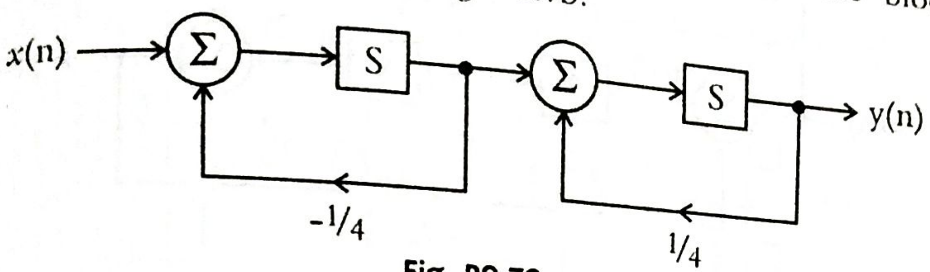

Fig. P2.73

Solution : Given :

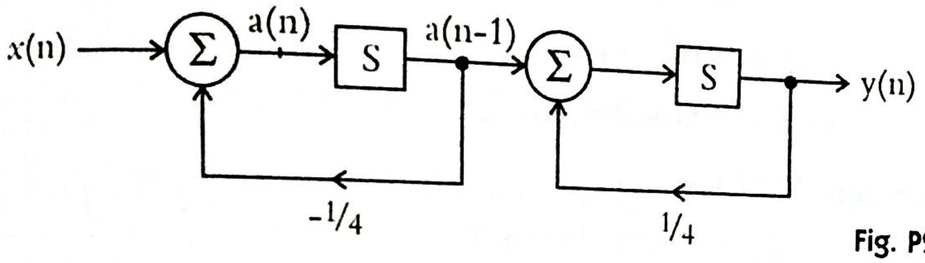

Fig. P2.73.1

From Fig. P2.73.1 we get,

$$a(n) = x(n) - \tfrac{1}{4}\, a(n-1) \qquad \ldots\ldots \quad P2.73.1$$

$$y(n) = a(n-2) + \tfrac{1}{4}\, y(n-1) \qquad \ldots\ldots \quad P2.73.2$$

From eqn. P2.73.1 we get,

$$\therefore\ a(n-1) = x(n-1) - \tfrac{1}{4}\, a(n-2)$$

$$a(n-2) = x(n-2) - \tfrac{1}{4}\, a(n-3) \qquad \ldots\ldots \quad P2.73.3$$

From eqn. P2.73.2 we get,

$$a(n-2) = y(n) - \tfrac{1}{4}\, y(n-1)$$

$$a(n-3) = y(n-1) - \tfrac{1}{4}\, y(n-2) \qquad \ldots\ldots \quad P2.73.4$$

Using eqn. P2.73.3 and eqn. P2.73.4 we get,

$$y(n) = x(n-2) - \tfrac{1}{4}\, a(n-3) + \tfrac{1}{4}\, y(n-1)$$

$$= x(n-2) - \tfrac{1}{4}\left[y(n-1) - \tfrac{1}{4}\, y(n-2)\right] + \tfrac{1}{4}\, y(n-1)$$

$$y(n) - \tfrac{1}{16}\, y(n-2) = x(n-2)$$

Example 2.74 Obtain the difference equation corresponding to the block diagram shown in Fig. P2.74.

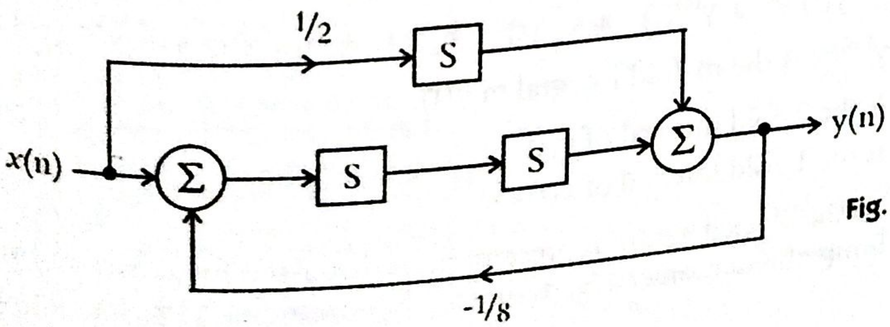

Fig. P2.74

Solution : Given :

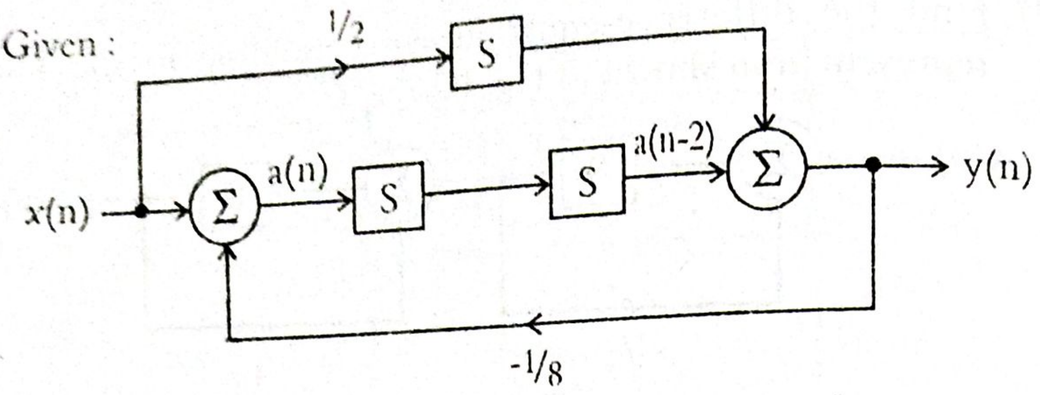

Fig. P2.74.1

$$a(n) = x(n) - {}^1\!/\!_8\, y(n) \qquad \text{...... P2.74.1}$$

$$y(n) = {}^1\!/\!_2 x(n-1) + a(n-2) \qquad \text{...... P2.74.2}$$

From eqn. P2.74.1 we get,

$$a(n-2) = x(n-2) - {}^1\!/\!_8\, y(n-2) \qquad \text{...... P2.74.3}$$

Substituting eqn. P2.74.3 in eqn. P2.74.2 we get,

$$y(n) = {}^1\!/\!_2 x(n-1) + x(n-2) - {}^1\!/\!_8\, y(n-2)$$

$$\therefore\ y(n) + {}^1\!/\!_8\, y(n-2) = {}^1\!/\!_2 x(n-1) + x(n-2)$$

2.5.2 For continuous-time system

Let us discuss the block diagram representation of a system described by differential equation. The general form of differential equation discription for a continuous-time system is,

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \qquad \text{...... (2.31)}$$

If $N \geq M$, by integrating eqn. 2.31 'N' times we get,

$$\sum_{k=0}^{N} a_k\, y^{(N-k)}(t) = \sum_{k=0}^{M} b_k\, x^{(N-k)}(t) \qquad \text{...... (2.32)}$$

For a second-order system (N=2), with $a_2=1$ we get,

$$y(t) = -a_1 y^{(1)}(t) - a_0 y^{(2)}(t) + b_2 x(t) + b_1 x^{(1)}(t) + b_0 x^{(2)}(t) \qquad \text{...... (2.33)}$$

where $y^{(m)}(t)$ is the m-fold integral of y(t).

Eg., $y^{(2)}(t)$ is the 2-fold integral of y(t)

$x^{(1)}(t)$ is the 1-fold integral of x(t) etc.

Similar to the block diagram representations discussed for a discrete-time system, the direct form I implementation can be drawn as shown in Fig. 2.19.

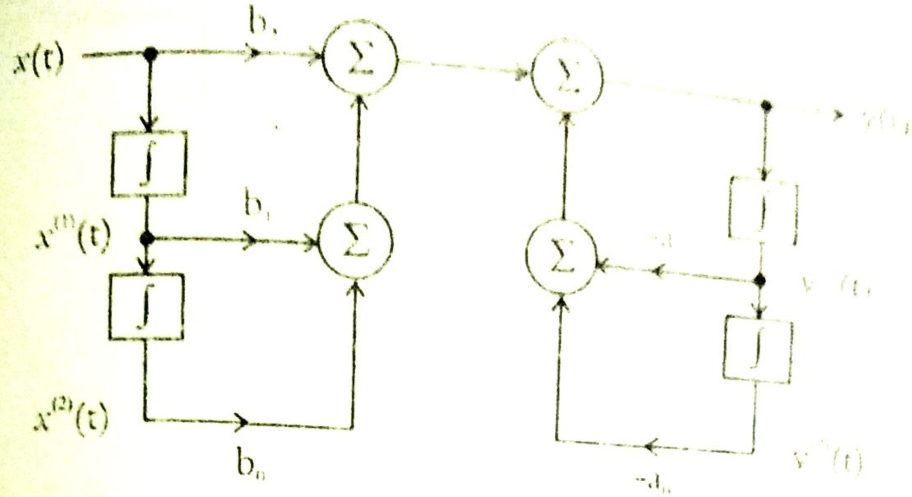

Fig. 2.19 Direct form I implementation

The direct form II implementation is shown in Fig. 2.20

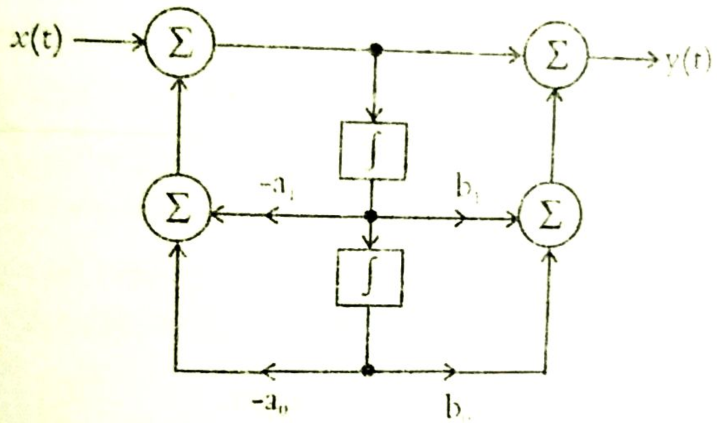

Fig. 2.20 Direct form II implementation

Examples

Example 2.75 By converting the differential equation to integral equation, draw direct form I and direct form II implementations for the system.

$$\frac{dy(t)}{dt} + 5\,y(t) = 3x(t)$$

Solution : Given : $\dfrac{dy(t)}{dt} + 5\,y(t) = 3x(t)$

Integrating both the sides, we get,

$$y(t) + 5y^{(1)}(t) = 3x^{(1)}(t)$$

$$y(t) = 3x^{(1)}(t) - 5y^{(1)}(t)$$

The direct form I and direct form II implementations are shown in Fig. P2.75.1 and Fig. P2.75.2 respectively.

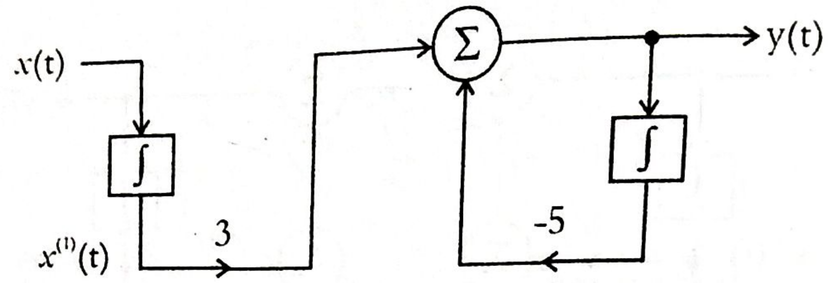

Fig. P2.75.1 Direct form I

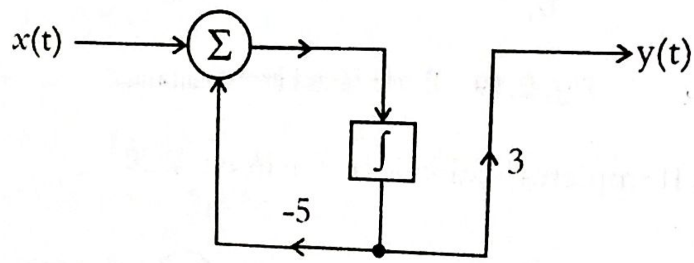

Fig. P2.75.2 Direct form II

Example 2.76 Repeat example 2.75 for,

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt}$$

Solution : Integrating twice on both the sides we get,

$$y(t) + 5\,y^{(1)}(t) + 4y^{(2)}(t) = x^{(1)}(t)$$

$$\therefore\ y(t) = x^{(1)}(t) - 5y^{(1)}(t) - 4y^{(2)}(t)$$

The direct form I and direct form II implementations are shown in Fig. P2.76.1 and Fig. P2.76.2 respectively.

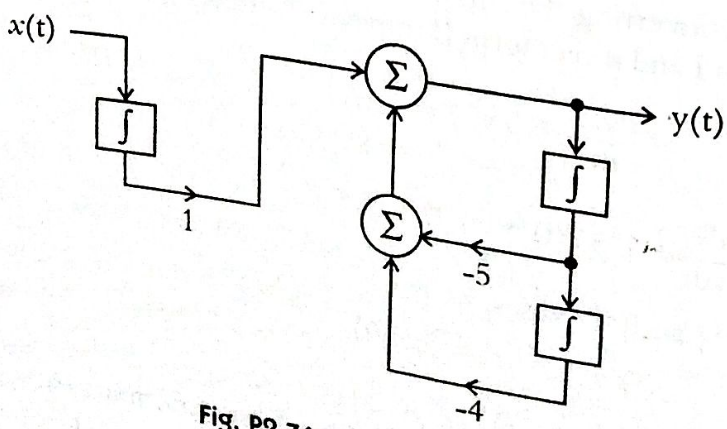

Fig. P2.76.1 Direct form I

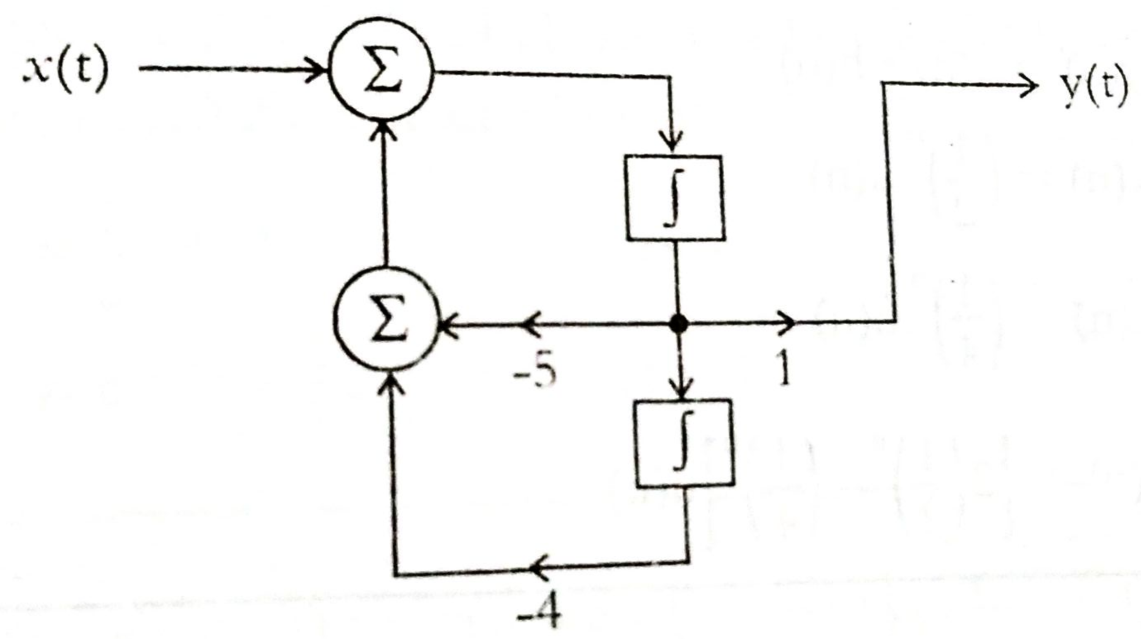

Fig. P2.76.2 Direct form II

2.6 EXERCISES

E2.1 Obtain the convolution of the sequences,

... $- 0.5\,\delta(n-3)$