Chapter 3

Fourier Representations of Signals

3.1 Introduction

The convolution sum and integral that we studied so far provide us with a convenient way to find the response of an LTI system if its impulse response is known. Also we know that an LTI system can be completely characterized by its impulse response.

In this chapter, we will study different alternate representations for signals and systems. We represent a signal as a weighted superposition of complex sinusoidals. If such a signal is applied to an LTI system, then the system output is a weighted superposition of the system outputs to each complex sinusoidals.

The representations of signals and systems using complex sinusoidals is called *Fourier representations* after Joseph Fourier for his contributions to this theory.

3.2 Fourier Representations for Signal Classes

Depending on the *periodic* nature of a signal there are four distinct Fourier representations for it. *Periodic* signals have Fourier series representations whereas *non-periodic* signals have Fourier transform representations. The *Fourier series* (FS) corresponds to continuous-time periodic signals whereas the *Discrete-time Fourier series* (DTFS) corresponds to discrete-time periodic signals. Similarly, if the signal is non-periodic continuous-time signal the representation is termed as *Fourier-transform* (FT) whereas for non-periodic discrete-time signal the representation is termed as *Discrete-time Fourier transform* (DTFT). Table. 3.1 gives the relationship between the time properties of a signal and the appropriate Fourier representation.

Table 3.1: *Time properties of a signal and its appropriate Fourier representation.*

Time Property	Periodic	Non-periodic
Continuous - time	Fourier Series (FS)	Fourier Transform (FT)
Discrete - time	Discrete-Time Fourier Series (DTFS)	Discrete-Time Fourier Transform (DTFT)

3.3 Orthogonality of Complex Sinusoidal Signals

In the previous sections, we studied that any signal can be expressed as a weighted superposition of complex sinusoidals. Now consider two continuous-time signals $x(t)$ and y(t). These two signals are said to be *orthogonal* over the interval (a, b) if,

$$\int_a^b x(t)y^*(t)dt = 0 \tag{3.1}$$

where $y^*(t)$ is the complex conjugate of y(t). For example, consider two continuous-time signals $x(t)$ and y(t) as shown in Fig. 3.1 below.

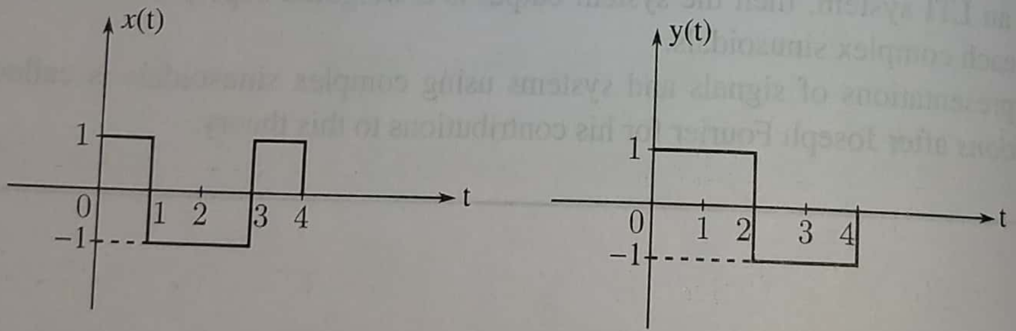


Fig. 3.1

Let us check for the orthogonality of $x(t)$ and y(t) over the interval (0, 4). Since y(t) is real $y^*(t) = y(t)$. Therefore to check the orthogonality we have to evaluate,

$$\int_0^4 x(t)y(t)dt$$

The signal $x(t)y(t)$ is shown in Fig. 3.2 below.

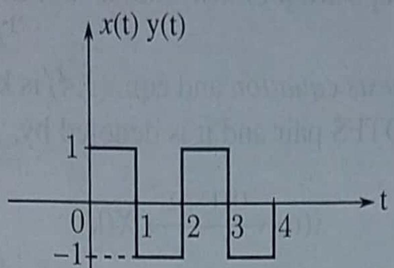


Fig. 3.2

$$\therefore \int_0^4 x(t)y(t)dt = \int_0^1 1dt - \int_1^2 1dt + \int_2^3 1dt - \int_3^4 1dt = 0$$

Therefore $x(t)$ and $y(t)$ are orthogonal over the interval (0, 4).

Similarly two discrete-time signals $x(n)$ and $y(n)$ are said to be *orthogonal* over the interval (N_1, N_2) if,

$$\sum_{n=N_1}^{N_2} x(n)y^*(n) = A_k \quad ; \; k = m \tag{3.2}$$

$$= 0 \quad ; \; k \neq m$$

where 'A_k' is a constant.

3.4 Discrete Time Periodic Signals : The Discrete-Time Fourier Series (DTFS)

A periodic discrete-time signal $x(n)$ can be expressed as,

$$x(n) = \sum_{k=<N>} X(k)e^{jk\Omega_o n} \tag{3.3}$$

$$\text{where} \quad X(k) = \frac{1}{N}\sum_{n=<N>} x(n)e^{-jk\Omega_o n} \tag{3.4}$$

where $x(n)$ has fundamental period 'N' and fundamental frequency $\Omega_o = \frac{2\pi}{N}$ (radians).

In the above equation, X(k) are known as *Discrete-Time Fourier Series coefficients* of $x(n)$. The coefficients specify a decomposition of $x(n)$ into a sum of 'N' harmonically related complex exponentials.

Eqn. (3.3) is known as *synthesis equation* and eqn. (3.4) is known as *analysis equation*. We say that $x(n)$ and $X(k)$ forms a DTFS pair and it is denoted by,

$$x(n) \xleftrightarrow{\text{DTFS}:\Omega_0} X(k)$$

where $x(n)$ is time-domain representation and $X(k)$ is frequency-domain representation.

When finding $x(n)$ and X(k) using eqn. (3.3) and eqn. (3.4) respectively, the starting values of the indices 'k' and 'n' are arbitrary because both $x(n)$ and $X(k)$ are *periodic* with period N.

The magnitude of $X(k)$ i.e., $|X(k)|$ is known as *magnitude spectrum* of $x(n)$ and phase of $X(k)$ i.e., arg $X(k)$ or $\measuredangle X(k)$ is known as *phase spectrum* of $x(n)$.

3.4.1 Properties of DTFS

In this section, we will discuss the different properties of DTFS which helps in solving examples.The different properties of DTFS are,

(a) Linearity

(b) Time shift

(c) Frequency shift

(d) Scaling

(e) Convolution

(f) Modulation

(g) Parseval's Theorem

(h) Duality

(i) Symmetry

(a) Linearity:

If $x(n) \xleftrightarrow{\text{DTFS}:\Omega_0} X(k) \quad ; \Omega_0 = \frac{2\pi}{N}$

and $y(n) \xleftrightarrow{\text{DTFS}:\Omega_0} Y(k)$

then $z(n) = ax(n) + by(n) \xleftrightarrow{\text{DTFS}:\Omega_0} Z(k) = aX(k) + bY(k).$

In this case, both $x(n)$ and y(n) are assumed to have the same fundamental period $N = \frac{2\pi}{\Omega_0}$

Proof. We know that,

$$X(k) = \frac{1}{N} \sum_{n=<N>} x(n)e^{-jk\Omega_o n}$$

$$Y(k) = \frac{1}{N} \sum_{n=<N>} y(n)e^{-jk\Omega_o n}$$

$$\therefore Z(k) = \frac{1}{N} \sum_{n=<N>} z(n)e^{-jk\Omega_o n}$$

$$= \frac{1}{N} \sum_{n=<N>} [ax(n) + by(n)]e^{-jk\Omega_o n}$$

$$= \frac{1}{N}a \sum_{n=<N>} x(n)e^{-jk\Omega_o n} + \frac{1}{N}b \sum_{n=<N>} y(n)e^{-jk\Omega_o n}$$

$$\therefore \quad Z(k) = aX(k) + bY(k)$$

Hence the proof.

(b) Time Shift: If $x(n) \xleftrightarrow{\text{DTFS}:\Omega_o} X(k) \quad ; \Omega_o = \frac{2\pi}{N}$

then $w(n) = x(n - n_o) \xleftrightarrow{\text{DTFS}:\Omega_o} W(k) = e^{-jk\Omega_o n_o} X(k)$

Proof. We have,

$$X(k) = \frac{1}{N} \sum_{n=<N>} x(n)e^{-jk\Omega_o n}$$

$$W(k) = \frac{1}{N} \sum_{n=<N>} w(n)e^{-jk\Omega_o n}$$

$$= \frac{1}{N} \sum_{n=<N>} x(n - n_o)e^{-jk\Omega_o n}$$

Put $n - n_o = m$, then

$$W(k) = \frac{1}{N} \sum_{m=<N>} x(m)e^{-jk\Omega_o (m+n_o)}$$

$$= e^{-jk\Omega_o n_o} \frac{1}{N} \sum_{m=<N>} x(m)e^{-jk\Omega_o m}$$

$$\therefore \quad W(k) = e^{-jk\Omega_o n_o} X(k).$$

Hence the proof.

(c) **Frequency Shift:**

If $x(n) \xleftrightarrow{\text{DTFS}:\Omega_o} X(k) \quad ; \Omega_o = \frac{2\pi}{N}$

then $g(n) = e^{jk_o\Omega_o n}x(n) \xleftrightarrow{\text{DTFS}:\Omega_o} G(k) = X(k - k_o)$

Proof. We have,

$$\begin{aligned} G(k) &= \frac{1}{N}\sum_{n=<N>} g(n)e^{-jk\Omega_o n} \\ &= \frac{1}{N}\sum_{n=<N>} e^{jk_o\Omega_o n}x(n)e^{-jk\Omega_o n} \\ &= \frac{1}{N}\sum_{n=<N>} x(n)e^{-j(k-k_o)\Omega_o n} \\ &= X(k - k_o). \end{aligned}$$

Hence the proof.

(d) **Scaling:** We studied that the scaling operation on a discrete-time signal discards information. Due to this loss of information it is not possible to express the DTFS of scaled signal in terms of DTFS of the original signal.

Consider a periodic discrete-time signal $x(n)$ with fundamental period N such that,

$$x(n) = 0 \qquad ;\text{unless } \frac{n}{p} \text{ is integer.}$$

then $z(n) = x(pn)$ has fundamental period N/p.

In this case, if

$$x(n) \xleftrightarrow{\text{DTFS}:\Omega_o} X(k)$$

then, $z(n) = x(pn) \xleftrightarrow{\text{DTFS}:\Omega_o} Z(k) = pX(k) \qquad \text{where } p > 0.$

The scaling operation changes the harmonic spacing from Ω_o to $p\Omega_o$ and amplifies the DTFS coefficients by 'p'.

(e) **Convolution:** If $x(n) \xleftrightarrow{\text{DTFS}:\Omega_o} X(k) \quad ; \Omega_o = \frac{2\pi}{N}$

and $y(n) \xleftrightarrow{\text{DTFS}:\Omega_o} Y(k)$

then $z(n) = x(n) \circledast y(n) \xleftrightarrow{\text{DTFS}:\Omega_o} Z(k) = NX(k)Y(k)$

where '$\circledast$' denotes periodic convolution.

Proof. We have,

$$X(k) = \frac{1}{N}\sum_{n=<N>} x(n)e^{-jk\Omega_0 n}$$

$$Y(k) = \frac{1}{N}\sum_{n=<N>} y(n)e^{-jk\Omega_0 n}$$

$$\therefore Z(k) = \frac{1}{N}\sum_{n=<N>} z(n)e^{-jk\Omega_0 n}$$

$$= \frac{1}{N}\sum_{n=<N>} [x(n) \circledast y(n)]\, e^{-jk\Omega_0 n}$$

Using definition of periodic convolution we get,

$$Z(k) = \frac{1}{N}\sum_{n=<N>}\left[\sum_{\ell=<N>} x(\ell)y(n-\ell)\right] e^{-jk\Omega_0 n}$$

Changing the order of summation we get,

$$Z(k) = \frac{1}{N}\left[\sum_{\ell=<N>} x(\ell)\sum_{n=<N>} y(n-\ell)e^{-jk\Omega_0 n}\right]$$

Put $n - \ell = m$ then,

$$Z(k) = \frac{1}{N}\left[\sum_{\ell=<N>} x(\ell)\sum_{m=<N>} y(m)e^{-jk\Omega_0(m+\ell)}\right]$$

$$= \frac{1}{N}\left[\sum_{\ell=<N>} x(\ell)\sum_{m=<N>} y(m)e^{-jk\Omega_0 m}e^{-jk\Omega_0 \ell}\right]$$

$$= \frac{1}{N}[NX(k).NY(k)]$$

$$\therefore \quad Z(k) = NX(k)Y(k)$$

$\therefore$ Convolution in time-domain is transformed to multiplication of DTFS coefficients.

(f) Modulation:

$$\text{If} \quad x(n) \xleftrightarrow{\text{DTFS}:\Omega_0} X(k) \qquad ; \Omega_0 = \frac{2\pi}{N}$$

$$\text{and} \quad y(n) \xleftrightarrow{\text{DTFS}:\Omega_0} Y(k)$$

$$\text{then} \qquad z(n) = x(n)\, y(n) \xleftrightarrow{\text{DTFS}:\Omega_0} Z(k) = X(k) \circledast Y(k)$$

Scanned by CamScanner

Proof. We have,

$$Z(k) = \frac{1}{N}\sum_{n=<N>} z(n)e^{-jk\Omega_o n}$$

$$= \frac{1}{N}\sum_{n=<N>} x(n)y(n)e^{-jk\Omega_o n} \quad (3.5)$$

From eqn. (3.3) we have,

$$x(n) = \sum_{\ell=<N>} X(\ell)e^{j\ell\Omega_o n} \quad (3.6)$$

Substituting eqn. (3.6) in eqn. (3.5) we get,

$$Z(k) = \frac{1}{N}\sum_{n=<N>}\left[\sum_{\ell=<N>} X(\ell)e^{j\ell\Omega_o n}\right] y(n)e^{-jk\Omega_o n}$$

Changing the order of summation we get,

$$Z(k) = \frac{1}{N}\sum_{\ell=<N>} X(\ell)\sum_{n=<N>} y(n)e^{-j(k-\ell)\Omega_o n}$$

$$= \sum_{\ell=<N>} X(\ell)Y(k-\ell)$$

$$= X(k) \circledast Y(k).$$

$\therefore$ Multiplication in time-domain is transformed to convolution of DTFS coefficients.

(g) Parseval's Theorem: If $x(n) \xleftrightarrow{DTFS:\Omega_o} X(k)$; $\Omega_o = \frac{2\pi}{N}$

then
$$\frac{1}{N}\sum_{n=<N>} |x(n)|^2 = \sum_{k=<N>} |X(k)|^2 \quad (3.7)$$

Proof. The LHS of eqn. (3.7) is the average power of a periodic discrete-time signal $x(n)$ with fundamental period N.

i.e.,
$$P = \frac{1}{N}\sum_{n=<N>} |x(n)|^2$$

This equation can be written as,

$$P = \frac{1}{N}\sum_{n=<N>} x(n)x^*(n)$$

$$= \frac{1}{N}\sum_{n=<N>} x(n)\left[\sum_{k=<N>} X^*(k)e^{-jk\Omega_o n}\right]$$

Changing the order of summation we get,

$$P = \sum_{k=<N>} X^*(k)\left[\frac{1}{N}\sum_{n=<N>} x(n)e^{-jk\Omega_o n}\right]$$

$$= \sum_{k=<N>} X^*(k)X(k)$$

$$= \sum_{k=<N>} |X(k)|^2$$

$$\therefore \frac{1}{N}\sum_{n=<N>} |x(n)|^2 = \sum_{k=<N>} |X(k)|^2 \tag{3.8}$$

In eqn. (3.8), the sequence $|X(k)|^2$ for $k = 0, 1, 2, \cdots\cdots N-1$ is the distribution of power as a function of frequency and it is called *power density spectrum* of the signal $x(n)$.

(h) **Duality:**

If $x(n) \xleftrightarrow{\text{DTFS}:\Omega_o} X(k)$; $\Omega_o = \frac{2\pi}{N}$

then $X(n) \xleftrightarrow{\text{DTFS}:\Omega_o} \frac{1}{N}x(-k)$

Proof. We have $x(n) = \sum_{k=<N>} X(k)e^{jk\Omega_o n}$

Replacing 'n' by '−n' we get,

$$x(-n) = \sum_{k=<N>} X(k)e^{-jk\Omega_o n}$$

Replacing 'n' by 'k' and 'k' by 'n' we get,

$$x(-k) = \sum_{n=<N>} X(n)e^{-jk\Omega_o n}$$

Multiplying both the sides by $\frac{1}{N}$ we get,

$$\frac{1}{N}x(-k) = \frac{1}{N}\sum_{n=<N>} X(n)e^{-jk\Omega_o n}$$

Comparing with eqn. (3.4) we get,

$$X(n) \xleftrightarrow{\text{DTFS}:\Omega_o} \frac{1}{N}x(-k)$$

Hence the proof.

(i) **Symmetry:**

If $x(n) \xleftrightarrow{DTFS:\Omega_0} X(k)$; $\Omega_0 = \frac{2\pi}{N}$

then (i) If $x(n)$ is real, then $X^*(k) = X(-k)$

(ii) If $x(n)$ is real and even, then $\text{Img}\{X(k)\} = 0$

(iii) If $x(n)$ is real and odd, then $\text{Re}\{X(k)\} = 0$.

EXAMPLES

Example 3.1 Determine the DFTS of the signal,

$$x(n) = \cos\left(\frac{\pi}{3}n\right)$$

and draw the spectrum.

Solution. We know that $x(n) = \cos(\Omega_o n)$ is periodic if Ω_o is an integer multiple of $\frac{2\pi}{N}$ where 'N' is the fundamental period. i.e., $\Omega_0 = \frac{2\pi}{N}m$

By comparing $x(n) = \cos\left(\frac{\pi}{3}n\right)$ with $x(n) = \cos(\Omega_o n)$ we have,

$$\Omega_0 = \frac{\pi}{3} = \frac{2\pi}{6}.1$$

$\therefore$ Fundamental period N = 6.

One possible way of finding X(k) is to use the equation,

$$X(k) = \frac{1}{N}\sum_{n=\langle N\rangle} x(n)e^{-jk\Omega_o n} \qquad ; k = 0, 1, 2, \cdots 5$$

But the given $x(n) = \cos\left(\frac{\pi}{3}n\right)$ can be expressed in the form of exponential so that we can obtain $X(k)$ directly using eqn. (3.3) by comparison.

$$\therefore x(n) = \cos\left(\frac{\pi}{3}n\right) = \frac{1}{2}e^{j(1)\left(\frac{\pi}{3}\right)n} + \frac{1}{2}e^{j(-1)\left(\frac{\pi}{3}\right)n}$$

By comparison with eqn. (3.3) we get, $x(n) = \sum_{k=\langle N\rangle} X(k)e^{jk\Omega_o n} = \sum_{k=\langle 6\rangle} X(k)e^{jk\left(\frac{\pi}{3}\right)n}$

$$X(1) = \frac{1}{2} \quad \text{and} \quad X(-1) = \frac{1}{2}$$

Since DTFS X(k) forms a periodic sequence of period N, we can write,

$$\ldots = X(-11) = X(-5) = X(1) = X(7) = X(13) = \ldots\ldots = \frac{1}{2}$$

$$\ldots = X(-7) = X(-1) = X(5) = X(11) = X(17) = \ldots\ldots = \frac{1}{2}$$

and other X(k)'s are equal to zero.

The spectrum is shown in Fig. E3.1.1

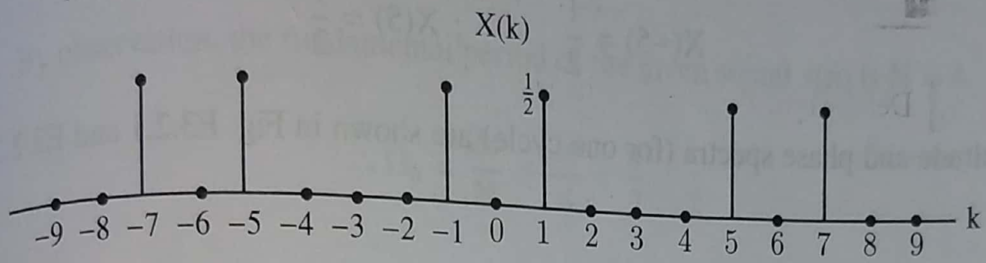


Fig. E3.1.1.

Example 3.2 Evaluate the DTFS representation for the signal,

$$x(n) = \sin\left(\frac{4\pi}{21}n\right) + \cos\left(\frac{10\pi}{21}n\right) + 1$$

Sketch the magnitude and phase spectra.

Solution. Given:

$$x(n) = \sin\left(\frac{4\pi}{21}n\right) + \cos\left(\frac{10\pi}{21}n\right) + 1 \qquad \text{(E3.2.1)}$$

In eqn. (E3.2.1), the I term has angular frequency $\Omega_{01} = \frac{4\pi}{21}$ and II term has angular frequency $\Omega_{02} = \frac{10\pi}{21}$. So effectively the angular frequency of the summation,

$$\Omega_o = \text{g.c.d. of } \Omega_{01} \text{ and } \Omega_{02}$$

$$= \text{g.c.d. of } \frac{4\pi}{21} \text{ and } \frac{10\pi}{21}$$

$$\Omega_o = \frac{2\pi}{21} \text{ and } N = 21.$$

Arranging eqn. (E3.2.1) as,

$$\therefore x(n) = \frac{e^{j\frac{4\pi}{21}n} - e^{-j\frac{4\pi}{21}n}}{2j} + \frac{e^{j\frac{10\pi}{21}n} + e^{-j\frac{10\pi}{21}n}}{2} + 1$$

$$= \frac{1}{2j}e^{j(2)(\frac{2\pi}{21})n} - \frac{1}{2j}e^{j(-2)(\frac{2\pi}{21})n} + \frac{1}{2}e^{j5(\frac{2\pi}{21})n} + \frac{1}{2}e^{j(-5)(\frac{2\pi}{21})n} + 1 \quad \text{(E3.2.2)}$$

Comparing eqn. (E3.2.2) with eqn. (3.3) we get,

$$X(0) = 1$$

$$X(-2) = -\frac{1}{2j} \quad ; \; X(2) = \frac{1}{2j}$$

$$X(-5) = \frac{1}{2} \quad ; \; X(5) = \frac{1}{2}$$

The magnitude and phase spectra (for one cycle) are shown in Fig. E3.2.1 and E3.2.2 respectively.

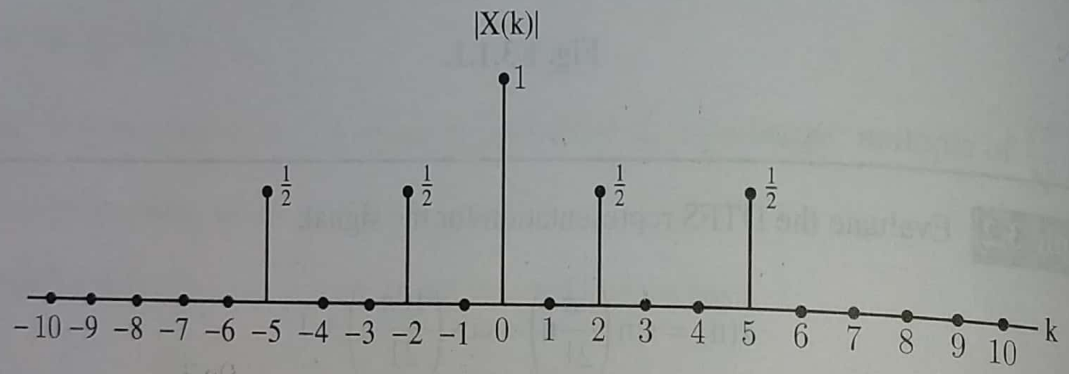


Fig. E3.2.1.

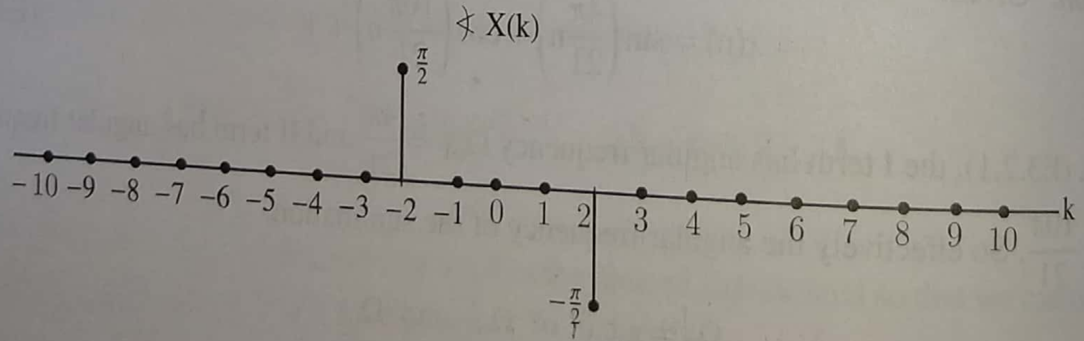


Fig. E3.2.2.

In the above figure the spectra is shown for only one period i.e., $k \in \{-10 \cdots 10\}$

Example 3.3 Evaluate the DTFS representation for the signal $x(n)$ shown in Fig. E3.3 and sketch the spectra. Also verify Parseval's identity.

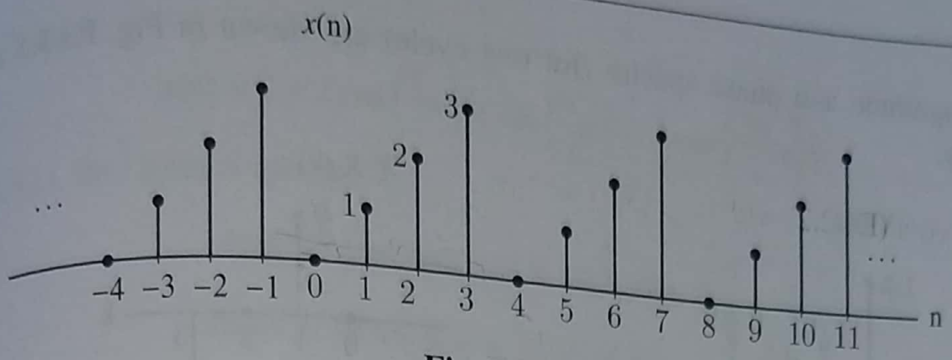


Fig. E3.3.

Solution. By observation, the fundamental period of the given signal $x(n)$ is N = 4.

$$\therefore \Omega_o = \frac{2\pi}{N} = \frac{2\pi}{4} = \frac{\pi}{2}$$

We have,

$$X(k) = \frac{1}{N} \sum_{n=<N>} x(n) e^{-jk\Omega_o n}$$

$$X(k) = \frac{1}{4} \sum_{n=0}^{3} x(n) e^{-jk(\frac{\pi}{2})n}$$

$$= \frac{1}{4}\left[x(0) + x(1)e^{-j\frac{\pi}{2}k} + x(2)e^{-j\pi k} + x(3)e^{-j\frac{3\pi}{2}k}\right]$$

$$= \frac{1}{4}\left[0 + e^{-j\frac{\pi}{2}k} + 2e^{-j\pi k} + 3e^{-j\frac{3\pi}{2}k}\right]$$

$$X(0) = \frac{1}{4}[0 + 1 + 2 + 3] = 1.5$$

$$X(1) = \frac{1}{4}\left[0 + e^{-j\frac{\pi}{2}} + 2e^{-j\pi} + 3e^{-j\frac{3\pi}{2}}\right] = -0.5 + j0.5$$

$-j(-1)\,3.$

$$X(2) = \frac{1}{4}\left[0 + e^{-j\pi} + 2e^{-j2\pi} + 3e^{-j3\pi}\right] = -0.5$$

$$X(3) = \frac{1}{4}\left[0 + e^{-j\frac{3\pi}{2}} + 2e^{-j3\pi} + 3e^{-j\frac{9\pi}{4}}\right] = -0.5 - j0.5$$

$\therefore \quad |X(0)| = 1.5 \qquad \angle X(0) = 0$

$\therefore \quad |X(1)| = 0.7.07 \qquad \angle X(1) = \frac{3\pi}{4}$

$\therefore \quad |X(2)| = 0.5 \qquad \angle X(2) = 0$

$\therefore \quad |X(3)| = 0.707 \qquad \angle X(3) = -\frac{3\pi}{4}$

Scanned by CamScanner

The magnitude and phase spectra (for one cycle) are shown in Fig. E3.3.1 and E3.3.2 respectively.

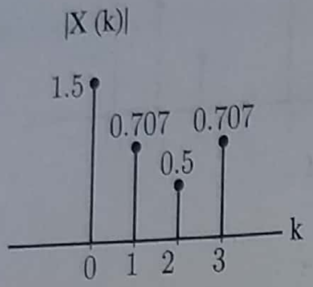


Fig. E3.3.1.

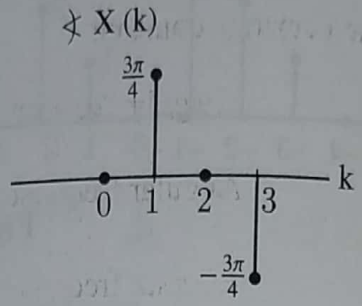


Fig. E3.3.2.

Verification of Parseval's identity:

From Parsevals theorem we have,

$$\frac{1}{N}\sum_{n=\langle N\rangle}|x(n)|^2 = \sum_{k=\langle N\rangle}|X(k)|^2$$

$$\text{L.H.S.} \rightarrow \frac{1}{N}\sum_{n=\langle N\rangle}|x(n)|^2 = \frac{1}{4}\sum_{n=0}^{3}|x(n)|^2$$

$$= \frac{1}{4}\left[|x(0)|^2 + |x(1)|^2 + |x(2)|^2 + |x(3)|^2\right]$$

$$= \frac{1}{4}[0+1+4+9]$$

$$= 3.5$$

$$\text{R.H.S.} \rightarrow \sum_{k=\langle N\rangle}|X(k)|^2 = \sum_{k=0}^{3}|X(k)|^2$$

$$= \left[|X(0)|^2 + |X(1)|^2 + |X(2)|^2 + |X(3)|^2\right]$$

$$= \left[(1.5)^2 + (0.707)^2 + (0.5)^2 + (0.707)^2\right]$$

$$= 3.5$$

Example 3.4 Consider the signal,

$$x(n) = 2 + 2\cos\left(\frac{\pi}{4}n\right) + \cos\left(\frac{\pi}{2}n\right) + \frac{1}{2}\cos\left(\frac{3\pi}{4}n\right)$$

(a) Determine and sketch its power density spectrum.

(b) Evaluate the power of the signal.

Solution. Given:

$$x(n) = 2 + 2\cos\left(\frac{\pi}{4}n\right) + \cos\left(\frac{\pi}{2}n\right) + \frac{1}{2}\cos\left(\frac{3\pi}{4}n\right) \qquad \text{(E3.4.1)}$$

In eqn. (E3.4.1), the I term is constant

$$\text{Angular frequency of II term } \Omega_{01} = \frac{\pi}{4}$$

$$\text{Angular frequency of III term } \Omega_{02} = \frac{\pi}{2}$$

$$\text{Angular frequency of IV term } \Omega_{03} = \frac{3\pi}{4}$$

$\therefore$ Angular frequency of the summation [i.e., of $x(n)$]

$$\Omega_o = \text{g.c.d. of } (\Omega_{01}, \Omega_{02}, \Omega_{03})$$

$$= \text{g.c.d. of } \left(\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right)$$

$$\Omega_o = \frac{\pi}{4} = \frac{2\pi}{8} \quad \text{and} \quad N = 8$$

$\therefore$ Fundamental period of $x(n)$ is N = 8

The given $x(n)$ can be written as,

$$x(n) = 2 + 2\left[\frac{e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}}{2}\right] + \left[\frac{e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}}{2}\right] + \frac{1}{2}\left[\frac{e^{j\frac{3\pi}{4}n} + e^{-j\frac{3\pi}{4}n}}{2}\right]$$

$$x(n) = 2 + e^{j(1)(\frac{2\pi}{8})n} + e^{j(-1)(\frac{2\pi}{8})n} + \frac{1}{2}e^{j(2)(\frac{2\pi}{8})n} + \frac{1}{2}e^{j(-2)(\frac{2\pi}{8})n} + \frac{1}{4}e^{j(3)(\frac{2\pi}{8})n} + \frac{1}{4}e^{j(-3)(\frac{2\pi}{8})n} \qquad \text{(E3.4.2)}$$

Comparing the above eqn. (E3.4.2) with eqn. (3.3) we get,

$X(0) = 2;\quad X(-1) = X(1) = 1 \quad \therefore\ X(-1+8) = X(7) = 1$

$X(-2) = X(2) = \frac{1}{2} \quad \therefore\ X(-2+8) = X(6) = \frac{1}{2}$

$X(-3) = X(3) = \frac{1}{4} \quad \therefore\ X(-3+8) = X(5) = \frac{1}{4}$

$X(-4) = X(4) = 0 \quad \therefore\ X(-4+8) = X(4) = 0$

(a) The power density spectrum $|X(k)|^2$ for one cycle i.e., $k \in (0......7)$ is shown in Fig. E3.4.1,

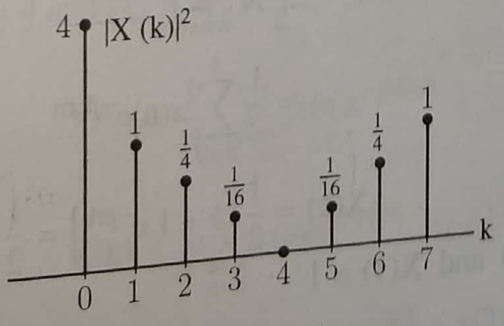


Fig. E3.4.1.

(b) The power of the signal, (from Parsavel's theorem).

$$P = \frac{1}{N}\sum_{n=<N>} |x(n)|^2 = \sum_{k=<N>} |X(k)|^2 \qquad k \in (0, \ldots\ldots, 7)$$

$$= 4 + 1 + \frac{1}{4} + \frac{1}{16} + 0 + \frac{1}{16} + \frac{1}{4} + 1$$

$$= \frac{53}{8}$$

Example 3.5 Determine and sketch the spectrum of the signal,

$$x(n) = (-1)^n \qquad ; -\infty < n < \infty$$

Solution. The signal $x(n) = (-1)^n \;\; ; -\infty < n < \infty$ is plotted in Fig. E3.5.1

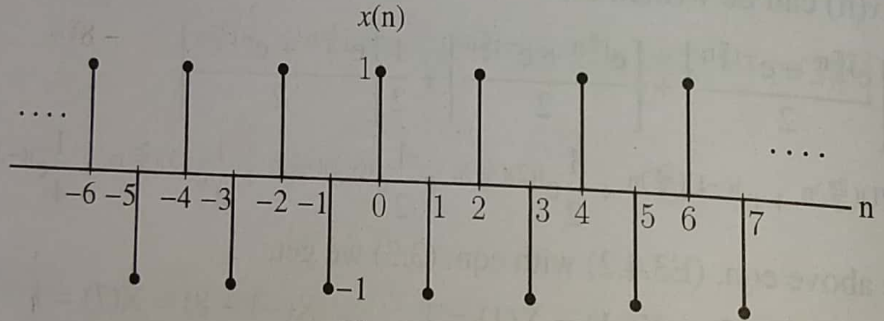


Fig. E3.5.1.

Fundamental period N = 2 $\quad \therefore \Omega_o = \frac{2\pi}{2} = \pi$

$$\therefore X(k) = \frac{1}{N}\sum_{n=<N>} x(n)e^{-jk\Omega_o n}$$

$$= \frac{1}{2}\sum_{n=0}^{1} x(n)e^{-jk\pi n}$$

$$X(k) = \frac{1}{2}\left[1 - 1.e^{-j\pi k}\right] = \frac{1}{2}\left[1 - (-1)^k\right]$$

$\therefore X(0) = 0$ and $X(1) = 1$

The spectrum (for 3 cycles) is shown in Fig. E3.5.2.

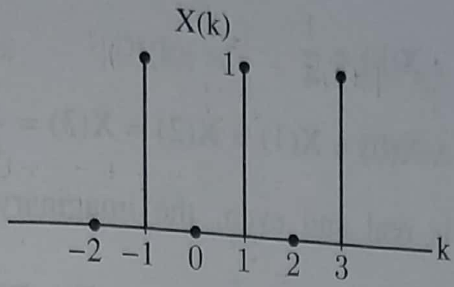


Fig. E3.5.2.

Example 3.6 Determine the DTFS coefficients of the periodic sequence given by,

$$x(n) = \sum_{m=-\infty}^{\infty} \delta(n-4m)$$

Draw the spectrum.

Solution. The expansion of given $x(n)$ is,

$$x(n) = \ldots\ldots + \delta(n+8) + \delta(n+4) + \delta(n) + \delta(n-4) + \delta(n-8) + \ldots\ldots$$

The sketch of $x(n)$ is shown in Fig. E3.6.1 below.

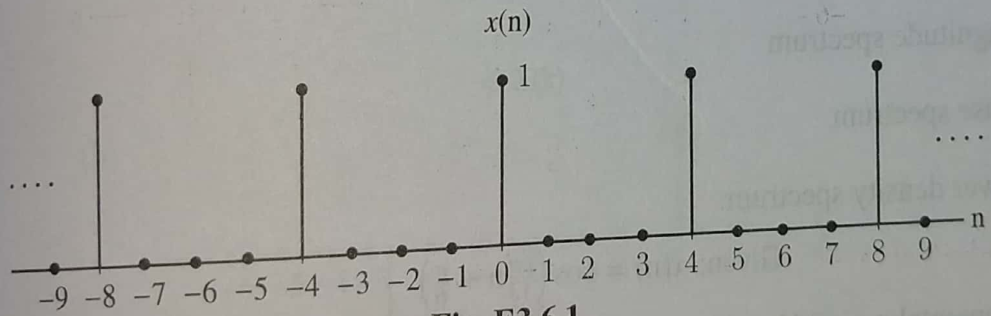


Fig. E3.6.1.

Fundamental period N = 4 and $\Omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

We have $$X(k) = \frac{1}{N}\sum_{n=<N>} x(n)e^{-jk\Omega_0 n}$$

$$\therefore X(k) = \frac{1}{4}\sum_{n=0}^{3} x(n)e^{-jk(\frac{\pi}{2})n}$$

$$= \frac{1}{4}[x(0) + x(1)e^{-jk\frac{\pi}{2}} + x(2)e^{-jk\pi} + x(3)e^{-jk\frac{3\pi}{2}}]$$

$$= \frac{1}{4}x(0) \qquad [\because x(1) = x(2) = x(3) = 0]$$

$$X(k) = \frac{1}{4}$$

$$\therefore X(0) = X(1) = X(2) = X(3) = \frac{1}{4}$$

Since the given signal $x(n)$ is real and even, the imaginary part of DTFS coefficients i.e., $\text{Img}\{X(k)\} = X_I(k) = 0$

The sketch of X(k) (for one cycle) is shown below in Fig. E3.6.2.

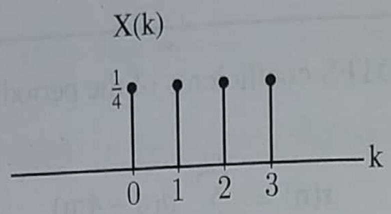


Fig. E3.6.2.

Example 3.7 Obtain the DTFS coefficients of.

$$x(n) = \cos\left(\frac{6\pi}{13}n + \frac{\pi}{6}\right)$$

Draw,

(i) Magnitude spectrum

(ii) Phase spectrum

(iii) Power density spectrum.

Solution. Given: $x(n) = \cos\left(\frac{6\pi}{13}n + \frac{\pi}{6}\right)$

Fundamental period N = 13 $\quad ;\Omega_0 = \frac{2\pi}{N} = \frac{2\pi}{13}$

Let us take $n, k \in (-6, -5 ....0, ...5, 6)$

$$x(n) = \frac{1}{2}e^{j\left(\frac{6\pi}{13}n+\frac{\pi}{6}\right)} + \frac{1}{2}e^{-j\left(\frac{6\pi}{13}n+\frac{\pi}{6}\right)}$$

$$= \frac{1}{2}e^{j\frac{\pi}{6}}.e^{j3\left(\frac{2\pi}{13}\right)n} + \frac{1}{2}e^{-j\frac{\pi}{6}}.e^{-j3\left(\frac{2\pi}{13}\right)n}$$

Comparing with eqn. 3.3 we get,

$$X(3) = \frac{1}{2}e^{j\frac{\pi}{6}} \text{ and } X(-3) = \frac{1}{2}e^{-j\frac{\pi}{6}}$$

(i) Magnitude spectrum:

$$|X(k)| = \frac{1}{2} \quad ; k = \pm 3$$
$$= 0 \quad ; k \neq \pm 3$$

(ii) Phase spectrum:

$$\measuredangle X(k) = \frac{\pi}{6} \quad ; k = 3$$
$$= -\frac{\pi}{6} \quad ; k = -3$$
$$= 0 \quad ; k \neq \pm 3$$

The magnitude and phase spectra (for one cycle) are shown in Fig. E3.7.1 (a) and (b) respectively.

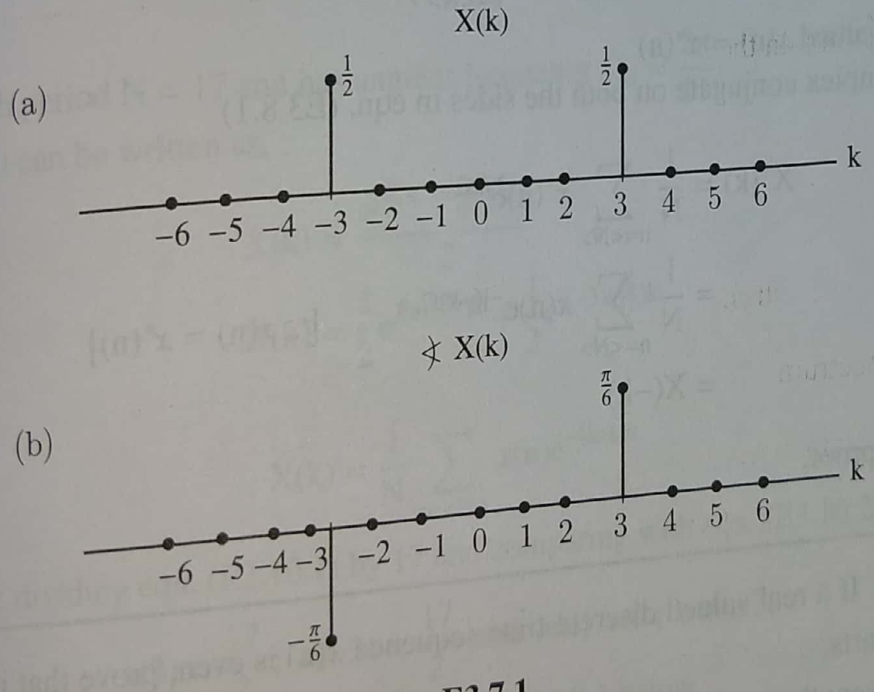


Fig. E3.7.1.

(iii) Power density spectrum:

$$|X(k)|^2 = \frac{1}{4} \quad ; k = \pm 3$$
$$= 0 \quad ; k \neq \pm 3$$

The power density spectrum (for one cycle) is shown in Fig. E3.7.2 below.

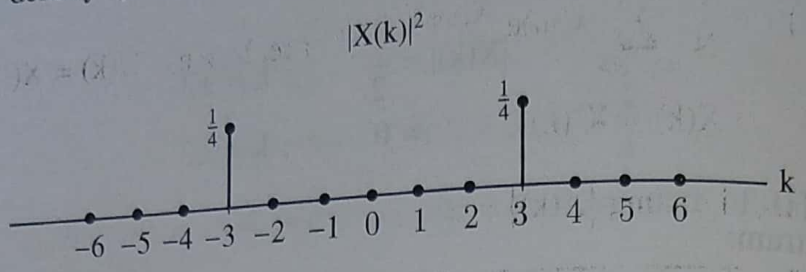


Fig. E3.7.2.

Example 3.8 If a real valued discrete-time periodic sequence $x(n)$ has DTFS $X(k)$, prove that $X^*(k) = X(-k)$

Solution. We have,

$$X(k) = \frac{1}{N}\sum_{n=<N>} x(n)e^{-jk\Omega_o n} \tag{E3.8.1}$$

If $x(n)$ is real valued $x(n) = x^*(n)$

Taking complex conjugate on both the sides in eqn. (E3.8.1)

$$X^*(k) = \frac{1}{N}\sum_{n=<N>} x^*(n)e^{jk\Omega_o n}$$

$$= \frac{1}{N}\sum_{n=<N>} x(n)e^{-j(-k)\Omega_o n} \qquad [\because x(n) = x^*(n)]$$

$$= X(-k)$$

Hence the proof.

Example 3.9 If a real valued discrete-time sequence $x(n)$ is even, prove that its DTFS $X(k)$ has only real parts.

If $x(n)$ is even; then $x(n) = x(-n)$

Replacing 'k' by '−k' in eqn. (E3.8.1) we get,

$$X(-k) = \frac{1}{N}\sum_{n=<N>} x(n)e^{jk\Omega_o n}$$

$$X(-k) = \frac{1}{N}\sum_{n=<N>} x(-n)e^{jk\Omega_o n} \qquad [\because x(n) = x(-n)]$$

Replacing 'n' by '–m' we get,

$$X(-k) = \frac{1}{N}\sum_{m=<N>} x(m)e^{-jk\Omega_o m} \qquad \text{[For real } x(n) : X(k) = X(-k)]$$

$$= X(k) = X^*(k)$$

$\therefore$ X(k) is real valued. i.e., $\text{Img}\{X(k)\} = 0$

Example 3.10 Find the time domain signal corresponding to the DTFS coefficients X(k) and sketch it.

$$X(k) = \cos\left(\frac{6\pi}{17}k\right)$$

Solution. We know that the DTFS coefficients X(k) of a periodic signal $x(n)$ with period N also periodic with period N.

$$\therefore \quad X(k) = \cos\left(\frac{6\pi}{17}k\right) = \cos\left(\frac{3}{17}2\pi k\right)$$

is periodic with period N = 17 and has angular frequency $\Omega_o = \frac{2\pi}{17}$.

Given X(k) can be written as,

$$X(k) = \frac{e^{j\frac{6\pi}{17}k} + e^{-j\frac{6\pi}{17}k}}{2} \qquad \text{(E3.10.1)}$$

$$X(k) = \frac{1}{2}e^{j(3)(\frac{2\pi}{17})k} + \frac{1}{2}e^{j(-3)(\frac{2\pi}{17})k}$$

We have,

$$X(k) = \frac{1}{N}\sum_{n=<N>} x(n)e^{-jk\Omega_o n} \qquad \text{(E3.10.2)}$$

Multiplying & dividing eqn. (E3.10.1) by 17 and comparing with eqn. (E3.10.2) we get,

$$x(3) = \frac{17}{2}\ ;\ x(-3) = \frac{17}{2} \text{ and } x(n) = 0\ ;\ n \neq \pm 3.$$

The sketch of $x(n)$ (for one cycle) is shown in Fig. E3.10.1 below.

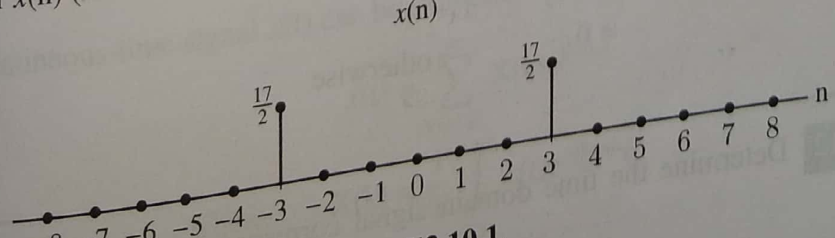


Fig. E3.10.1.

Example 3.11 Find the time domain signal corresponding to the DTFS coefficients,

$$X(k) = \cos\left(\frac{10\pi}{21}k\right) + j\sin\left(\frac{4\pi}{21}k\right)$$

Solution. Given:

$$X(k) = \cos\left(\frac{10\pi}{21}k\right) + j\sin\left(\frac{4\pi}{21}k\right)$$

$$= \left(\frac{e^{j\frac{10\pi}{21}k} + e^{-j\frac{10\pi}{21}k}}{2}\right) + j\left(\frac{e^{j\frac{4\pi}{21}k} - e^{-j\frac{4\pi}{21}k}}{2j}\right)$$

$$= \left(\frac{e^{-j(-5)\frac{2\pi}{21}k} + e^{-j(5)\frac{2\pi}{21}k}}{2}\right) + \left(\frac{e^{-j(-2)\frac{2\pi}{21}k} - e^{-j(2)\frac{2\pi}{21}k}}{2}\right)$$

$$N = 21 \;;\; \Omega_o = \frac{2\pi}{21} \tag{E3.11.1}$$

Multiplying & dividing eqn. (E3.11.1) by 21 and comparing with equation,

$$X(k) = \frac{1}{N}\sum_{<N>} x(n)e^{-j\Omega_o kn}$$

we get,

$$x(n) = \frac{21}{2} \quad ; n = \pm 5 \;\text{ and }\; n = -2$$

$$= -\frac{21}{2} \quad ; n = 2$$

$$= 0 \quad ; \text{otherwise.}$$

Example 3.12 Determine the time domain signal corresponding to the spectra shown in Fig. E3.12.

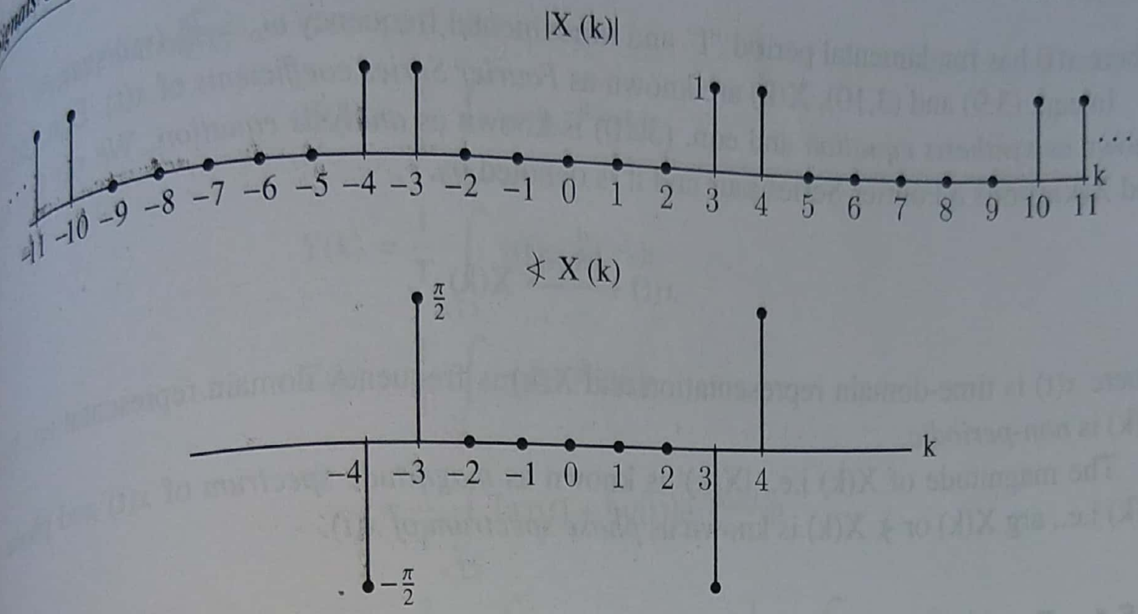


Fig. E3.12.

Solution. $N = 7$; $\Omega_o = \dfrac{2\pi}{7}$

$$x(n) = \sum_{k=<N>} X(k)e^{jk\Omega_o n}$$

Taking $n, k \in (-3.....3)$

$$x(n) = \sum_{k=-3}^{3} X(k)e^{jk\frac{2\pi}{7}n}$$

From Fig. E3.12 we get,

$$x(n) = 1.e^{j\frac{\pi}{2}}.e^{j(-3)(\frac{2\pi}{7})n} + 1.e^{-j\frac{\pi}{2}}e^{j(3)(\frac{2\pi}{7})n}$$

$$= je^{-(\frac{6\pi}{7})n} - je^{j(\frac{6\pi}{7})n}$$

$$x(n) = 2\sin\left(\frac{6\pi n}{7}\right) \qquad n \in (-3, ..., 3)$$

3.5 Continuous-Time Periodic Signals : The Fourier Series (FS)

A periodic continuous-time signal $x(t)$ can be expressed as,

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_0 t} \tag{3.9}$$

where
$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}dt \tag{3.10}$$

where $x(t)$ has fundamental period 'T' and fundamental frequency $\omega_o = \frac{2\pi}{T}$ (rad/sec)

In eqn. (3.9) and (3.10), X(k) are known as *Fourier Series coefficients* of $x(t)$. Eqn. (3.9) is known as *synthesis equation* and eqn. (3.10) is known as *analysis equation.* We say that $x(t)$ and X(k) forms a Fourier Series pair and it is denoted by,

$$x(t) \xleftrightarrow{FS;\omega_o} X(k)$$

where $x(t)$ is time-domain representation and X(k) is frequency domain representation. Here X(k) is *non-periodic*.

The magnitude of X(k) i.e., |X(k)| is known as *magnitude spectrum* of $x(t)$ and phase of X(k) i.e., arg X(k) or $\measuredangle$ X(k) is known as *phase spectrum of* $x(t)$.

3.5.1 Properties of FS

In this section, we will discuss the different properties of Fourier Series. They are,

(a) Linearity

(b) Time shift

(c) Frequency shift

(d) Scaling

(e) Time-differentiation

(f) Convolution

(g) Modulation

(h) Parseval's Theorem

(i) Symmetry

(a) Linearity:

If $x(t) \xleftrightarrow{FS;\omega_o} X(k)$

and $y(t) \xleftrightarrow{FS;\omega_o} Y(k)$

then $z(t) = ax(t) + by(t) \xleftrightarrow{FS;\omega_o} Z(k) = aX(k) + bY(k)$

In this case, both $x(t)$ and y(t) are assumed to have the same fundamental period $T = \frac{2\pi}{\omega_o}$.

Proof. We have,

$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}dt$$

$$Y(k) = \frac{1}{T}\int_{<T>} y(t)e^{-jk\omega_0 t}dt$$

$$\therefore \quad Z(k) = \frac{1}{T}\int_{<T>} z(t)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T}\int_{<T>} [ax(t) + by(t)]e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T}\, a\int_{<T>} x(t)\ e^{-jk\omega_0 t}dt + \frac{1}{T}\, b\int_{<T>} y(t)e^{-jk\omega_0 t}dt$$

$$\therefore \quad Z(k) = aX(k) + bY(k).$$

Hence the proof.

(b) Time Shift:

$$\text{If} \quad x(t) \xleftrightarrow{FS;\omega_0} X(k)$$

$$\text{then} \quad w(t) = x(t - t_0) \xleftrightarrow{FS;\omega_0} W(k) = e^{-jk\omega_0 t_0}X(k)$$

Proof. We have,

$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}dt$$

$$\therefore W(k) = \frac{1}{T}\int_{<T>} w(t)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T}\int_{<T>} x(t - t_0)e^{-jk\omega_0 t}dt$$

Put $t - t_0 = m$, then

$$W(k) = \frac{1}{T}\int_{<T>} x(m)e^{-jk\omega_0 (m+t_0)}dm$$

$$= e^{-jk\omega_0 t_0}\frac{1}{T}\int_{<T>} x(m)e^{-jk\omega_0 m}dm$$

$$= e^{-jk\omega_0 t_0}X(k)$$

Hence the proof.

(c) Frequency Shift: If $x(t) \xleftrightarrow{FS;\omega_o} X(k)$

then $g(t) = e^{jk_o\omega_o t}x(t) \xleftrightarrow{FS;\omega_o} G(k) = X(k - k_o)$

Proof. We have,

$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_o t}dt$$

$$\therefore G(k) = \frac{1}{T}\int_{<T>} g(t)e^{-jk\omega_o t}dt$$

$$= \frac{1}{T}\int e^{jk_o\omega_o t}x(t)e^{-jk\omega_o t}dt$$

$$= \frac{1}{T}\int_{<T>} x(t)e^{-j\omega_o(k-k_o)t}dt$$

$$= X(k - k_o)$$

Hence the proof.

(d) Scaling: If $x(t) \xleftrightarrow{FS;\omega_o} X(k)$.

then $z(t) = x(at) \xleftrightarrow{FS;a\omega_o} Z(k) = X(k) \quad ; a > 0$

Proof. If $x(t)$ is periodic, then $z(t) = x(at)$ is also periodic. If $x(t)$ has fundamental period T, then $z(t) = x(at)$ has fundamental period T/a.

Alternatively, if the fundamental frequency of $x(t)$ is ω_o (rad/sec), then the fundamental frequency of $z(t) = x(at)$ is $a\omega_o$ (rad/sec).

$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_o t}dt$$

$$\therefore Z(k) = \frac{1}{(T/a)}\int_{<T/a>} z(t)e^{-jka\,\omega_o t}dt$$

$$= \frac{a}{T}\int_{<T/a>} x(at)e^{-jka\omega_o t}dt$$

Put $at = m$, then $dt = \frac{1}{a}dm$

$$\therefore Z(k) = \frac{a}{T}\int_{<T/a>} x(m)e^{-jk\omega_o m}\frac{1}{a}dm$$

$$\therefore Z(k) = X(k)$$

Therefore, the FS coefficients of $x(t)$ and $x(at)$ are identical but the harmonic spacing changes from 'ω_o' to '$a\omega_o$'.

(e) **Time-differentation:**

$$\text{If } x(t) \xleftrightarrow{FS;\omega_o} X(k)$$

$$\text{then } \frac{dx(t)}{dt} \xleftrightarrow{FS;\omega_o} jk\omega_o X(k)$$

Proof. From eqn. (3.9) we have

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_o t}$$

Differentiating both the sides w.r.t time 't' we get,

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left[\sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_o t}\right]$$

Changing the order of differentiation and summation,

$$\frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} X(k)\frac{\partial}{\partial t}e^{jk\omega_o t}$$

$$= \sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_o t}.(jk\omega_o)$$

$$\frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} jk\omega_o X(k)e^{jk\omega_o t}$$

Comparing with eqn. (3.9) we get,

$$\frac{dx(t)}{dt} \xleftrightarrow{FS;\omega_o} jk\omega_o X(k)$$

Hence the proof.

(f) **Convolution:**

$$\text{If } x(t) \xleftrightarrow{FS;\omega_o} X(k)$$

$$\text{and } y(t) \xleftrightarrow{FS;\omega_o} Y(k)$$

$$\text{then } z(t) = x(t) \circledast y(t) \xleftrightarrow{FS;\omega_o} Z(k) = TX(k)Y(k)$$

where '$\circledast$' denotes periodic convolution.

Proof. We have,

$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_o t}dt$$

$$Y(k) = \frac{1}{T}\int_{<T>} y(t)e^{-jk\omega_o t}dt$$

$$\therefore \quad Z(k) = \frac{1}{T}\int_{<T>} z(t)e^{-jk\omega_o t}dt$$

$$= \frac{1}{T}\int_{<T>} [x(t) \circledast y(t)]e^{-jk\omega_o t}dt$$

Using the definition of periodic convolution we get,

$$Z(k) = \frac{1}{T}\int_{<T>}\left[\int_{\ell=<T>} x(\ell)y(t-\ell)d\ell\right]e^{-jk\omega_o t}dt$$

Changing the order of integration we get,

$$Z(k) = \frac{1}{T}\int_{\ell=<T>} x(\ell)\int_{t=<T>} y(t-\ell)e^{-jk\omega_o t}d\ell dt$$

Put $t - \ell = m$, $\therefore$ $dt = dm$, then,

$$Z(k) = \frac{1}{T}\int_{\ell=<T>} x(\ell)\int_{m=<T>} y(m)e^{-jk\omega_o (m+\ell)}d\ell dm$$

$$= \frac{1}{T}\int_{\ell=<T>} x(\ell)e^{-jk\omega_o \ell}d\ell \int_{m=<T>} y(m)e^{-jk\omega_o m}dm$$

$$= \frac{1}{T}[TX(k)TY(k)]$$

$$\therefore \quad Z(k) = TX(k)Y(k)$$

Hence the proof.

$\therefore$ Convolution in time domain is transformed to multiplication of FS coefficients.

(g) Modulation:

$$\text{If} \quad x(t) \xleftrightarrow{FS;\omega_o} X(k)$$

$$\text{and} \quad y(t) \xleftrightarrow{FS;\omega_o} Y(k)$$

$$\text{then} \quad z(t) = x(t)y(t) \xleftrightarrow{FS;\omega_o} Z(k) = X(k) * Y(k)$$

Proof. We have,

$$Z(k) = \frac{1}{T}\int_{<T>} z(t)e^{-jk\omega_o t}dt$$

$$= \frac{1}{T}\int_{<T>} x(t)y(t)e^{-jk\omega_o t}dt \quad (3.11)$$

From eqn. (3.9) we have,

$$x(t) = \sum_{\ell=-\infty}^{\infty} X(\ell)e^{j\ell\omega_o t} \quad (3.12)$$

Substituting eqn. (3.12) in eqn. (3.11) we get,

$$Z(k) = \frac{1}{T}\int_{<T>}\left[\sum_{\ell=-\infty}^{\infty} X(\ell)e^{j\ell\omega_o t}\right] y(t)e^{-jk\omega_o t}dt$$

Changing the order of summation and integration we get,

$$Z(k) = \frac{1}{T}\left[\sum_{\ell=-\infty}^{\infty} X(\ell)\int_{<T>} y(t)e^{-j(k-\ell)\omega_o t}dt\right]$$

$$= \sum_{\ell=-\infty}^{\infty} X(\ell)Y(k-\ell)$$

$$\therefore Z(k) = X(k) * Y(k).$$

Hence the proof.

(h) Parseval's Theorem: If $x(t) \xleftrightarrow{FS;\omega_o} X(k)$

then $$\frac{1}{T}\int_{<T>} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X(k)|^2 \quad (3.13)$$

Proof. The LHS of eqn. (3.13) is the average power of a periodic continuous-time signal $x(t)$ with fundamental period T.

i.e., $$P = \frac{1}{T}\int_{<T>} |x(t)|^2 dt$$

This equation can be written as,

$$P = \frac{1}{T}\int_{<T>} x(t)x^*(t)dt$$

$$= \frac{1}{T}\int_{<T>} x(t)\left[\sum_{k=-\infty}^{\infty} X^*(k)e^{-jk\omega_o t}\right]dt$$

Changing the order of summation and integration we get,

$$P = \frac{1}{T}\sum_{k=-\infty}^{\infty} X^*(k)\int_{<T>} x(t)e^{-jk\omega_o t}dt$$

$$= \sum_{k=-\infty}^{\infty} X^*(k)X(k)$$

$$= \sum_{k=-\infty}^{\infty} |X(k)|^2$$

$$\therefore \quad \frac{1}{T}\int_{<T>} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X(k)|^2 \tag{3.14}$$

In eqn. (3.14), the sequence $|X(k)|^2$ for k = 0, 1, 2....... is the distribution of power as a function of frequency and it is called *power density spectrum* of the signal $x(t)$.

(i) Symmetry: If $x(t) \xleftrightarrow{FS;\omega_o} X(k)$

then (i) If $x(t)$ is real, then $X^*(k) = X(-k)$

(ii) If $x(t)$ is real and even, then $\text{Img}\{X(k)\} = 0$

(iii) If $x(t)$ is real and odd, then $\text{Re}\{X(k)\} = 0$.

EXAMPLES

Example 3.13 For the signal $x(t) = \sin(\omega_o t)$, find the Fourier series and draw its spectrum.

Solution. Given:

$e^{j\theta} = \cos\theta + j\sin\theta$

$$x(t) = \sin\omega_o t$$

$e^{-j\theta} = \cos\theta - j\sin\theta$

$$\therefore x(t) = \frac{1}{2j}e^{j\omega_o t} - \frac{1}{2j}e^{-j\omega_o t} \tag{E3.13.1}$$

$e^{j\theta} + e^{-j\theta} = 2\cos\theta$

$$= \frac{1}{2j}e^{j(1)\omega_o t} - \frac{1}{2j}e^{j(-1)\omega_o t}$$

$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$

Comparing eqn. (E3.13.1) with eqn. (3.9), i.e., with,

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_o t}$$

we get $X(1) = \frac{1}{2j}$; $X(-1) = -\frac{1}{2j}$

and $X(k) = 0$ for $k \neq \pm 1$.

The magnitude and phase spectra are shown in Fig. E3.13.1 and E3.13.2 respectively.

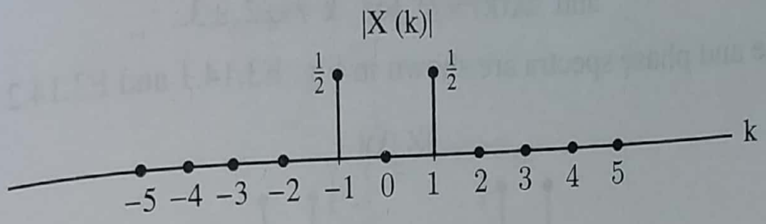


Fig. E3.13.1.

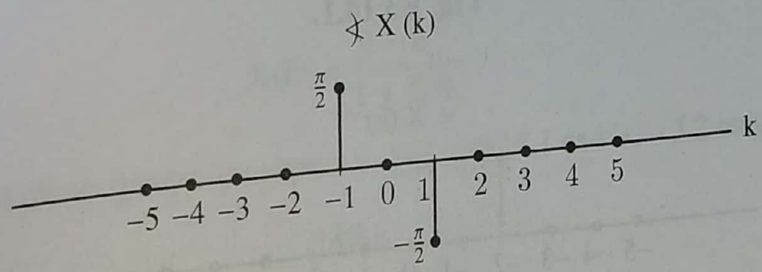


Fig. E3.13.2.

Example 3.14 Evaluate the FS representation for the signal,

$$x(t) = \sin(2\pi t) + \cos(3\pi t) \tag{E3.14.1}$$

Solution. Given:

$$x(t) = \sin(2\pi t) + \cos(3\pi t)$$

The first term has angular frequency $\omega_{o1} = 2\pi$

The second term has angular frequency $\omega_{o2} = 3\pi$

$\therefore$ The angular frequency of the summation i.e., of $x(t)$

$$\omega_o = \text{g.c.d}\,(\omega_{o1}, \omega_{o2})$$

$$= \text{g.c.d}\,(2\pi, 3\pi)$$

$$\omega_o = \pi$$

Eqn. (E3.14.1) can be written as,

$$x(t) = \frac{1}{2j}e^{j2\pi t} - \frac{1}{2j}e^{-j2\pi t} + \frac{1}{2}e^{j3\pi t} + \frac{1}{2}e^{-j3\pi t}$$

Comparing with eqn. (3.9) we get,

$$X(2) = \frac{1}{2j} \quad ; \quad X(-2) = \frac{-1}{2j}$$

$$X(3) = \frac{1}{2} \quad ; \quad X(-3) = \frac{1}{2}$$

$$\text{and} \quad X(k) = 0 \ \text{ for } \ k \neq \pm 2, \pm 3.$$

The magnitude and phase spectra are shown in Fig. E3.14.1 and E3.14.2 respectively.

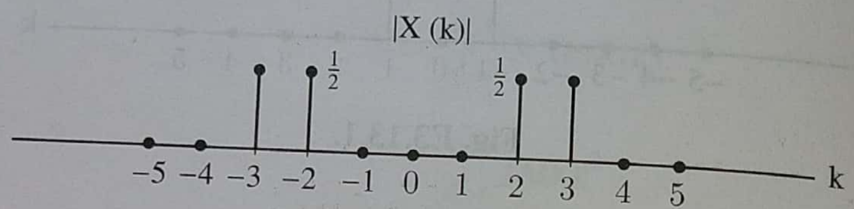


Fig. E3.14.1.

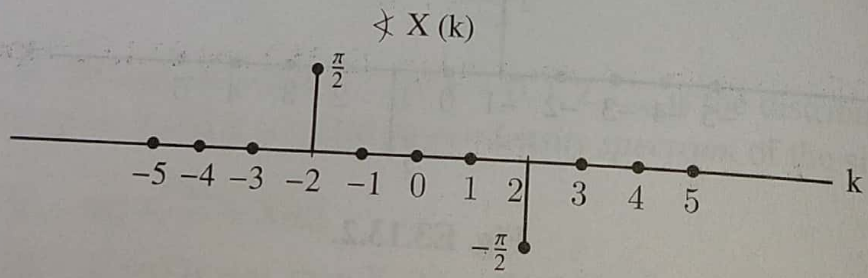


Fig. E3.14.2.

Example 3.15 For the signal $x(t)$ shown below in Fig. E3.15, find the FS representation and draw its magnitude and phase spectra.

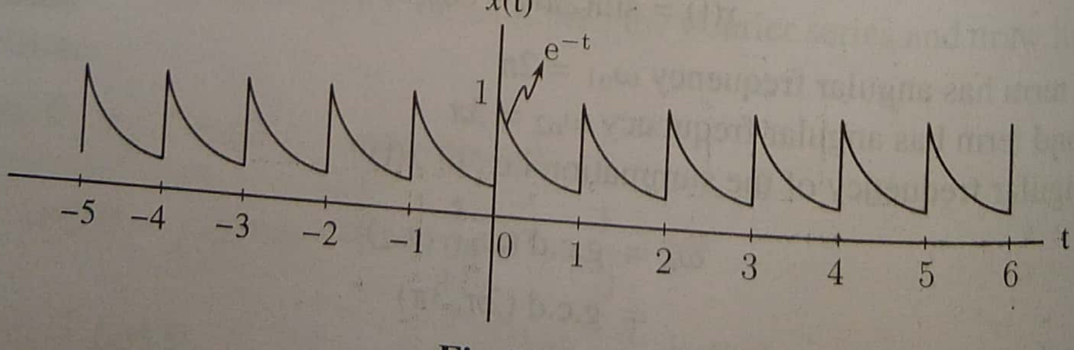


Fig. E3.15.

Solution. By observing Fig. E3.15 we have,

$$T = 1 \text{ and } \omega_0 = \frac{2\pi}{T} = 2\pi$$

From eqn. (3.10) we have,

$$X(k) = \frac{1}{T} \int_{<T>} x(t) e^{-jk\omega_0 t} dt$$

$$\therefore \quad X(k) = \int_{t=0}^{1} e^{-t} e^{-jk(2\pi)t} dt$$

$$= \int_{t=0}^{1} e^{-(1+j2\pi k)t} dt$$

$$= -\frac{e^{-(1+j2\pi k)t}}{(1+j2\pi k)}\Bigg|_0^1$$

$$X(k) = \frac{1 - e^{-1}}{1 + j2\pi k}$$

The magnitude and phase spectra are shown in Fig. E3.15.1 and E3.15.2 respectively.

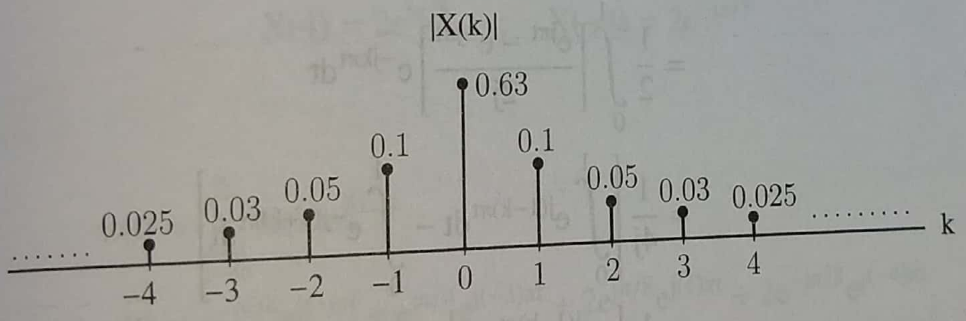


Fig. E3.15.1.

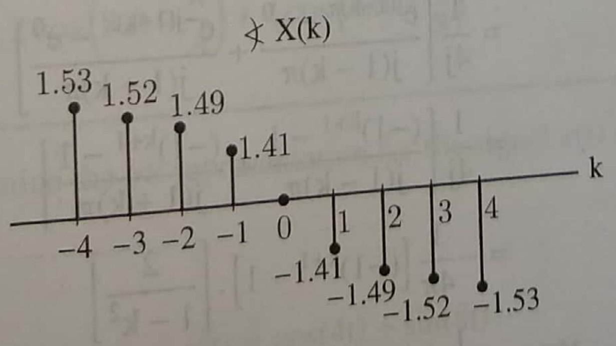


Fig. E3.15.2.

Example 3.16 Find the FS coefficients for the signal $x(t)$ shown in Fig. E3.16.

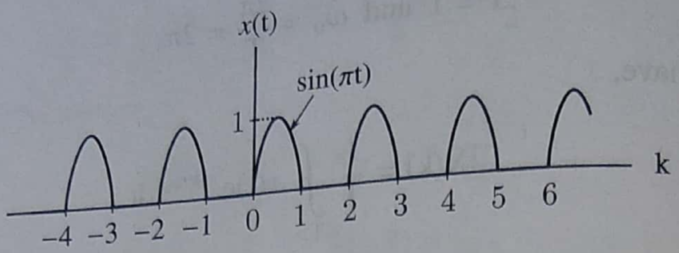


Fig. E3.16.

Solution. By observing Fig. E3.16 we have,

$$T = 2 \text{ and } \omega_0 = \frac{2\pi}{T} = \pi$$

From eqn. (3.10) we have,

$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}dt$$

$$\therefore X(k) = \frac{1}{2}\int_0^1 \sin(\pi t)e^{-jk\pi t}dt$$

$$= \frac{1}{2}\int_0^1 \left[\frac{e^{j\pi t} - e^{-j\pi t}}{2j}\right]e^{-jk\pi t}dt$$

$$= \frac{1}{4j}\left[\int_0^1 e^{j(1-k)\pi t}dt - \int_0^1 e^{-j(1+k)\pi t}dt\right]$$

$$= \frac{1}{4j}\left[\frac{e^{j(1-k)\pi t}}{j(1-k)\pi}\Bigg|_0^1 + \frac{e^{-j(1+k)\pi t}}{j(1+k)\pi}\Bigg|_0^1\right]$$

$$= \frac{1}{4j}\left[\frac{e^{j(1-k)\pi} - e^0}{j(1-k)\pi} + \frac{e^{-j(1+k)\pi} - e^0}{j(1+k)\pi}\right]$$

$$= \frac{1}{4j}\left[\frac{(-1)^{k+1} - 1}{j(1-k)\pi} + \frac{(-1)^{k+1} - 1}{j(1+k)\pi}\right]$$

$$= -\frac{1}{4\pi}\left[(-1)^{k+1} - 1\right].\left[\frac{2}{1-k^2}\right]$$

$$X(k) = \frac{1}{2\pi}\left[1 - (-1)^{k+1}\right].\left[\frac{1}{1-k^2}\right]$$

Example 3.17 Determine the time domain signal corresponding to the magnitude and phase spectra shown in Fig. E3.17 with $\omega_o = \pi$.

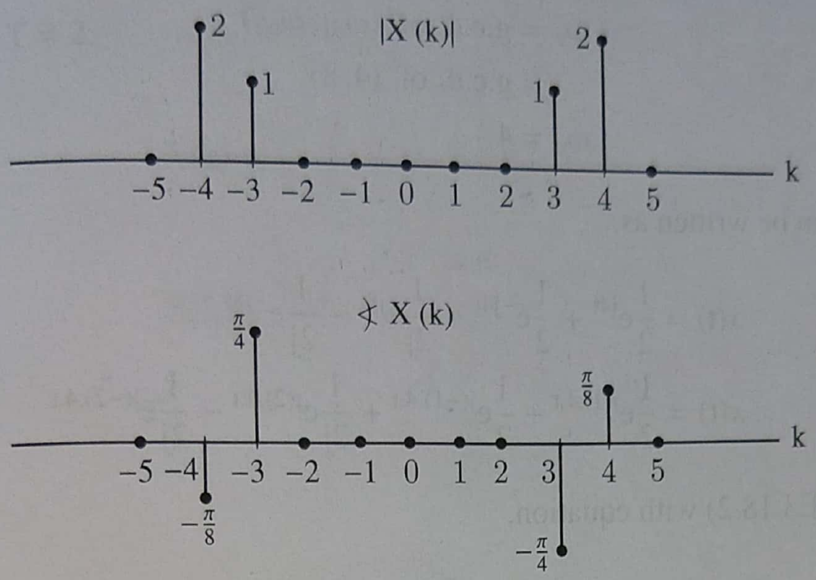


Fig. E3.17.

Solution. From Fig. E3.17 we have,

$$X(3) = 1e^{-j\pi/4} \quad ; \quad X(-3) = 1e^{j\pi/4}$$

$$X(4) = 2e^{j\pi/8} \quad ; \quad X(-4) = 2e^{-j\pi/8}$$

We have,

$$x(t) = \sum_{k-\infty}^{\infty} X(k)e^{jk\omega_o t}$$

$$\therefore x(t) = e^{-j\pi/4}e^{j(3)\pi t} + e^{j\pi/4}e^{j(-3)\pi t} + 2e^{j\pi/8}e^{j(4)\pi t} + 2e^{-j\pi/8}e^{j(-4)\pi t}$$

$$= \left[e^{j(3\pi t-\pi/4)} + e^{-j(3\pi t-\pi/4)}\right] + 2\left[e^{j(4\pi t+\pi/8)} + e^{-j(4\pi t+\pi/8)}\right]$$

$$\therefore x(t) = 2\cos\left(3\pi t - \frac{\pi}{4}\right) + 4\cos\left(4\pi t + \frac{\pi}{8}\right)$$

Example 3.18 Determine the FS representation for the signal $x(t) = \cos 4t + \sin 8t$ and draw the spectrum.

Solution. Given:

$$x(t) = \cos(4t) + \sin(8t) \tag{E3.18.1}$$

Angular frequency of first term is $\omega_{o1} = 4$

Angular frequency of second term is $\omega_{o2} = 8$

$\therefore$ Angular frequency of summation [i.e., of $x(t)$] is,

$$\omega_o = \text{g.c.d. of } (\omega_{o1}, \omega_{o2})$$
$$= \text{g.c.d. of } (4, 8)$$
$$\therefore \quad \omega_o = 4$$

Eqn. (E3.18.1) can be written as,

$$x(t) = \frac{1}{2}e^{j4t} + \frac{1}{2}e^{-j4t} + \frac{1}{2j}e^{j8t} - \frac{1}{2j}e^{-j8t}$$

$$\therefore x(t) = \frac{1}{2}e^{j(1).4.t} + \frac{1}{2}e^{j(-1).4.t} + \frac{1}{2j}e^{j(2).4.t} - \frac{1}{2j}e^{j(-2).4.t} \tag{E3.18.2}$$

Comparing eqn. (E3.18.2) with equation,

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_o t}$$

we get,

$$X(-1) = \frac{1}{2} \quad ; \; X(1) = \frac{1}{2}$$
$$X(-2) = -\frac{1}{2j} \quad ; \; X(2) = \frac{1}{2j}$$
$$\& \;\; X(k) = 0 \;\; \text{for} \;\; k \neq \pm 1, \pm 2.$$

The spectrum is shown in Fig. E3.18.1 below.

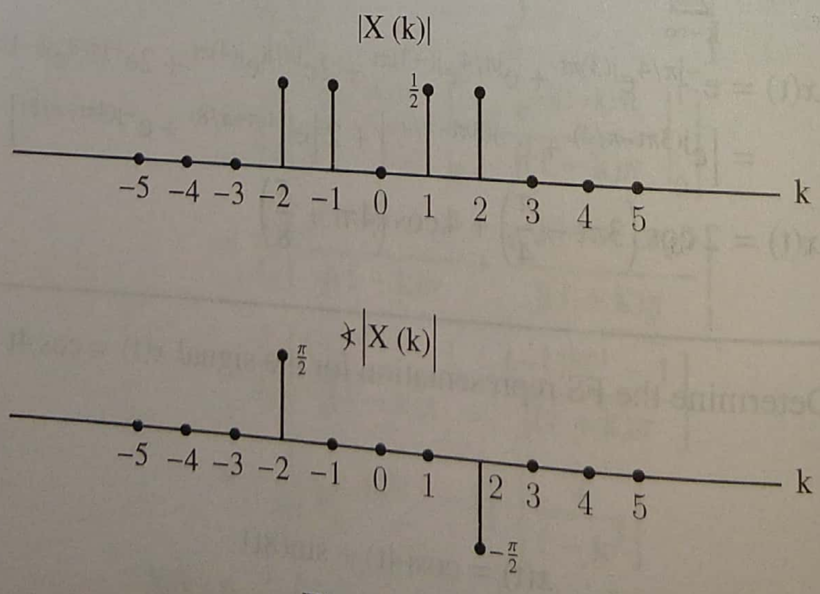


Fig. E3.18.1.

Example 3.19 Find the FS coefficients for the periodic signal $x(t)$ with period 2 given by,

$$x(t) = e^{-t} \ ; \text{for} \ -1 < t < 1.$$

Solution. Given: T = 2. $\quad \therefore \omega_0 = \frac{2\pi}{T} = \pi.$

$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}dt$$

$$\therefore \quad X(k) = \frac{1}{2}\int_{-1}^{1} e^{-t}e^{-jk\pi t}dt$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+jk\pi)t}dt$$

$$= \frac{1}{2}\left[\frac{e^{-(1+jk\pi)t}}{-(1+jk\pi)}\right]_{-1}^{1}$$

$$= -\frac{1}{2(1+jk\pi)}\left[e^{-(1+jk\pi)} - e^{(1+jk\pi)}\right]$$

$$= \frac{(-1)^k}{2(1+jk\pi)}(e - e^{-1}) \qquad ; \text{for all } k.$$

Example 3.20 If a real valued continuous time periodic signal $x(t)$ has FS X(k), prove that,

$$X^*(k) = X(-k)$$

Solution. We have,

$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}dt \tag{E3.20.1}$$

If $x(t)$ is real valued, $x(t) = x^*(t)$

Taking complex conjugate on both the sides in eqn. (E3.20.1),

$$X^*(k) = \frac{1}{T}\int_{<T>} x^*(t)e^{jk\omega_0 t}dt$$

$$= \frac{1}{T}\int_{<T>} x(t)e^{-j(-k)\omega_0 t}dt \qquad [\because x(t) = x^*(t)]$$

$$= X(-k)$$

$$\therefore \quad x(t) : \text{real} \xleftrightarrow{FS;\omega_0} X^*(k) = X(-k)$$

Example 3.21 Find the Fourier series coefficients for the periodic signal $x(t)$ shown in Fig. E3.21 below.

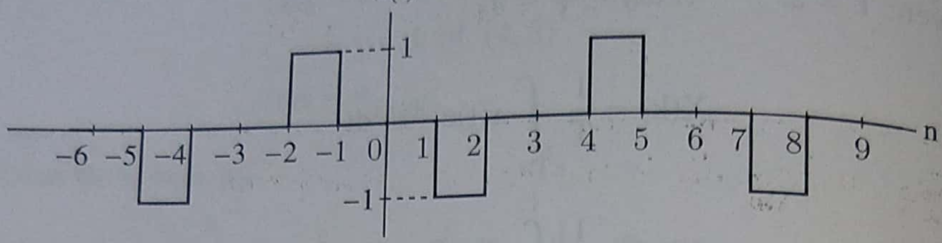


Fig. E3.21.

Solution. By observation, T = 6 $\quad \therefore \omega_o = \frac{2\pi}{T} = \frac{\pi}{3}$

We have,

$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_o t}dt$$

$$\therefore X(k) = \frac{1}{6}\int_{-3}^{3} x(t)e^{-jk(\frac{\pi}{3})t}dt$$

$$= \frac{1}{6}\left[\int_{-2}^{-1} e^{-jk(\frac{\pi}{3})t}dt - \int_{1}^{2} e^{-jk(\frac{\pi}{3})t}dt\right]$$

$$= \frac{1}{6}\left[\frac{e^{-jk(\frac{\pi}{3})t}}{-jk(\frac{\pi}{3})}\Bigg|_{-2}^{-1} - \frac{e^{-jk(\frac{\pi}{3})t}}{-jk(\frac{\pi}{3})}\Bigg|_{1}^{2}\right]$$

$$X(k) = \frac{1}{3}\left[\frac{\cos\left(\frac{2\pi}{3}k\right) - \cos\left(\frac{\pi}{3}k\right)}{\left(jk\frac{\pi}{3}\right)}\right]$$

Example 3.22 Find the time domain signal whose FS coefficient is given by,

$$X(k) = j\delta(k-1) - j\delta(k+1) + \delta(k-3) + \delta(k+3) \qquad ;\omega_o = \pi$$

Solution. We have,

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_o t}$$

$$\therefore x(t) = \sum_{k=-\infty}^{\infty} X(k)e^{jk\pi t}$$

$$= je^{j\pi t} - je^{-j\pi t} + e^{j3\pi t} + e^{-j3\pi t}$$

$$x(t) = -2\sin(\pi t) + 2\cos(3\pi t)$$

Example 3.23 Find the time domain signal corresponding to,

$$X(k) = \left(-\frac{1}{2}\right)^{|k|} \quad ; \omega_o = 1$$

Solution. We have,

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_o t}$$

$$\therefore x(t) = \sum_{k=-\infty}^{-1} \left(-\frac{1}{2}\right)^{-k} e^{jkt} + \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^{k} e^{jkt}$$

$$= \sum_{k=1}^{\infty} \left(-\frac{1}{2}e^{-jt}\right)^{k} + \sum_{k=0}^{\infty} \left(-\frac{1}{2}e^{jt}\right)^{k}$$

$$= \frac{-\frac{1}{2}e^{-jt}}{1 + \frac{1}{2}e^{-jt}} + \frac{1}{1 + \frac{1}{2}e^{jt}}$$

$$\therefore x(t) = \frac{\frac{3}{4}}{\frac{5}{4} + \cos(t)}$$

Example 3.24 Find the Fourier series coefficients X(k) for the signal,

$$x(t) = \sum_{m=-\infty}^{\infty} \left[\delta\left(t - \frac{1}{2}m\right) + \delta\left(t - \frac{3}{2}m\right)\right]$$

Also sketch the amplitude and phase spectra.

Solution. The sketch of $x(t)$ is shown in Fig. E3.24.1 below.

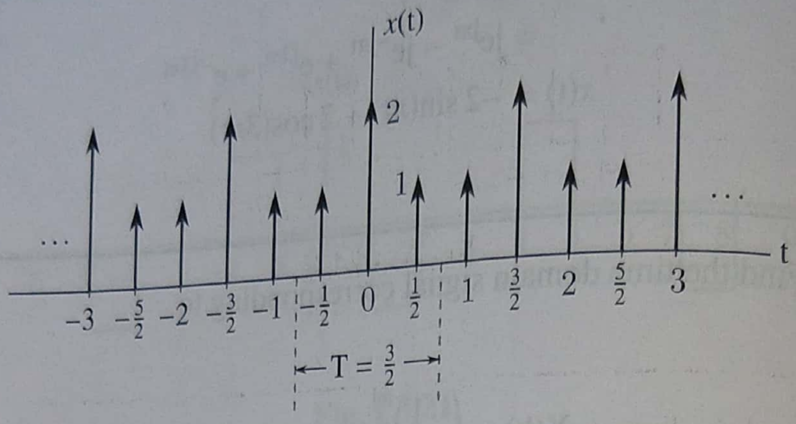


Fig. E3.24.1.

$$T = \frac{3}{2} \qquad ; \omega_o = \frac{2\pi}{T} = \frac{4\pi}{3}$$

The expression for $x(t)$ over one period is,

$$x(t) = \delta\left(t + \frac{1}{2}\right) + 2\delta(t) + \delta\left(t - \frac{1}{2}\right)$$

We have,

$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_o t}dt$$

$$\therefore X(k) = \frac{1}{3/2}\int_{<T>} \left[\delta\left(t + \frac{1}{2}\right) + 2\delta(t) + \delta\left(t - \frac{1}{2}\right)\right] e^{-jk\omega_o t}dt$$

Using sifting property we get,

$$X(k) = \frac{2}{3}\left[e^{jk\omega_o \cdot \frac{1}{2}} + 2 + e^{-jk\omega_o \cdot \frac{1}{2}}\right]$$

$$= \frac{4}{3}\left[1 + \cos\left(\frac{1}{2}k\omega_o\right)\right]$$

$$X(k) = \frac{4}{3}\left[1 + \cos\left(\frac{2\pi}{3}k\right)\right]$$

The magnitude spectrum is shown in Fig. E3.24.2 below.

Since the given signal $x(t)$ is real and even, the phase spectrum is null spectrum.

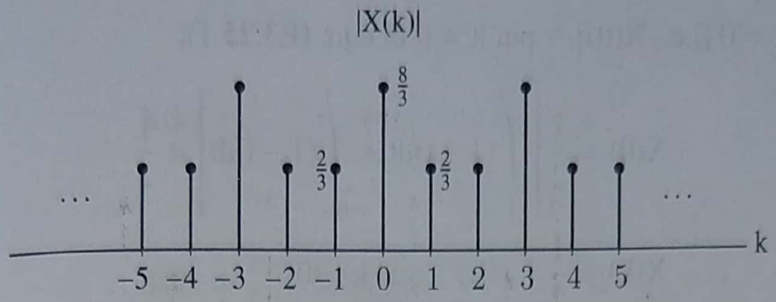


Fig. E3.24.2.

Example 3.25 Find the Fourier series coefficients for the signal $x(t)$ shown in Fig. E3.25 and draw the spectra.

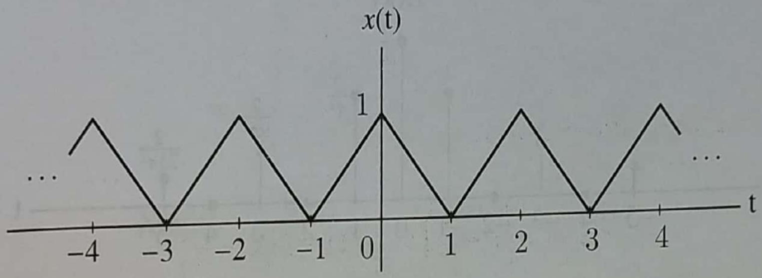


Fig. E3.25.

Solution. By observation T = 2 $\quad \therefore \omega_0 = \frac{2\pi}{T} = \pi$

The expression for $x(t)$ is given by,

$$x(t) = 1 + t \quad ; \; -1 < t < 0$$
$$= 1 - t \quad ; \; 0 < t < 1$$

We have,

$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{2}\left[\int_{-1}^{0}(1+t)e^{-jk\pi t}dt + \int_{0}^{1}(1-t)e^{-jk\pi t}dt\right] \qquad \text{(E3.25.1)}$$

$$X(k) = \frac{1}{\pi^2 k^2}\left[1-(-1)^k\right] \qquad ; k \neq 0$$

To find X(k) for k = 0 [i.e., X(0)], put k = 0 in eqn. (E3.25.1),

$$X(0) = \frac{1}{2}\left[\int_{-1}^{0}(1+t)dt + \int_{0}^{1}(1-t)dt\right] = \frac{1}{2}$$

$$\therefore \quad X(k) = \frac{1}{2} \quad ; \; k = 0$$

$$= 0 \quad ; \; k = \pm 2, \pm 4, \pm 6, \pm 8 \ldots\ldots$$

$$= \frac{2}{\pi^2 k^2} \quad ; \; k = \pm 1, \pm 3, \pm 5, \pm 7 \ldots\ldots$$

The spectrum is shown in Fig. E3.25.1 below.

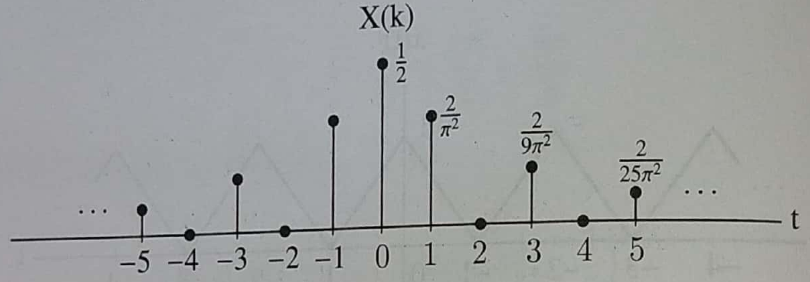


Fig. E3.25.1.

The phase spectrum is null spectrum because the given signal $x(t)$ is real and even.

Example 3.26 Plot and find the Fourier series coefficients for the periodic signal,

$$x(t) = |\sin(2\pi t)|$$

Solution. The plot of the signal is shown in Fig. E3.26.1 below,

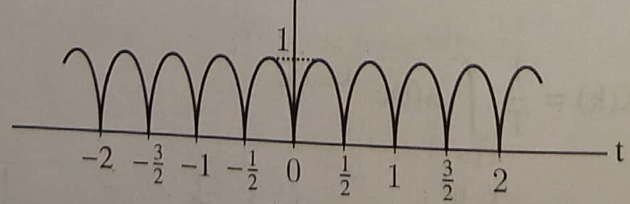


Fig. E3.26.1.

$$T = \frac{1}{2} \quad ; \omega_0 = \frac{2\pi}{T} = 4\pi$$

We have,

$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_o t}$$

$$\therefore X(k) = \frac{1}{\frac{1}{2}}\int_0^{\frac{1}{2}} \left|\sin(2\pi t)\right| e^{-jk(4\pi)t}dt$$

$$= 2\int_0^{\frac{1}{2}}\left[\frac{e^{j2\pi t} - e^{-j2\pi t}}{2j}\right]e^{-j4\pi kt}dt$$

$$= -j\int_0^{\frac{1}{2}}\left[e^{j2\pi(1-2k)t} + e^{-j2\pi(1+2k)t}\right]dt$$

$$= -j\left[\frac{e^{j2\pi(1-2k)t}}{j2\pi(1-2k)}\Bigg|_0^{\frac{1}{2}} - \frac{e^{-j2\pi(1+2k)t}}{j2\pi(1+2k)}\Bigg|_0^{\frac{1}{2}}\right]$$

$$X(k) = \frac{1-e^{j\pi(1-2k)}}{2\pi(1-2k)} - \frac{1-e^{-j\pi(1+2k)}}{2\pi(1+2k)}$$

Example 3.27 Find the FS representation for a periodic impulse train (Dirac comb) shown in Fig. E3.27 below and draw the spectrum.

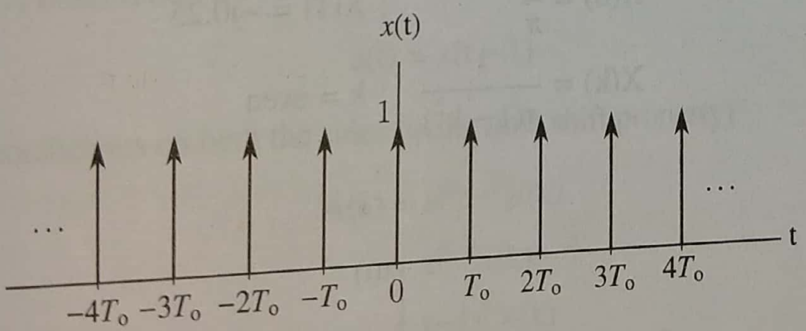


Fig. E3.27.

Solution.

$$T = T_o \quad ; \quad \omega_o = \frac{2\pi}{T} = \frac{2\pi}{T_o}$$

We have,

$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_o t}dt$$

$$= \frac{1}{T_0}\int_0^{T_0} \delta(t) e^{-jk\left(\frac{2\pi}{T_0}\right)t} dt$$

$$= \frac{1}{T_0} e^{-jk\left(\frac{2\pi}{T_0}\right)t}\Big|_{t=0} \quad [\because \text{ sifting property}]$$

$$= \frac{1}{T_0}$$

The spectrum is shown in Fig. E3.27.1.

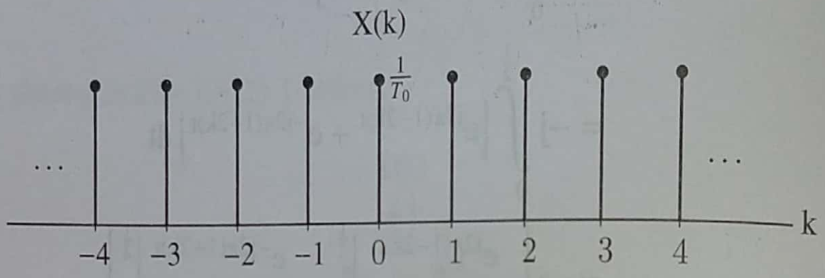


Fig. E3.27.1.

Example 3.28 The Fourier Series coefficients for the signal $x(t)$ shown in Fig. E3.28 is given by,

$$X(0) = \frac{1}{\pi} \quad ; X(1) = -j0.25$$

$$X(k) = \frac{1}{\pi(1-k^2)} \quad ; k = \text{even}$$

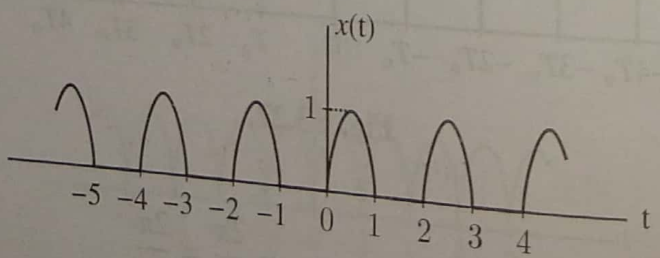
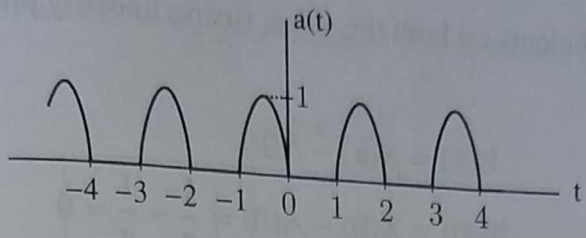


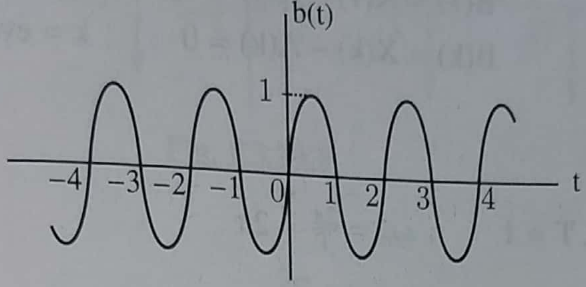
Fig. E3.28.

Using the appropriate properties, determine the FS coefficients of the following signals shown in Fig. E3.28.1 (a), (b) & (c) respectively.

(a)



(b)



(c)

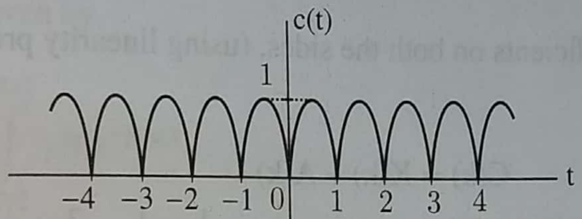


Fig. E3.28.1.

Solution. (a) By observation we have, T = 2 ; $\omega_o = \frac{2\pi}{T} = \pi$

$$a(t) = x(t-1)$$

Taking FS coefficients on both the sides (using time shift property)

$$A(k) = e^{jk\omega_o.1}X(k)$$
$$= e^{jk\pi}X(k)$$
$$= (-1)^k X(k)$$

$$\therefore A(0) = \frac{1}{\pi} \quad ; A(1) = j0.25$$

When k = even,

$$A(k) = X(k) = \frac{1}{\pi(1-k^2)}$$

(b) By observation we have, T = 2 ; $\omega_o = \frac{2\pi}{T} = \pi$

& $b(t) = x(t) - a(t)$

Taking FS coefficients on both the sides, (using linearity property)

$$B(k) = X(k) - A(k)$$

$$\therefore B(0) = X(0) - A(0) = \frac{1}{\pi} - \frac{1}{\pi} = 0$$

$$B(1) = X(1) - A(1) = -j0.25 - j0.25 = -j0.5$$

$$B(k) = X(k) - A(k) = 0 \quad ; k = \text{even}$$

(c) By observation, T = 1 $\quad ; \omega_0 = \frac{2\pi}{T} = 2\pi$

& $\quad c(t) = x(t) + a(t)$

Taking FS coefficients on both the sides, (using linearity property)

$$C(k) = X(k) + A(k)$$

$$\therefore C(0) = X(0) + A(0) = \frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi}$$

$$C(1) = X(1) + A(1) = -j0.25 + j0.25 = 0$$

$$C(k) = X(k) + A(k) = \frac{2}{\pi(1-k^2)} \quad ; k = \text{even}$$

Example 3.29 For the signal shown in Fig. E3.29, obtain the Fourier Series representation using time differentiation property.

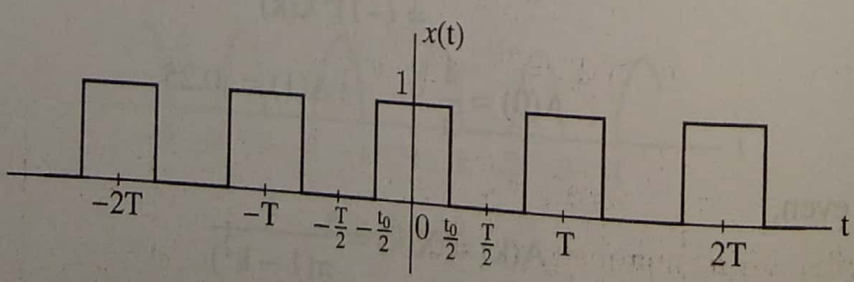


Fig. E3.29.

Solution. Differentiating $x(t)$ w.r.t time we get the signal y(t) as shown in Fig. E3.29.1 below.

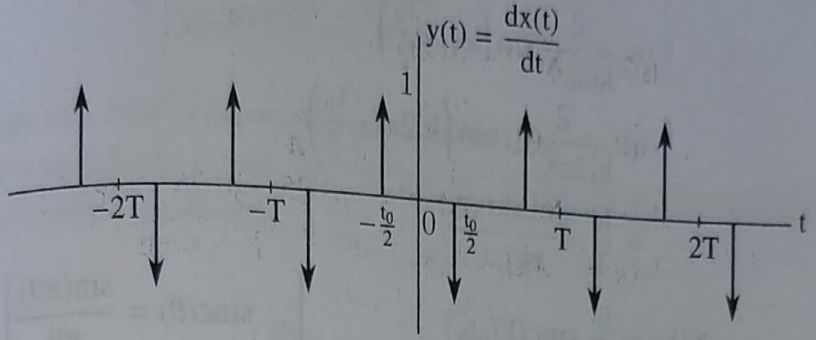


Fig. E3.29.1.

$$\omega_0 = \frac{2\pi}{T}$$

Fourier series of y(t) is given by,

$$Y(k) = \frac{1}{T}\int_{-T/2}^{T/2} y(t)e^{-jk\omega_0 t}$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \left[\delta\left(t+\frac{t_0}{2}\right) - \delta\left(t-\frac{t_0}{2}\right)\right] e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T}\left[\int_{-T/2}^{T/2} \delta\left(t+\frac{t_0}{2}\right)e^{-jk\omega_0 t}dt - \int_{-T/2}^{T/2} \delta\left(t-\frac{t_0}{2}\right)e^{-jk\omega_0 t}dt\right]$$

Using sifting property,

$$Y(k) = \frac{1}{T}\left[e^{jk\omega_0 \frac{t_0}{2}} - e^{-jk\omega_0 \frac{t_0}{2}}\right]$$

$$Y(k) = \frac{2j}{T}\sin\left(k\omega_0 \cdot \frac{t_0}{2}\right)$$

We have, $$y(t) = \frac{dx(t)}{dt}$$

Using time differentiation property we get,

$$Y(k) = jk\omega_0 X(k)$$

$$\therefore \quad X(k) = \frac{1}{jk\omega_0} Y(k)$$

$$= \frac{1}{jk\omega_o}\cdot\frac{2j}{T}\sin\left(k\omega_o\cdot\frac{t_o}{2}\right)$$

$$= \frac{2}{k\omega_o T}\sin\left(k\omega_o\cdot\frac{t_o}{2}\right)$$

$$= \frac{2}{k.2\pi f_o T}.\sin\left(k.2\pi f_o\cdot\frac{t_o}{2}\right)$$

$$= \frac{t_o}{T}\cdot\frac{\sin(\pi k f_o t_o)}{\pi k f_o t_o}$$

$$\therefore X(k) = \frac{t_o}{T}\,\text{sinc}(kf_o t_o) \qquad \left[\because\ \text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}\right]$$

3.6 Discrete-Time Non-Periodic Signals : The Discrete-Time Fourier Transform (DTFT)

A non-periodic discrete-time signal $x(n)$ can be expressed as,

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$$

$$\text{or} \qquad x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega \tag{3.15}$$

$$\text{where} \qquad X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n} \tag{3.16}$$

$X(e^{j\Omega})$ is known as *Discrete-time Fourier Transform* (DTFT) of the signal $x(n)$. Alternatively, we say that $X(e^{j\Omega})$ and $x(n)$ forms a DTFT pair which can be expressed as,

$$x(n) \xleftrightarrow{\text{DTFT}} X(e^{j\Omega})$$

Here $X(e^{j\Omega})$ is the frequency domain representation of the time-domain signal $x(n)$. $X(e^{j\Omega})$ is also known as *spectrum* of $x(n)$.

Eqn. (3.15) is called *synthesis equation* and eqn. (3.16) is called *analysis equation.*

Let us consider,

$$X(e^{j(\Omega+2k\pi)}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j(\Omega+2k\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}\cancel{e^{-j2k\pi n}} \rightarrow 1$$

$$= \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$X(e^{j(\Omega+2k\pi)}) = X(e^{j\Omega})$$

This indicates that $X(e^{j\Omega})$ is *periodic* with period 2π.

Eqn. (3.16) is an infinite series. This series will converge or alternatively $X(e^{j\Omega})$ will exist only if $x(n)$ is *absolutely summable.*

$$\text{i.e.,} \quad \sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

This is only the sufficient but not the necessary condition for $X(e^{j\Omega})$ to exist.

3.6.1 Properties of DTFT

In this section, we will discuss the different properties of DTFT. They are,

(a) Linearity

(b) Time shift

(c) Frequency shift

(d) Scaling

(e) Frequency differentiation

(f) Summation

(g) Convolution

(h) Modulation

(i) Parseval's Theorem

(j) Symmetry

(a) Linearity:

$$\text{If} \quad x(n) \xleftrightarrow{\text{DTFT}} X(e^{j\Omega})$$

$$\text{and} \quad y(n) \xleftrightarrow{\text{DTFT}} Y(e^{j\Omega})$$

$$\text{then} \quad z(n) = ax(n) + by(n) \xleftrightarrow{\text{DTFT}} Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e^{j\Omega}).$$

Proof. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$Y(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} y(n)e^{-j\Omega n}$$

$$\therefore \quad Z(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} z(n)e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} [ax(n) + by(n)]e^{-j\Omega n}$$

$$= a\sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n} + b\sum_{n=-\infty}^{\infty} y(n)e^{-j\Omega n}$$

$$\therefore \quad Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e^{j\Omega}).$$

Hence the proof.

(b) Time Shift: If $x(n) \xleftrightarrow{\text{DTFT}} X\left(e^{j\Omega}\right)$

then $y(n) = x(n - n_o) \xleftrightarrow{\text{DTFT}} Y\left(e^{j\Omega}\right) = e^{-j\Omega n_o} X\left(e^{j\Omega}\right)$

Proof. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$\therefore \quad Y(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} y(n)e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n - n_o)e^{-j\Omega n}$$

Put $m = n - n_o$ then

$$Y\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x(m)e^{-j\Omega(m+n_o)}$$

$$= e^{-j\Omega n_o} \sum_{m=-\infty}^{\infty} x(m)e^{-j\Omega m}$$

$$\therefore Y\left(e^{j\Omega}\right) = e^{-j\Omega n_o} X\left(e^{j\Omega}\right)$$

Hence the proof.

(c) **Frequency Shift:**

$$\text{If} \quad x(\text{n}) \xleftrightarrow{\text{DTFT}} \text{X}\left(\text{e}^{\text{j}\Omega}\right)$$

$$\text{then} \quad \text{y(n)} = \text{e}^{\text{j}\beta\text{n}} x(\text{n}) \xleftrightarrow{\text{DTFT}} \text{Y}\left(\text{e}^{\text{j}\Omega}\right) = \text{X}\left(\text{e}^{\text{j}(\Omega-\beta)}\right)$$

Proof. We have,

$$\text{X}\left(\text{e}^{\text{j}\Omega}\right) = \sum_{\text{n}=-\infty}^{\infty} x(\text{n})\text{e}^{-\text{j}\Omega\text{n}}$$

$$\therefore \quad \text{Y}(\text{e}^{\text{j}\Omega}) = \sum_{\text{n}=-\infty}^{\infty} \text{y(n)}\text{e}^{-\text{j}\Omega\text{n}}$$

$$= \sum_{\text{n}=-\infty}^{\infty} \text{e}^{\text{j}\beta\text{n}} x(\text{n})\text{e}^{-\text{j}\Omega\text{n}}$$

$$= \sum_{\text{n}=-\infty}^{\infty} x(\text{n})\text{e}^{-\text{j}(\Omega-\beta)\text{n}}$$

$$\therefore \quad \text{Y}\left(\text{e}^{\text{j}\Omega}\right) = \text{X}\left(\text{e}^{\text{j}(\Omega-\beta)}\right)$$

Hence the proof.

(d) **Scaling:** Since scaling of a discrete-time signal discards information, it it not possible to express the DTFT of the scaled signal in terms of the DTFT of the original signal.

But consider a non-periodic sequence $x(\text{n})$ such that,

$$x(\text{n}) = 0 \quad ; \text{ unless } \frac{\text{n}}{\text{p}} \text{ is integer} \quad ; \ \text{p} > 1$$

then $\text{z(n)} = x(\text{pn})$ is also non-periodic.

In this case,

$$\text{if} \quad x(\text{n}) \xleftrightarrow{\text{DTFT}} \text{X}\left(\text{e}^{\text{j}\Omega}\right)$$

$$\text{then} \quad \text{z(n)} = x(\text{pn}) \xleftrightarrow{\text{DTFT}} \text{Z}\left(\text{e}^{\text{j}\Omega}\right) = \text{X}\left(\text{e}^{\text{j}\Omega/\text{p}}\right).$$

Proof. We have,

$$\text{X}(\text{e}^{\text{j}\Omega}) = \sum_{\text{n}=-\infty}^{\infty} x(\text{n})\text{e}^{-\text{j}\Omega\text{n}}$$

$$\therefore \quad \text{Z}(\text{e}^{\text{j}\Omega}) = \sum_{\text{n}=-\infty}^{\infty} \text{z(n)}\text{e}^{-\text{j}\Omega\text{n}}$$

$$= \sum_{\text{n}=-\infty}^{\infty} x(\text{pn})\text{e}^{-\text{j}\Omega\text{n}}$$

Put pn = m, then

$$Z(e^{j\Omega}) = \sum_{m=-\infty}^{\infty} x(m)e^{-j(\Omega/p)m}$$

$$\therefore \quad Z(e^{j\Omega}) = X(e^{j\Omega/p})$$

Hence the proof.

(e) Frequency-differentiation: If $x(n) \xleftrightarrow{\text{DTFT}} X\left(e^{j\Omega}\right)$

then $-jnx(n) \xleftrightarrow{\text{DTFT}} \dfrac{d}{d\Omega}X\left(e^{j\Omega}\right)$

Proof. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

Differentiating both the sides with respect to 'Ω' we get,

$$\frac{d}{d\Omega}X\left(e^{j\Omega}\right) = \frac{d}{d\Omega}\left[\sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}\right]$$

Changing the order of differentiation and summation we get,

$$\frac{d}{d\Omega}X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x(n)\left(\frac{\partial}{\partial\Omega}e^{-j\Omega n}\right)$$

$$= \sum_{n=-\infty}^{\infty} x(n)(-jn)e^{-j\Omega n}$$

$$\frac{d}{d\Omega}X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} [-jnx(n)]e^{-j\Omega n}$$

Comparing wih eqn. $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$ we get,

$$-jnx(n) \xleftrightarrow{\text{DTFT}} \frac{d}{d\Omega}X(e^{j\Omega})$$

Hence the proof.

(f) Summation: If $x(n) \xleftrightarrow{\text{DTFT}} X\left(e^{j\Omega}\right)$

$$y(n) = \sum_{k=-\infty}^{n} x(k) \xleftrightarrow{\text{DTFT}} Y\left(e^{j\Omega}\right) = \frac{X\left(e^{j\Omega}\right)}{1 - e^{-j\Omega}} + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$$

Proof. We know that summation is the reverse process of differencing. The summation operation on $x(n)$ yields $y(n)$ whereas the difference operation on $y(n)$ yields $x(n)$.

$$\text{i.e.,} \qquad x(n) = y(n) - y(n-1)$$

Taking DTFT on both the sides we get, (using time shift property),

$$X\left(e^{j\Omega}\right) = Y\left(e^{j\Omega}\right) - e^{-j\Omega} Y(e^{j\Omega})$$

$$\therefore \quad Y\left(e^{j\Omega}\right) = \frac{X\left(e^{j\Omega}\right)}{1 - e^{-j\Omega}} \qquad (3.17)$$

From eqn. (3.17), we cannot determine $Y(e^{j0})$. Therefore we add an impulse to account for a non zero average value in $x(k)$, to get the exact relationship as,

$$Y\left(e^{j\Omega}\right) = \frac{X\left(e^{j\Omega}\right)}{1 - e^{-j\Omega}} + \pi X\left(e^{j0}\right)\delta(\Omega) \quad ; \ -\pi < \Omega < \pi$$

where the first term assumed to be zero for $\Omega = 0$. Since $Y\left(e^{j\Omega}\right)$ is periodic with period 2π, we have,

$$Y(e^{j\Omega}) = \frac{X\left(e^{j\Omega}\right)}{1 - e^{-j\Omega}} + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$$

Hence the proof.

(g) Convolution: If $x(n) \xleftrightarrow{\text{DTFT}} X\left(e^{j\Omega}\right)$

and $y(n) \xleftrightarrow{\text{DTFT}} Y\left(e^{j\Omega}\right)$

then $z(n) = x(n) * y(n) \xleftrightarrow{\text{DTFT}} Z(e^{j\Omega}) = X\left(e^{j\Omega}\right) Y\left(e^{j\Omega}\right)$

Proof. We have,

$$X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$Y(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} y(n) e^{-j\Omega n}$$

$$\therefore \quad Z\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} z(n)\ e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} [x(n) * y(n)]e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty}\left[\sum_{\ell=-\infty}^{\infty} x(\ell)y(n-\ell)\right]e^{-j\Omega n}$$

Changing the order of summation we get,

$$Z\left(e^{j\Omega}\right) = \sum_{\ell=-\infty}^{\infty} x(\ell) \sum_{n=-\infty}^{\infty} y(n-\ell)e^{-j\Omega n}$$

Put $n - \ell = m$ then,

$$Z\left(e^{j\Omega}\right) = \sum_{\ell=-\infty}^{\infty} x(\ell) \sum_{m=-\infty}^{\infty} y(m)e^{-j\Omega(m+\ell)}$$

$$= \sum_{\ell=-\infty}^{\infty} x(\ell)e^{-j\Omega\ell} \sum_{m=-\infty}^{\infty} y(m)e^{-j\Omega m}$$

$$\therefore Z\left(e^{j\Omega}\right) = X\left(e^{j\Omega}\right)Y\left(e^{j\Omega}\right)$$

Therefore convolution in time-domain is equivalent to multiplication in frequency domain.

(h) Modulation: If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X\left(e^{j\Omega}\right)$

and $y(n) \overset{\text{DTFT}}{\longleftrightarrow} Y\left(e^{j\Omega}\right)$

then $z(n) = x(n)y(n) \overset{\text{DTFT}}{\longleftrightarrow} Z(e^{j\Omega}) = \frac{1}{2\pi}\left[X\left(e^{j\Omega}\right) \circledast Y\left(e^{j\Omega}\right)\right]$

where '⊛' indicates periodic convolution.

Proof. We have,

$$X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$Y\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} y(n)e^{-j\Omega n}$$

$$\therefore \quad Z\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} z(n)e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)y(n)e^{-j\Omega n} \tag{3.18}$$

Substituting the following equation,

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\beta}\right)e^{j\beta n}d\beta \quad \text{in eqn. 3.18}$$

we get,

$$Z\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} y(n)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\beta}\right)e^{j\beta n}d\beta\right]e^{-j\Omega n}$$

Interchanging the order of integration and summation we get,

$$Z\left(e^{j\Omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\beta}\right)\sum_{n=-\infty}^{\infty} y(n)e^{j\beta n}e^{-j\Omega n}d\beta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\beta}\right)\sum_{n=-\infty}^{\infty} y(n)e^{-j(\Omega-\beta)n}d\beta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\beta})Y\left(e^{j(\Omega-\beta)}\right)d\beta$$

$$= \frac{1}{2\pi}\left[X\left(e^{j\Omega}\right) \circledast Y\left(e^{j\Omega}\right)\right]$$

Hence the proof.

(i) **Parseval's Theorem:** If $x(n) \xleftrightarrow{\text{DTFT}} X\left(e^{j\Omega}\right)$

then
$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{2\pi} \left|X\left(e^{j\Omega}\right)\right|^2 d\Omega \tag{3.19}$$

In eqn. (3.19), $|X(e^{j\Omega})|^2$ is known as *energy density spectrum* of the signal $x(n)$.

We know that the LHS of eqn. (3.19) is the energy of the signal $x(n)$.

Proof. We have,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} x(n) x^*(n)$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left[\frac{1}{2\pi} \int_{2\pi} X^*\left(e^{j\Omega}\right) e^{-j\Omega n} d\Omega \right]$$

Changing the order of summation and integration we get,

$$E = \frac{1}{2\pi} \int_{2\pi} X^*(e^{j\Omega}) \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} d\Omega$$

$$= \frac{1}{2\pi} \int_{2\pi} X^*(e^{j\Omega}) X(e^{j\Omega}) d\Omega$$

$$E = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\Omega})|^2 d\Omega$$

$$\therefore \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\Omega})|^2 d\Omega$$

Hence the proof.

(j) Symmetry: If $x(n) \xleftrightarrow{\text{DTFT}} X(e^{j\Omega})$

then, (i) If $x(n)$ is real, then $X^*(e^{j\Omega}) = X(e^{-j\Omega})$

(ii) If $x(n)$ is real and even, then $\text{Img}\left\{X(e^{j\Omega})\right\} = 0$ [i.e., $X(e^{j\Omega})$ is purely real]

(iii) If $x(n)$ is real and odd, then $\text{Re}\left\{X(e^{j\Omega})\right\} = 0$ [i.e., $X(e^{j\Omega})$ is purely imaginary]

Proof. If $x(n)$ is real, $x(n) = x^*(n)$

Let $x(n) = x_e(n) + x_o(n) \xleftrightarrow{\text{DTFT}} X(e^{j\Omega}) = X_R(e^{j\Omega}) + jX_I(e^{j\Omega})$ (3.20)

We have $$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega \quad (3.21)$$

Taking complex conjugate on both the sides,

$$x^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(e^{j\Omega}) e^{-j\Omega n} d\Omega \quad (3.22)$$

$$x^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(e^{j\Omega}) e^{j(-\Omega)n} d\Omega \quad (3.23)$$

Comparing eqn. (3.22) with eqn. (3.23) we get,

$$x(n) \xleftrightarrow{\text{DTFT}} X^*(e^{-j\Omega}) \qquad [\because x(n) = x^*(n)] \tag{3.24}$$

From eqn. (3.20) and eqn. (3.24) we get,

$$X(e^{j\Omega}) = X^*(e^{-j\Omega})$$

Taking complex conjugate on both the sides,

$$X^*(e^{j\Omega}) = X(e^{-j\Omega})$$

From eqn. (3.21) we get,

$$x(-n) = x_e(-n) + x_o(-n) = x_e(n) - x_o(n)$$

$$\therefore x(-n) = x_e(n) - x_o(n) \xleftrightarrow{\text{DTFT}} X(e^{-j\Omega}) = X^*(e^{j\Omega}) = X_R(e^{j\Omega}) - jX_I(e^{j\Omega}) \tag{3.25}$$

Adding eqn. (3.21) and eqn. (3.25) we get,

$$2x_e(n) \xleftrightarrow{\text{DTFT}} 2X_R(e^{j\Omega})$$

$$\therefore \quad x_e(n) \xleftrightarrow{\text{DTFT}} X_R(e^{j\Omega})$$

Therefore, DTFT of real and even sequence is purely real.

Subtracting eqn. (3.25) from eqn. (3.21) we get,

$$2x_o(n) \xleftrightarrow{\text{DTFT}} 2jX_I(e^{j\Omega})$$

$$\therefore \quad x_o(n) \xleftrightarrow{\text{DTFT}} jX_I(e^{j\Omega})$$

Therefore, DTFT of real and odd sequence is purely imaginary.

EXAMPLES

Example 3.30 Find the DTFT of the signal,

$$x(n) = \alpha^n u(n) \qquad ; |\alpha| < 1$$

Draw the magnitude spectrum.

Solution. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^n e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-j\Omega})^n$$

$$\therefore X(e^{j\Omega}) = \frac{1}{1-\alpha e^{-j\Omega}}$$

The magnitude spectra for $0 < \alpha < 1$ and $-1 < \alpha < 0$ are shown in Fig. E3.30 (a) & (b) respectively. The spectra is periodic with period 2π.

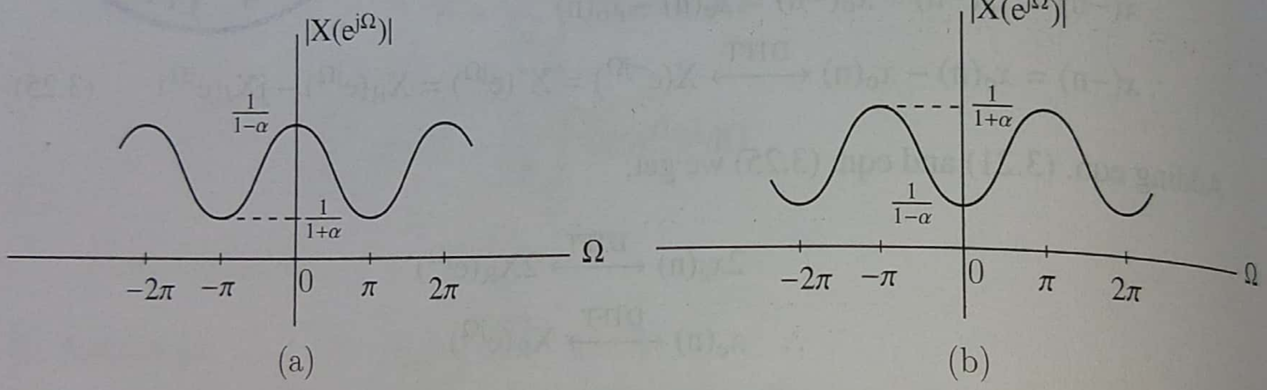


Fig. E3.30.

Example 3.31 Find the DTFT of the signal,

$$x(n) = \{1, 3, \underset{\uparrow}{5}, 3, 1\}$$

and evaluate $X(e^{j\Omega})$ at $\Omega = 0$.

Solution. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$= x(-2)e^{j2\Omega} + x(-1)e^{j\Omega} + x(0) + x(1)e^{-j\Omega} + x(2)e^{-j2\Omega}$$

$$= 1e^{j2\Omega} + 3e^{j\Omega} + 5 + 3e^{-j\Omega} + 1e^{-j2\Omega}$$

$$\therefore X(e^{j\Omega}) = 5 + 6\cos(\Omega) + 2\cos(2\Omega)$$

Also $X(e^{j\Omega})\big|_{\Omega=0} = X(e^{j0}) = 13$

Example 3.32 Find the DTFT of $\delta(n)$ and draw the spectrum.

Solution. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n)e^{-j\Omega n}$$

But $\delta(n) = 1 \quad ; n = 0$

$\qquad\quad = 0 \quad ; n \neq 0$

$$\therefore X(e^{j\Omega}) = \delta(0)e^{0}$$

$$= 1$$

The spectrum is shown in Fig. E3.32 below.

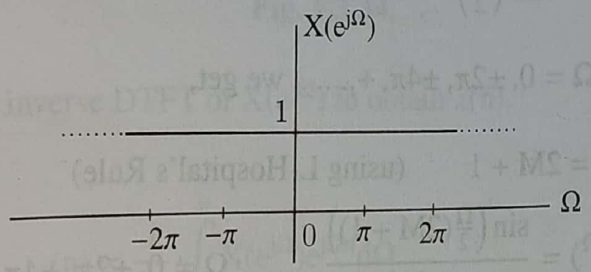


Fig. E3.32.

Example 3.33 Obtain the DTFT of a rectangular pulse which is defined as,

$$x(n) = 1 \quad ; |n| \leq M$$

$$= 0 \quad ; |n| > M$$

Draw its spectrum.

Solution. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$X(e^{j\Omega}) = \sum_{n=-M}^{M} 1.e^{-j\Omega n}$$

Put $\ell = n + M$ then,

$$X(e^{j\Omega}) = \sum_{\ell=0}^{2M} 1.e^{-j\Omega(\ell-M)}$$

$$= e^{j\Omega M}\left[\sum_{\ell=0}^{2M} e^{-j\Omega\ell}\right] \qquad \left[\because \sum_{n=0}^{N-1}\alpha^n = \frac{1-\alpha^N}{1-\alpha} \quad ; \alpha \neq 1 \atop = N \quad ; \alpha = 1\right]$$

$$= e^{j\Omega M}\left[\frac{1-e^{-j\Omega(2M+1)}}{1-e^{-j\Omega}}\right]$$

$$= e^{j\Omega M}.\frac{e^{-j\Omega(2M+1)/2}}{e^{-j\Omega/2}}\left[\frac{e^{j\Omega(2M+1)/2} - e^{-j\Omega(2M+1)/2}}{e^{j\Omega/2} - e^{-j\Omega/2}}\right]$$

$$\therefore X(e^{j\Omega}) = \frac{\sin\left(\frac{\Omega}{2}(2M+1)\right)}{\sin\left(\frac{\Omega}{2}\right)} \qquad \text{(E3.33.1)}$$

In eqn. (E3.33.1), when $\Omega = 0, \pm 2\pi, \pm 4\pi, +$........ we get,

$$X(e^{j\Omega}) = 2M + 1 \qquad \text{(using L-Hospital's Rule)}$$

$$\therefore \begin{cases} X(e^{j\Omega}) = \dfrac{\sin\left(\frac{\Omega}{2}(2M+1)\right)}{\sin\left(\frac{\Omega}{2}\right)} & ; \Omega \neq 0, \pm 2\pi, \pm 4\pi, \pm\ldots\ldots \\ \quad = 2M+1 & ; \Omega = 0, \pm 2\pi, \pm 4\pi, \pm\ldots\ldots\ldots \end{cases}$$

The spectrum is shown in Fig. E3.33 below.

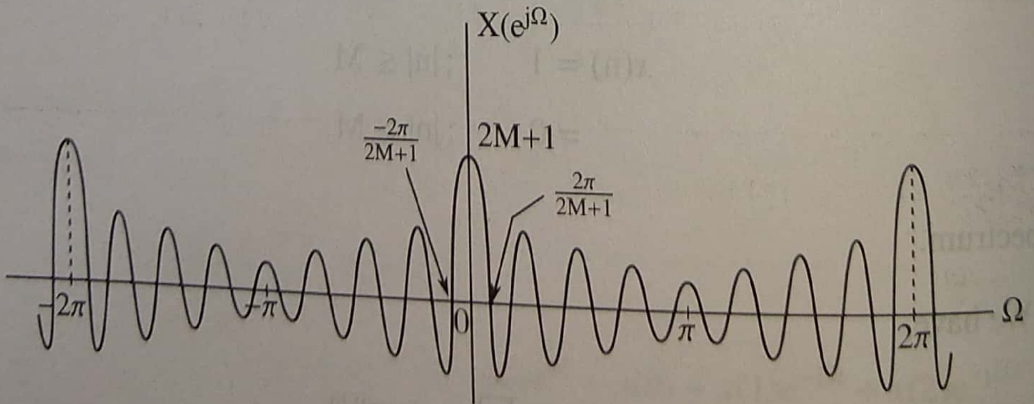


Fig. E3.33.

Example 3.34 Show that the DTFT of,

$$x(n) = 1 \quad ; -\infty < n < \infty$$

is given by,

$$X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega + 2\pi k)$$

Solution. The plot of $X(e^{j\Omega})$ is shown in Fig. E3.34 below.

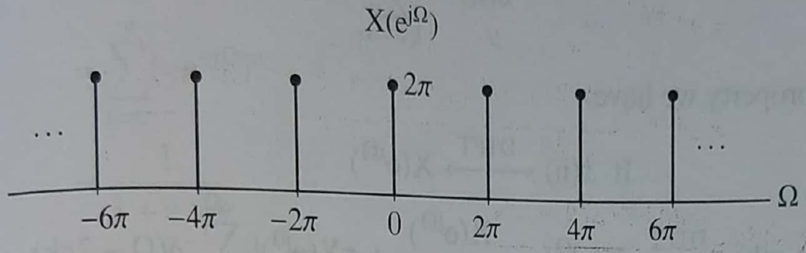


Fig. E3.34.

Now let us find the inverse DTFT of $X(e^{j\Omega})$ to obtain $x(n)$.

We have,

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$$

$$= \frac{1}{2\pi}.2\pi e^{j(0)n}$$

$$= 1.$$

$$\therefore x(n) = 1 \xleftrightarrow{\text{DTFT}} X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega + 2\pi k)$$

Example 3.35 Find the DTFT of unit step sequence.

Solution. Given:

$$x(n) = u(n)$$

$$= \delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3) + \ldots\ldots\ldots$$

$$= \sum_{k=0}^{\infty} \delta(n-k)$$

Put $n - k = m$

We have,

$$x(n) = u(n) = \sum_{m=n}^{-\infty} \delta(m)$$

$$u(n) = \sum_{m=-\infty}^{n} \delta(m)$$

Since 'm' is a dummy variable we can write.

$$u(n) = \sum_{k=-\infty}^{n} \delta(k)$$

By summation property we have,

$$\text{If } x(n) \xleftrightarrow{\text{DTFT}} X(e^{j\Omega})$$

$$\text{then } y(n) = \sum_{k=-\infty}^{n} x(k) \xleftrightarrow{\text{DTFT}} Y(e^{j\Omega}) = \frac{X(e^{j\Omega})}{1 - e^{-j\Omega}} + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$$

We know that,

$$\delta(n) \xleftrightarrow{\text{DTFT}} 1$$

Using summation property we have,

$$u(n) = \sum_{k=-\infty}^{n} \delta(k) \xleftrightarrow{\text{DTFT}} \frac{1}{1 - e^{-j\Omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$$

For one cycle,

$$\therefore \quad u(n) \xleftrightarrow{\text{DTFT}} \frac{1}{1 - e^{-j\Omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega)$$

Example 3.36 Show that the DTFT of the sequence $x(n) = e^{j\Omega_o n}$ is given by,

$$X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_o + 2\pi k)$$

Solution. We know that,

$$1 \xleftrightarrow{\text{DTFT}} \sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega + 2\pi k) \qquad \text{[Refer to example 3.34]}$$

Using frequency shift property we get,

$$e^{j\Omega_o n} \xleftrightarrow{\text{DTFT}} \sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_o + 2\pi k)$$

Example 3.37 Find the DTFT of the signal,

$$x(n) = (-1)^n u(n)$$

Solution. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} (-1)^n e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} (-e^{-j\Omega})^n$$

$$= \frac{1}{1+e^{-j\Omega}}$$

$$= \frac{1}{e^{-j\Omega/2}(e^{j\Omega/2}+e^{-j\Omega/2})}$$

$$X(e^{j\Omega}) = \frac{e^{j\Omega/2}}{2\cos(\Omega/2)} \quad ; \Omega \neq 2\pi\left(k+\frac{1}{2}\right), \text{ where } k = 0, 1, \ldots\ldots$$

Example 3.38 Find the DTFT of the signal,

$$x(n) = u(n) - u(n-6)$$

Solution. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$= \sum_{n=0}^{5} 1.e^{-j\Omega n}$$

$$X(e^{j\Omega}) = \frac{1-e^{-j6\Omega}}{1-e^{-j\Omega}}$$

Example 3.39 Evaluate the DTFT of the signal,

$$x(n) = \left(\frac{1}{2}\right)^n u(n-4)$$

Also find the expression for magnitude and phase spectra.

Solution. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$\therefore X(e^{j\Omega}) = \sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\Omega n}$$

$$= \sum_{n=4}^{\infty} \left(\frac{1}{2}e^{-j\Omega}\right)^n$$

$$X(e^{j\Omega}) = \frac{\left(\frac{1}{2}e^{-j\Omega}\right)^4}{1-\frac{1}{2}e^{-j\Omega}}$$

$$\therefore \text{Magnitude of } X(e^{j\Omega}) = |X(e^{j\Omega})| = \frac{\left(\frac{1}{2}\right)^4}{\sqrt{\left(1-\frac{1}{2}\cos(\Omega)\right)^2 + \left(\frac{1}{2}\sin(\Omega)\right)^2}}$$

$$\& \text{ Phase of } X(e^{j\Omega}) = \measuredangle X(e^{j\Omega}) = -4\Omega - \tan^{-1}\left[\frac{\sin(\Omega)}{2-\cos(\Omega)}\right]$$

Example 3.40 Compute the DTFT of the signal,

$$x(n) = 2^n u(-n)$$

Solution. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$\therefore X(e^{j\Omega}) = \sum_{n=-\infty}^{0} 2^n e^{-j\Omega n}$$

Put m = −n, then

$$\therefore X(e^{j\Omega}) = \sum_{m=\infty}^{0} 2^{-m}e^{j\Omega m}$$

$$= \sum_{m=0}^{\infty} (2^{-1}e^{j\Omega})^m$$

$$= \frac{1}{1-2^{-1}e^{j\Omega}}$$

$$\therefore X(e^{j\Omega}) = \frac{2}{2-e^{j\Omega}}$$

Example 3.41 Find the DTFT of the signal,

$$x(n) = \left(\frac{1}{4}\right)^n u(n+4)$$

Solution. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$= \sum_{n=-4}^{\infty} \left(\frac{1}{4}\right)^n e^{-j\Omega n}$$

Put m = n + 4 then,

$$X(e^{j\Omega}) = \sum_{m=0}^{\infty} \left(\frac{1}{4}\right)^{m-4} e^{-j\Omega(m-4)}$$

$$= 4^4 . e^{j\Omega 4}\left[\sum_{m=0}^{\infty}\left(\frac{1}{4}e^{-j\Omega}\right)^m\right]$$

$$\therefore \quad X(e^{j\Omega}) = \frac{256.e^{j4\Omega}}{1-\frac{1}{4}e^{-j\Omega}}$$

Example 3.42 Find the DTFT of the signal,

$$x(n) = a^{|n|} \quad ; |a| < 1$$

Solution. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^{|n|}e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n}e^{-j\Omega n} + \sum_{n=0}^{\infty} a^n e^{-j\Omega n}$$

$$= \sum_{n=1}^{\infty}(ae^{j\Omega})^n + \sum_{n=0}^{\infty}(ae^{-j\Omega})^n$$

$$= \frac{ae^{j\Omega}}{1-ae^{j\Omega}} + \frac{1}{1-ae^{-j\Omega}}$$

$$\therefore X(e^{j\Omega}) = \frac{1-a^2}{1-2a\cos(\Omega)+a^2}$$

Example 3.43 Find the DTFT of the signal,

$$x(n) = \alpha^n \sin(\Omega_o n)u(n) \qquad ; |\alpha| < 1$$

Solution. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$\therefore X(e^{j\Omega}) = \sum_{n=0}^{\infty} \alpha^n \sin(\Omega_o n)\, e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^n \left[\frac{e^{j\Omega_o n} - e^{-j\Omega_o n}}{2j}\right] e^{-j\Omega n}$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty}\left(\alpha e^{-j(\Omega-\Omega_o)}\right)^n - \sum_{n=0}^{\infty}\left(\alpha e^{-j(\Omega+\Omega_o)}\right)^n\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1-\alpha e^{-j(\Omega-\Omega_o)}} - \frac{1}{1-\alpha e^{-j(\Omega+\Omega_o)}}\right]$$

$$X(e^{j\Omega}) = \frac{\alpha \sin(\Omega_o)e^{-j\Omega}}{1 - 2\alpha\cos(\Omega_o)e^{-j\Omega} + \alpha^2 e^{-j2\Omega}}$$

Example 3.44 Evaluate the DTFT of the signal,

$$x(n) = \delta(6 - 3n)$$

Sketch its magnitude and phase spectra.

Solution. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$\therefore X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \delta(6-3n)e^{-j\Omega n}$$

We know that $\delta(6 - 3n)$ is an impulse function. Its time of occurance is obtained by equating $(6 - 3n)$ to zero.

i.e., $6 - 3n = 0 \qquad \therefore n = 2$

$$X(e^{j\Omega}) = \delta(2)e^{-j2\Omega} = e^{-j2\Omega}$$

$$\therefore |X(e^{j\Omega})| = 1$$

$$\measuredangle X(e^{j\Omega}) = -2\Omega$$

The magnitude and phase spectra (for one cycle) are shown in Fig. E3.44.1 and E3.44.2 respectively.

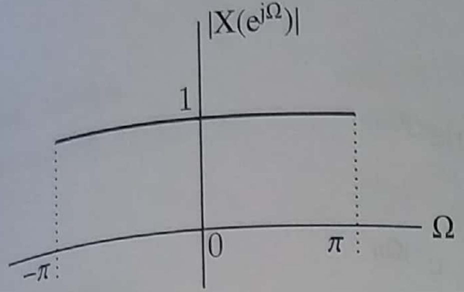


Fig. E3.44.1.

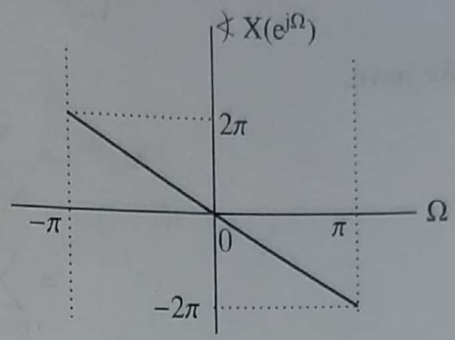


Fig. E3.44.2.

Example 3.45 Evaluate the DTFT for the signal $x(n)$ shown in Fig. E3.45. Find the expression for magnitude and phase spectra.

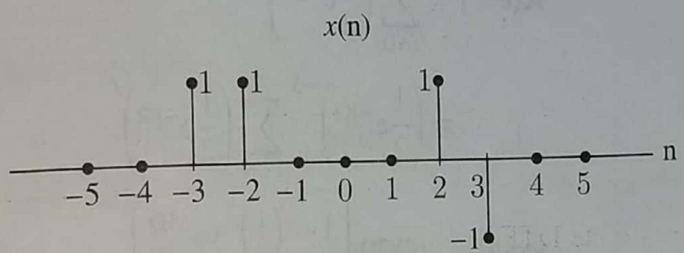


Fig. E3.45.

Solution. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

In the given sequence $x(n)$, there are only 4 non-zero samples.

$$\therefore\ X(e^{j\Omega}) = x(-3)e^{j3\Omega} + x(-2)e^{j2\Omega} + x(2)e^{-j2\Omega} + x(3)e^{-j3\Omega}$$

$$= 1.e^{j3\Omega} + 1.e^{j2\Omega} + 1.e^{-j2\Omega} + (-1)e^{-j3\Omega}$$

$$X(e^{j\Omega}) = 2\cos(2\Omega) + 2j\sin(3\Omega).$$

$$\therefore\ |X(e^{j\Omega})| = 2\sqrt{\cos^2(2\Omega) + \sin^2(3\Omega)}$$

and $$\measuredangle X(e^{j\Omega}) = \tan^{-1}\left[\frac{\sin(3\Omega)}{\cos(2\Omega)}\right]$$

Example 3.46 Compute the DTFT of the signal,

$$x(n) = \left(\frac{1}{2}\right)^n \{u(n+3) - u(n-2)\}$$

Solution. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$= \sum_{n=-3}^{1} \left(\frac{1}{2}\right)^n e^{-j\Omega n}$$

$$= \sum_{n=-3}^{1} \left(\frac{1}{2}e^{-j\Omega}\right)^n$$

Put $\ell = n + 3$

$$X(e^{j\Omega}) = \sum_{\ell=0}^{4} \left(\frac{1}{2}e^{-j\Omega}\right)^{\ell-3}$$

$$= \left(\frac{1}{2}e^{-j\Omega}\right)^{-3} \sum_{\ell=0}^{4} \left(\frac{1}{2}e^{-j\Omega}\right)^{\ell}$$

$$\therefore X(e^{j\Omega}) = 8e^{j3\Omega}\left[\frac{1-\left(\frac{1}{2}\right)^5 e^{-j5\Omega}}{1-\left(\frac{1}{2}\right)e^{-j\Omega}}\right]$$

Example 3.47 Find the DTFT of the signal,

$$x(n) = n\left(\frac{1}{2}\right)^{|n|}$$

Solution. From example 3.42 we have,

$$a^{|n|} \xleftrightarrow{\text{DTFT}} \frac{1-a^2}{1-2a\cos(\Omega)+a^2} \quad ; |a| < 1$$

$$\therefore \left(\frac{1}{2}\right)^{|n|} \xleftrightarrow{\text{DTFT}} \frac{1-\frac{1}{4}}{1-2.\frac{1}{2}\cos(\Omega)+\frac{1}{4}} = \frac{\frac{3}{4}}{\frac{5}{4}-\cos(\Omega)}$$

Using frequency differentiation property we get,

$$n\left(\frac{1}{2}\right)^{|n|} \xleftrightarrow{\text{DTFT}} j\frac{d}{d\Omega}\left[\frac{\frac{3}{4}}{\frac{5}{4}-\cos(\Omega)}\right] = -j\left[\frac{\frac{3}{4}\sin(\Omega)}{\left(\frac{5}{4}-\cos(\Omega)\right)^2}\right]$$

Example 3.48 Using the appropriate properties, find the DTFT of the following signal,

$$x(n) = \left(\frac{1}{2}\right)^n u(n-2)$$

Solution. Given:

$$x(n) = \left(\tfrac{1}{2}\right)^n u(n-2)$$

This can be written as,

$$x(n) = \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{n-2} u(n-2)$$

We know that,

$$\left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{\text{DTFT}} \frac{1}{1-\frac{1}{2}e^{-j\Omega}}$$

Using time-shift property we get,

$$\left(\frac{1}{2}\right)^{n-2} u(n-2) \xleftrightarrow{\text{DTFT}} e^{-j2\Omega}\frac{1}{1-\frac{1}{2}e^{-j\Omega}}$$

Using linearity property we get,

$$\left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{n-2} u(n-2) \xleftrightarrow{\text{DTFT}} \left(\frac{1}{2}\right)^2 .e^{-j2\Omega}\frac{1}{1-\frac{1}{2}e^{-j\Omega}}$$

$$\therefore x(n) = \left(\frac{1}{2}\right)^n u(n-2) \xleftrightarrow{\text{DTFT}} \frac{1}{4}.e^{-j2\Omega}.\frac{1}{1-\frac{1}{2}e^{-j\Omega}}$$

$$\therefore X(e^{j\Omega}) = \left(\frac{1}{4}\right)\frac{e^{-j2\Omega}}{1-\frac{1}{2}e^{-j\Omega}}$$

Example 3.49 Using the appropriate properties, find the DTFT of the following signal.

$$x(n) = \sin\left(\frac{\pi}{4}n\right)\left(\frac{1}{4}\right)^n u(n-1)$$

Solution. Given:

$$x(n) = \sin\left(\frac{\pi}{4}n\right)\left(\frac{1}{4}\right)^n u(n-1)$$

$$= \left[\frac{e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}}{2j}\right]\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^{n-1} u(n-1)$$

$$\therefore x(n) = \frac{1}{8j}\left[e^{j\frac{\pi}{4}n}\left(\frac{1}{4}\right)^{n-1} u(n-1) - e^{-j\frac{\pi}{4}n}\left(\frac{1}{4}\right)^{n-1} u(n-1)\right]$$

We know that,

$$\left(\frac{1}{4}\right)^n u(n) \xleftrightarrow{\text{DTFT}} \frac{1}{1-\frac{1}{4}e^{-j\Omega}}$$

Using time-shift property we get,

$$\left(\frac{1}{4}\right)^{n-1} u(n-1) \xleftrightarrow{\text{DTFT}} e^{-j\Omega}\frac{1}{1-\frac{1}{4}e^{-j\Omega}} = \frac{1}{e^{j\Omega}-\frac{1}{4}}$$

Using frequency shift property we get,

$$e^{j\frac{\pi}{4}n}\left(\frac{1}{4}\right)^{n-1} u(n-1) \xleftrightarrow{\text{DTFT}} \frac{1}{e^{j(\Omega-\frac{\pi}{4})}-\frac{1}{4}}$$

Using linearity property we get,

$$\frac{1}{8j}e^{j\frac{\pi}{4}n}\left(\frac{1}{4}\right)^{n-1} u(n-1) \xleftrightarrow{\text{DTFT}} \frac{1}{8j}\frac{1}{e^{j(\Omega-\frac{\pi}{4})}-\frac{1}{4}}$$

Similarly, $$\frac{1}{8j}e^{-j\frac{\pi}{4}n}\left(\frac{1}{4}\right)^{n-1} u(n-1) \xleftrightarrow{\text{DTFT}} \frac{1}{8j}\frac{1}{e^{j(\Omega+\frac{\pi}{4})}-\frac{1}{4}}$$

$$\therefore X(e^{j\Omega}) = \frac{1}{8j}\left[\frac{1}{e^{j(\Omega-\frac{\pi}{4})}-\frac{1}{4}} - \frac{1}{e^{j(\Omega+\frac{\pi}{4})}-\frac{1}{4}}\right]$$

Example 3.50 Find the DTFT of the sequence,

$$x(n) = \alpha^n \cos(\Omega_o n + \phi)u(n)$$

Solution. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^n \cos(\Omega_o n + \phi)e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^n \left(\frac{e^{j\Omega_o n}e^{j\phi} + e^{-j\Omega_o n}e^{-j\phi}}{2}\right)e^{-j\Omega n}$$

$$= \frac{1}{2}e^{j\phi}\sum_{n=0}^{\infty}\left(\alpha e^{j\Omega_o}e^{-j\Omega}\right)^n + \frac{1}{2}e^{-j\phi}\sum_{n=0}^{\infty}(\alpha e^{-j\Omega_o}e^{-j\Omega})^n$$

$$X(e^{j\Omega}) = \frac{1}{2}e^{j\phi}\frac{1}{1-\alpha e^{-j(\Omega-\Omega_o)}} + \frac{1}{2}e^{-j\phi}\frac{1}{1-\alpha e^{-j(\Omega+\Omega_o)}}$$

Scanned by CamScanner

Example 3.51 Let $x(n)$ be a sequence given by,

$$x(n) = \{3, 0, 1, \underset{\uparrow}{-2}, -3, 4, 1, 0, -1\}$$

with DTFT $X(e^{j\Omega})$. Evaluate the following functions of $X(e^{j\Omega})$ without computing $X(e^{j\Omega})$.

(a) $X(e^{j0})$ (b) $X(e^{j\pi})$

(c) $\int_{-\pi}^{\pi} X(e^{j\Omega})d\Omega$ (d) $\int_{-\pi}^{\pi} |X(e^{j\Omega})|^2 d\Omega$

(e) $\int_{-\pi}^{\pi} \left|\frac{dX(e^{j\Omega})}{d\Omega}\right|^2 d\Omega$

Solution. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

(a) $$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x(n)$$

$$= x(-3) + x(-2) + x(-1) + x(0) + x(1) + x(2) + x(3) + x(4) + x(5)$$

$$= 3 + 0 + 1 - 2 - 3 + 4 + 1 + 0 - 1$$

$$\therefore X(e^{j0}) = 3.$$

(b) $$X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\pi n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)(-1)^n$$

$$= -3 + 0 - 1 - 2 + 3 + 4 - 1 + 0 + 1$$

$$\therefore X(e^{j\pi}) = 1$$

(c) We have, $$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$$

Put $n = 0$,

$$x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})d\Omega$$

$$\therefore \int_{-\pi}^{\pi} X(e^{j\Omega})d\Omega = 2\pi x(0)$$

$$= 2\pi(-2)$$
$$= -4\pi$$

(d) From Parseval's theorem we have,

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\Omega})|^2 d\Omega$$

$$\therefore \int_{-\pi}^{\pi} |X(e^{j\Omega})|^2 d\Omega = 2\pi \sum_{n=-\infty}^{\infty} |x(n)|^2$$
$$= 2\pi(9 + 0 + 1 + 4 + 9 + 16 + 1 + 0 + 1)$$
$$= 82\pi$$

(e) From Parseval's theorem we have,

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\Omega})\right|^2 d\Omega$$

Using frequency differentiation property we get,

$$\sum_{n=-\infty}^{\infty} \left|-jnx(n)\right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{dX(e^{j\Omega})}{d\Omega}\right|^2 d\Omega$$

$$\int_{-\pi}^{\pi} \left|\frac{dX(e^{j\Omega})}{d\Omega}\right|^2 = 2\pi \sum_{n=-\infty}^{\infty} |nx(n)|^2$$
$$= 2\pi(81 + 0 + 1 + 0 + 9 + 64 + 9 + 0 + 25)$$
$$= 378\pi.$$

Example 3.52 Find the DTFT of the signal,

$$x(n) = \cos\left(\frac{\pi}{5}n\right) + j\sin\left(\frac{\pi}{5}n\right) \quad ; |n| \le 10$$
$$= 0 \quad ; \text{otherwise}$$

Solution. Given:

$$x(n) = \cos\left(\frac{\pi}{5}n\right) + j\sin\left(\frac{\pi}{5}n\right) \quad ; |n| \le 10$$
$$\therefore x(n) = e^{j\frac{\pi}{5}n}$$

We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

$$= \sum_{n=-10}^{10} e^{j\frac{\pi}{5}n}e^{-j\Omega n}$$

$$= \sum_{n=-10}^{10} e^{j(\frac{\pi}{5}-\Omega)n}$$

Put $m = n + 10$

$$\therefore X(e^{j\Omega}) = \sum_{m=0}^{20} e^{j(\frac{\pi}{5}-\Omega)(m-10)}$$

$$= e^{-j(\frac{\pi}{5}-\Omega)10} \sum_{m=0}^{20} e^{j(\frac{\pi}{5}-\Omega)m}$$

$$= e^{-j(\frac{\pi}{5}-\Omega)10} \frac{1 - e^{j(\frac{\pi}{5}-\Omega).21}}{1 - e^{j(\frac{\pi}{5}-\Omega)}}$$

$$\therefore \quad X(e^{j\Omega}) = \frac{\sin\left(\frac{21}{2}\left(\frac{\pi}{5}-\Omega\right)\right)}{\sin\left(\frac{1}{2}\left(\frac{\pi}{5}-\Omega\right)\right)}$$

Example 3.53 Find the DTFT of the signal,

$$x(n) = \cos\left(\frac{8\pi}{7}n\right) + \sin(2n)$$

$$x(n) = \frac{1}{2}e^{j(8\pi/7)n} + \frac{1}{2}e^{-j(8\pi/7)n} + \frac{1}{2j}e^{j2n} - \frac{1}{2j}e^{-j2n}$$

we know that,

$$e^{j\Omega_o n} \xleftrightarrow{\text{DTFT}} \sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_o + 2\pi k)$$ [Refer to example 3.36]

$\therefore$ Using this relation we get,

$$X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} \pi\delta\left(\Omega - \frac{8\pi}{7} + 2\pi k\right) + \sum_{k=-\infty}^{\infty} \pi\delta\left(\Omega + \frac{8\pi}{7} + 2\pi k\right)$$

$$-j\sum_{k=-\infty}^{\infty} \pi\delta(\Omega - 2 + 2\pi k) + j\sum_{k=-\infty}^{\infty} \pi\delta(\Omega + 2 + 2\pi k)$$

$$\therefore X(e^{j\Omega}) = \pi \sum_{k=-\infty}^{\infty} \left[\delta(\Omega - \frac{8\pi}{7} + 2\pi k) + \delta\left(\Omega + \frac{8\pi}{7} + 2\pi k\right) - j\delta(\Omega - 2 + 2\pi k) + j\delta\left(\Omega + 2 + 2\pi k\right)\right]$$

Example 3.54 Find the DTFT of the signal,

$$x(n) = (-1)^n a^n u(n) \quad ; \ |a| < 1$$

using frequency shifting property.

Solution. Give:

$$x(n) = (-1)^n a^n u(n) \quad ; \ |a| < 1$$

$$= e^{jn\pi} a^n u(n)$$

We know that $a^n u(n) \overset{\text{DTFT}}{\longleftrightarrow} \dfrac{1}{1 - ae^{-j\Omega}}$

Using frequency shifting property,

$$e^{jn\pi} a^n u(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{1 - ae^{-j(\Omega-\pi)}} = \frac{1}{1 + ae^{-j\Omega}}$$

$$\therefore \quad (-1)^n a^n u(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{1 + ae^{-j\Omega}}$$

Example 3.55 Determine the time-domain signal having the following DTFT.

$$X(e^{j\Omega}) = 0 \quad ; \ 0 \le |\Omega| \le \Omega_o$$

$$= 1 \quad ; \ \Omega_o < |\Omega| \le \pi$$

Solution. We have,

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega$$

$$\therefore x(n) = \frac{1}{2\pi} \int_{-\pi}^{-\Omega_o} 1.e^{j\Omega n} d\Omega + \frac{1}{2\pi} \int_{\Omega_o}^{\pi} 1.e^{j\Omega n} d\Omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\Omega n}}{jn} \Bigg|_{-\pi}^{-\Omega_o} + \frac{e^{j\Omega n}}{jn} \Bigg|_{\Omega_o}^{\pi} \right]$$

$$= \frac{1}{2jn\pi} \left[(e^{-j\Omega_o n} - e^{-jn\pi}) + (e^{jn\pi} - e^{j\Omega_o n}) \right]$$

$$= \frac{1}{n\pi}\left[\frac{\{e^{jn\pi} - e^{-jn\pi}\}}{2j} - \frac{\{e^{j\Omega_o n} - e^{-j\Omega_o n}\}}{2j}\right]$$

$$= \frac{1}{n\pi}\left[\sin(n\pi) - \sin(\Omega_o n)\right]$$

$$\therefore x(n) = \frac{-\sin(\Omega_o n)}{n\pi} \qquad [\because \sin n\pi = 0]$$

Example 3.56 Find the time-domain signal corresponding to the following DTFT.

$$X(e^{j\Omega}) = \cos^2(\Omega)$$

Solution. We have,

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$$

$$\therefore x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2(\Omega)e^{j\Omega n}d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{e^{j\Omega} + e^{-j\Omega}}{2}\right]^2 e^{j\Omega n}d\Omega$$

$$= \frac{1}{8\pi}\int_{-\pi}^{\pi}(e^{j2\Omega} + e^{-j2\Omega} + 2)e^{j\Omega n}d\Omega$$

$$= \frac{1}{8\pi}\left[\int_{-\pi}^{\pi} e^{j2\Omega}e^{j\Omega n}d\Omega + \int_{-\pi}^{\pi} e^{-j2\Omega}e^{j\Omega n}d\Omega + \int_{-\pi}^{\pi} 2e^{j\Omega n}d\Omega\right]$$

Using time-shift property we get,

$$x(n) = \frac{1}{4}\left[\delta(n+2) + \delta(n-2) + 2\delta(n)\right]$$

Example 3.57 Determine the time-domain signal corresponding to the DTFT,

$$X(e^{j\Omega}) = \cos(\Omega) + j\sin(\Omega).$$

Solution. Given :

$$X(e^{j\Omega}) = \cos(\Omega) + j\sin(\Omega)$$

$$\therefore\ X(e^{j\Omega}) = e^{j\Omega}.$$

Using time-shift property we have,

$$\text{if} \quad \delta(n) \xleftrightarrow{\text{DTFT}} 1$$

$$\delta(n - n_o) \xleftrightarrow{\text{DTFT}} e^{-jn_o\Omega}$$

$$\therefore \quad x(n) = \delta(n + 1)$$

Example 3.58 Find the time-domain signal if,

$$|X(e^{j\Omega})| = 1 \quad ; \frac{\pi}{2} < |\Omega| < \pi$$

$$= 0 \quad ; \text{otherwise}$$

$$\arg\left\{X(e^{j\Omega})\right\} = -4\Omega$$

Solution. We have,

$$X(e^{j\Omega}) = \left|X(e^{j\Omega})\right| e^{j\measuredangle X(e^{j\Omega})}$$

$$\therefore \quad X(e^{j\Omega}) = 1 \cdot e^{-j4\Omega} \quad ; \frac{\pi}{2} < |\Omega| < \pi$$

$$= 0 \quad ; \text{otherwise.}$$

$$\therefore \quad x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/2} e^{-j4\Omega}e^{j\Omega n}d\Omega + \int_{\pi/2}^{\pi} e^{-j4\Omega}e^{j\Omega n}d\Omega\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/2} e^{j\Omega(n-4)}d\Omega + \int_{\pi/2}^{\pi} e^{j\Omega(n-4)}d\Omega\right]$$

$$x(n) = \frac{-\sin(\frac{\pi}{2}n)}{\pi(n-4)} \quad ; n \neq 4$$

When $n = 4$; $\quad x(4) = \frac{1}{2}$

Example 3.59 Determine the signal $x(n)$ if its spectrum is as shown in Fig. E3.59.

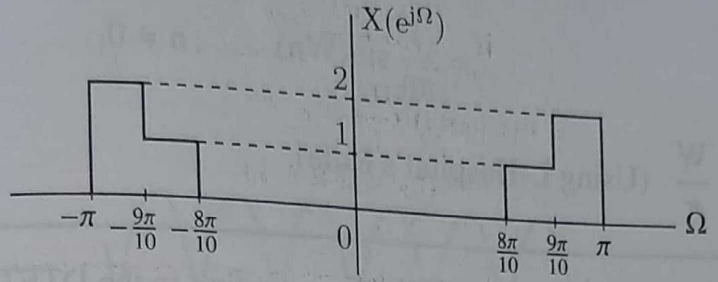


Fig. E3.59.

Solution. We have,

$$\therefore \quad x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$$

$$\therefore \quad x(n) = \left[\frac{1}{2\pi}\int_{-\pi}^{\frac{-9\pi}{10}} 2\cdot e^{j\Omega n}d\Omega + \int_{-\frac{9\pi}{10}}^{\frac{-8\pi}{10}} 1\cdot e^{j\Omega n}d\Omega + \int_{\frac{8\pi}{10}}^{\frac{9\pi}{10}} 1\cdot e^{j\Omega n}d\Omega + \int_{\frac{9\pi}{10}}^{\pi} 2\cdot e^{j\Omega n}d\Omega\right]$$

$$= \frac{1}{2\pi}\left[2\left(\frac{e^{j\Omega n}}{jn}\right)_{-\pi}^{\frac{-9\pi}{10}} + \left(\frac{e^{j\Omega n}}{jn}\right)_{\frac{-9\pi}{10}}^{\frac{-8\pi}{10}} + \left(\frac{e^{j\Omega n}}{jn}\right)_{\frac{8\pi}{10}}^{\frac{9\pi}{10}} + 2\left(\frac{e^{j\Omega n}}{jn}\right)_{\frac{9\pi}{10}}^{\pi}\right]$$

$$= \frac{1}{n\pi}\left[2\sin(n\pi) - \sin\left(\frac{8\pi n}{10}\right) - \sin\left(\frac{9\pi n}{10}\right)\right]$$

$$x(n) = -\frac{1}{n\pi}\left[\sin\left(\frac{8\pi n}{10}\right) + \sin\left(\frac{9\pi n}{10}\right)\right] \qquad [\because \sin(n\pi) = 0]$$

Example 3.60 Find the time-domain signal if the DTFT is,

$$X(e^{j\Omega}) = 1 \quad ; |\Omega| \le W$$
$$= 0 \quad ; W < |\Omega| < \pi$$

Solution. We have

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$$

$$\therefore \quad x(n) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\Omega n}d\Omega$$

$$= \frac{1}{2\pi nj} e^{j\Omega n}\Big|_{-W}^{W}$$

$$= \frac{1}{\pi n} \sin(Wn) \qquad ; n \neq 0.$$

If n = 0 $\quad x(n) = \frac{W}{\pi}$ (Using L-Hospital's Rule).

Example 3.61 Find the time-domain signal corresponding to the DTFT shown in Fig. E3.61 below.

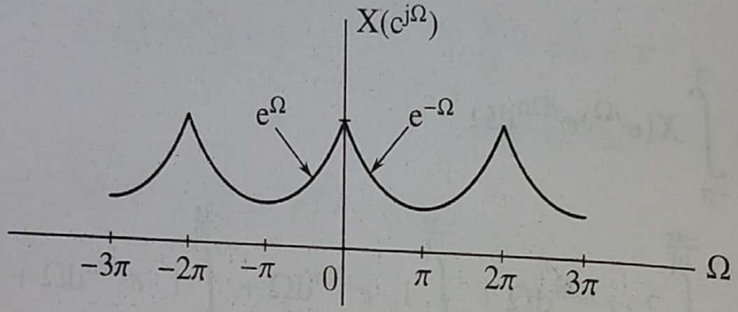


Fig. E3.61.

Solution. From Fig. E3.61 we have,

$$X(e^{j\Omega}) = e^{\Omega} \qquad ; -\pi < \Omega < 0$$

$$= e^{-\Omega} \qquad ; 0 < \Omega < \pi$$

$$\therefore \quad x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n} d\Omega$$

$$\therefore \quad = \frac{1}{2\pi}\left[\int_{-\pi}^{0} e^{\Omega}e^{j\Omega n} d\Omega + \int_{0}^{\pi} e^{-\Omega}e^{j\Omega n} d\Omega\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} e^{(jn+1)\Omega} d\Omega + \int_{0}^{\pi} e^{(jn-1)\Omega} d\Omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{(jn+1)\Omega}}{(jn+1)}\Big|_{-\pi}^{0} + \frac{e^{(jn-1)\Omega}}{(jn-1)}\Big|_{0}^{\pi}\right]$$

$$\therefore \quad x(n) = \frac{1+(-1)^n}{\pi(n^2+1)} \cdot e^{-\pi}$$

Example 3.62 Determine the time-domain signal corresponding to the spectrum shown in Fig. E3.62.

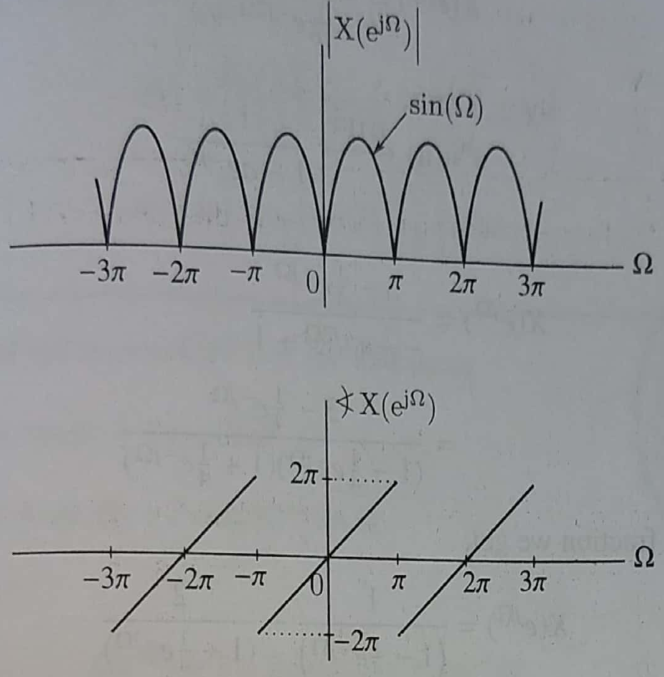


Fig. E3.62.

Solution. From Fig. E3.62, we have,

$$X(e^{j\Omega}) = -\sin(\Omega)e^{j2\Omega} \qquad ; \ -\pi < \Omega < 0$$

$$= \sin(\Omega)e^{j2\Omega} \qquad ; \ 0 < \Omega < \pi$$

$$\therefore \quad x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-\sin(\Omega)e^{j2\Omega}e^{j\Omega n}d\Omega + \int_{0}^{\pi}\sin(\Omega)e^{j2\Omega}e^{j\Omega n}d\Omega\right]$$

$$= \frac{1}{2\pi}\left[-\int_{-\pi}^{0}\frac{e^{j(n+3)\Omega} - e^{j(n+1)\Omega}}{2j}d\Omega + \int_{0}^{\pi}\frac{e^{j(n+3)\Omega} - e^{j(n+1)\Omega}}{2j}d\Omega\right]$$

$$x(n) = \left[1-(-1)^{n+1}\right]\left[\frac{-1}{\pi(n+1)(n+3)}\right] \qquad ; \ n \neq -1, -3$$

and $\quad x(-1) = 0$

$\quad x(-3) = 0$

Example 3.63 Using partial fraction expansion determine the inverse DTFT of the signal,

$$X(e^{j\Omega}) = \frac{3 - \frac{1}{4}e^{-j\Omega}}{-\frac{1}{16}e^{-j2\Omega} + 1}$$

Solution. We have,

$$a^n u(n) \xleftrightarrow{\text{DTFT}} \frac{1}{1 - ae^{-j\Omega}} \tag{E3.63.1}$$

Given :

$$X(e^{j\Omega}) = \frac{3 - \frac{1}{4}e^{-j\Omega}}{-\frac{1}{16}e^{-j2\Omega} + 1}$$

$$= \frac{3 - \frac{1}{4}e^{-j\Omega}}{(1 - \frac{1}{4}e^{-j\Omega})(1 + \frac{1}{4}e^{-j\Omega})}$$

Expanding by partial fraction we get,

$$X(e^{j\Omega}) = \frac{1}{\left(1 - \frac{1}{4}e^{-j\Omega}\right)} + \frac{2}{(1 + \frac{1}{4}e^{-j\Omega})}$$

Taking the inverse DTFT (using relation E3.63.1) we get,

$$x(n) = \left(\frac{1}{4}\right)^n u(n) + 2\left(-\frac{1}{4}\right)^n u(n)$$

$$x(n) = \left[\left(\frac{1}{4}\right)^n + 2\left(-\frac{1}{4}\right)^n\right] u(n)$$

Example 3.64 Find the inverse DTFT of,

$$X(e^{j\Omega}) = \frac{6}{e^{-j2\Omega} - 5e^{-j\Omega} + 6}$$

Solution. Given :

$$X(e^{j\Omega}) = \frac{6}{e^{-j2\Omega} - 5e^{-j\Omega} + 6}$$

$$= \frac{6}{(e^{-j\Omega} - 2)(e^{-j\Omega} - 3)}$$

By partial fraction expansion we get,

$$X(e^{j\Omega}) = \frac{-6}{(e^{-j\Omega} - 2)} + \frac{6}{(e^{-j\Omega} - 3)}$$

$$X(e^{j\Omega}) = \frac{3}{\left(1-\frac{1}{2}e^{-j\Omega}\right)} + \frac{(-2)}{\left(1-\frac{1}{3}e^{-j\Omega}\right)}$$

Taking inverse DTFT we get,

$$x(n) = \left[3\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{3}\right)^n u(n)\right]$$

$$x(n) = \left[3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n\right] u(n)$$

Example 3.65 Find the inverse DTFT of the following.

(i) $X(e^{j\Omega}) = 1 + 2\cos(\Omega) + 3\cos 2(\Omega)$

(ii) $Y(e^{j\Omega}) = j[3 + 4\cos(\Omega) + 2\cos(2\Omega)]\sin\Omega$

Solution. (i) Given :

$$X(e^{j\Omega}) = 1 + 2\cos(\Omega) + 3\cos(2\Omega)$$

$$= 1 + 2\left[\frac{e^{j\Omega} + e^{-j\Omega}}{2}\right] + 3\left[\frac{e^{j2\Omega} + e^{-j2\Omega}}{2}\right]$$

$$= 1 + e^{j\Omega} + e^{-j\Omega} + \frac{3}{2}e^{j2\Omega} + \frac{3}{2}e^{-j2\Omega}$$

Taking inverse DTFT we get,

$$x(n) = \delta(n) + \delta(n+1) + \delta(n-1) + \frac{3}{2}\delta(n+2) + \frac{3}{2}\delta(n-2)$$

$$\therefore \quad x(n) = \left\{\frac{3}{2}, 1, \underset{\uparrow}{1}, 1, \frac{3}{2}\right\}$$

(ii) Given :

$$Y(e^{j\Omega}) = j[3 + 4\cos(\Omega) + 2\cos(2\Omega)]\sin(\Omega)$$

$$= j(3 + 2e^{j\Omega} + 2e^{-j\Omega} + e^{j2\Omega} + e^{-j2\Omega})\left[\frac{e^{j\Omega} - e^{-j\Omega}}{2j}\right]$$

$$Y(e^{j\Omega}) = \left(\frac{1}{2}e^{j3\Omega} + e^{j2\Omega} + e^{j\Omega} - e^{-j\Omega} - e^{-j2\Omega} - \frac{1}{2}e^{-j3\Omega}\right)$$

Taking inverse DTFT we get,

$$y(n) = \frac{1}{2}\delta(n+3) + \delta(n+2) + \delta(n+1) - \delta(n-1) - \delta(n-2) - \frac{1}{2}\delta(n-3)$$

$$\therefore \quad y(n) = \left\{\frac{1}{2}, 1, 1, \underset{\uparrow}{0}, -1, -1, -\frac{1}{2}\right\}$$

Example 3.66 Determine the time-domain signal corresponding to,

(a) $X(e^{j\Omega}) = \cos(2\Omega) + 1$

(b) $X(e^{j\Omega}) = \left[\dfrac{\sin\left(\frac{15}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)}\right] \circledast \left[e^{-j3\Omega}\dfrac{\sin\left(\frac{7}{2}\Omega\right)}{\sin\left(\frac{\Omega}{2}\right)}\right]$

Solution. (a) Given :

$$X(e^{j\Omega}) = \cos(2\Omega) + 1 = \frac{1}{2}e^{j2\Omega} + \frac{1}{2}e^{-j2\Omega} + 1 \qquad \text{(E3.66.1)}$$

We know that $\delta(n) \xleftrightarrow{\text{DTFT}} 1$

$$e^{j\Omega_o n} \xleftrightarrow{\text{DTFT}} \delta(\Omega - \Omega_o)$$

Taking inverse DTFT of eqn. (E3.66.1) we get,

$$x(n) = \frac{1}{2}\delta(n+2) + \frac{1}{2}\delta(n-2) + \delta(n)$$

(b) Given :

$$X(e^{j\Omega}) = \underbrace{\left[\frac{\sin\left(\frac{15}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)}\right]}_{A(e^{j\Omega})} \circledast \underbrace{\left[e^{-j3\Omega}\underbrace{\frac{\sin\left(\frac{7}{2}\Omega\right)}{\sin\left(\frac{\Omega}{2}\right)}}_{B_1(e^{j\Omega})}\right]}_{B(e^{j\Omega})}$$

Referring to Example 3.33 we get,

$$\left.\begin{aligned} a(n) &= 1 \quad ; |n| \le 7 \\ &= 0 \quad ; \text{otherwise} \end{aligned}\right\} \xleftrightarrow{\text{DTFT}} \frac{\sin\left(\frac{15}{2}\Omega\right)}{\sin\left(\frac{\Omega}{2}\right)}$$

$$\left.\begin{aligned} b_1(\mathrm{n}) &= 1 \quad &&; |\mathrm{n}| \le 3 \\ &= 0 &&; \text{otherwise} \end{aligned}\right\} \xleftrightarrow{\text{DTFT}} \frac{\sin\left(\frac{7}{2}\Omega\right)}{\sin\left(\frac{\Omega}{2}\right)}$$

$\therefore$ Using time shifting property

$$\begin{aligned} b(\mathrm{n}) = b_1(\mathrm{n}-3) &= 1 \quad &&; 0 \le \mathrm{n} \le 6 \\ &= 0 &&; \text{otherwise} \end{aligned}$$

From modulation property we have

$$a(\mathrm{n}) \cdot b(\mathrm{n}) \xleftrightarrow{\text{DTFT}} \frac{1}{2\pi}[A(e^{j\Omega}) \circledast B(e^{j\Omega})]$$

$$x(\mathrm{n}) = 2\pi a(\mathrm{n}) \cdot b(\mathrm{n}) \xleftrightarrow{\text{DTFT}} A(e^{j\Omega}) \circledast B(e^{j\Omega})$$

$$\begin{aligned} \therefore \quad x(\mathrm{n}) &= 2\pi \quad &&; 0 \le n \le 6 \\ &= 0 &&; \text{otherwise} \end{aligned}$$

Example 3.67 Given that,

$$x(\mathrm{n}) = n\left(-\frac{1}{2}\right)^{\mathrm{n}} u(\mathrm{n}) \xleftrightarrow{\text{DTFT}} X(e^{j\Omega})$$

Without evaluating $X(e^{j\Omega})$, find $y(\mathrm{n})$ if $Y(e^{j\Omega})$ is given by,

(a) $Y(e^{j\Omega}) = e^{j3\Omega}X(e^{j\Omega})$

(b) $Y(e^{j\Omega}) = \dfrac{d}{d\Omega}X(e^{j\Omega})$

(c) $Y(e^{j\Omega}) = \dfrac{d}{d\Omega}\left\{e^{-j2\Omega}\left[X(e^{j(\Omega+\frac{\pi}{4})}) - X(e^{j(\Omega-\frac{\pi}{4})})\right]\right\}$

Solution. (a) Given :

$$Y(e^{j\Omega}) = e^{j3\Omega}X(e^{j\Omega})$$

Taking inverse DTFT we get,

$$y(\mathrm{n}) = x(\mathrm{n}+3) \qquad \text{(Time shift property is used)}$$

$$\therefore \quad y(\mathrm{n}) = (\mathrm{n}+3)\left(-\frac{1}{2}\right)^{\mathrm{n}+3} u(\mathrm{n}+3)$$

(b) Given :

$$Y(e^{j\Omega}) = \frac{d}{d\Omega}X(e^{j\Omega})$$

Taking inverse DTFT we get,

$$y(n) = -jnx(n)$$

$$\therefore \quad y(n) = -jn^2\left(-\frac{1}{2}\right)^n u(n)$$

(c) Given :

$$Y(e^{j\Omega}) = \frac{d}{d\Omega}\left\{e^{-j2\Omega}(X(e^{j(\Omega+\frac{\pi}{3})}) - X(e^{j(\Omega-\frac{\pi}{3})})\right\}$$

$$X(e^{j\Omega}) \xleftrightarrow{\text{DTFT}} x(n)$$

$$X(e^{j(\Omega+\frac{\pi}{4})}) \xleftrightarrow{\text{DTFT}} e^{-j\frac{\pi}{4}n}x(n) \quad \text{(Frequency shifting property is used)}$$

$$\therefore \quad \left[X(e^{j(\Omega+\frac{\pi}{4})}) - X(e^{j(\Omega-\frac{\pi}{4})})\right] \xleftrightarrow{\text{DTFT}} e^{-j\frac{\pi}{4}n}x(n) - e^{j\frac{\pi}{4}n}x(n)$$

$$e^{-j2\Omega}\left[X(e^{j(\Omega+\frac{\pi}{4})}) - X(e^{j(\Omega-\frac{\pi}{4})})\right] \xleftrightarrow{\text{DTFT}} e^{-j\frac{\pi}{4}(n-2)}x(n-2) - e^{j\frac{\pi}{4}(n-2)}x(n-2)$$

(Time shifting property is used)

$$\frac{d}{d\Omega}\left\{e^{-j2\Omega}[X(e^{j(\Omega+\frac{\pi}{4})}) - X(e^{(j\Omega-\frac{\pi}{4})})]\right\} \xleftrightarrow{\text{DTFT}} -jn[e^{-j\frac{\pi}{4}(n-2)}x(n-2) - e^{j\frac{\pi}{4}(n-2)}x(n-2)]$$

(Frequency differentiation property is used)

$$\therefore \quad y(n) = -jn\left[e^{-j\frac{\pi}{4}(n-2)}(n-2)\left(-\frac{1}{2}\right)^{n-2}u(n-2) - e^{j\frac{\pi}{4}(n-2)}(n-2)\left(-\frac{1}{2}\right)^{n-2}u(n-2)\right]$$

$$= jn(n-2)\left(-\frac{1}{2}\right)^{n-2} 2\cdot\cos\left(\frac{\pi}{4}(n-2)\right)\cdot u(n-2)$$

Example 3.68 A signal $x(n)$ has the DTFT,

$$X(e^{j\Omega}) = \frac{1}{1-ae^{-j\Omega}}$$

Determine the DTFT of the following.

(i) $x_1(n) = x(2n+1)$

(ii) $x_2(n) = e^{\frac{\pi}{2}n}x(n+2)$

(iii) $x_3(n) = x(n) * x(n-1)$

(iv) $x_4(n) = x(-2n)$

Solution. We have,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

(i) $$X_1(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_1(n)e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(2n+1)e^{-j\Omega n}$$

Put $m = 2n + 1$

$$= \sum_{m=-\infty}^{\infty} x(m)e^{-j\Omega(\frac{m-1}{2})}$$

$$= \sum_{m=-\infty}^{\infty} x(m)e^{-j\frac{\Omega}{2}m}e^{j\frac{\Omega}{2}}$$

$$= e^{j\frac{\Omega}{2}} \sum_{m=-\infty}^{\infty} x(m)e^{-j\frac{\Omega}{2}m}$$

$$= e^{j\frac{\Omega}{2}} X(e^{j\frac{\Omega}{2}})$$

$$\therefore \quad X_1(e^{j\Omega}) = \frac{e^{j\frac{\Omega}{2}}}{1 - ae^{-j\frac{\Omega}{2}}}$$

(ii) $$X_2(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_2(n)e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} e^{\frac{\pi}{2}n}x(n+2)e^{-j\Omega n}$$

Put $m = n + 2$

$$X_2(e^{j\Omega}) = \sum_{m=-\infty}^{\infty} x(m)e^{\frac{\pi}{2}(m-2)}e^{-j\Omega(m-2)}$$

$$= e^{-\pi} \cdot e^{j2\Omega} \sum_{m=-\infty}^{\infty} x(m)e^{-j(\Omega+j\frac{\pi}{2})m}$$

$$= e^{j2\Omega-\pi} X(e^{j(\Omega+j\frac{\pi}{2})})$$

$$X_2(e^{j\Omega}) = \frac{e^{j2\Omega-\pi}}{1 - ae^{-j(\Omega+j\frac{\pi}{2})}}$$

(iii) From convolution property,

$$x_3(n) = x(n) * x(n-1) \xleftrightarrow{\text{DTFT}} X_3(e^{j\Omega}) = X(e^{j\Omega}) \cdot e^{-j\Omega} X(e^{j\Omega})$$

$$\therefore \quad X_3(e^{j\Omega}) = e^{-j\Omega} \cdot X^2(e^{j\Omega})$$

$$= \frac{e^{-j\Omega}}{(1 - ae^{-j\Omega})^2}$$

(iv) $X_4(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_4(n)e^{-j\Omega n}$

$$= \sum_{n=-\infty}^{\infty} x(-2n)e^{-j\Omega n}$$

Put m = −2n

$$= \sum_{m=\infty}^{-\infty} x(m)e^{j\Omega\frac{m}{2}}$$

$$= \sum_{m=-\infty}^{\infty} x(m)e^{-j(-\frac{\Omega}{2})m}$$

$$= X(e^{-j\frac{\Omega}{2}})$$

$$\therefore \quad X_4(e^{j\Omega}) = \frac{1}{1 - ae^{j\frac{\Omega}{2}}}$$

3.7 Continuous-Time Non-Periodic Signals: The Fourier Transform (FT)

A non-periodic continuous-time signal $x(t)$ can be expressed as,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega \tag{3.26}$$

where $$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \tag{3.27}$$

$X(j\omega)$ is known as *Fourier Transform* (FT) of $x(t)$. Alternatively, we say that $X(j\omega)$ and $x(t)$ forms a FT pair, which can be expressed as,

$$x(t) \overset{\text{FT}}{\longleftrightarrow} X(j\omega)$$

where $X(j\omega)$ is the frequency domain representation of the time-domain signal $x(t)$. $X(j\omega)$ is also known as *spectrum of* $x(t)$. $X(j\omega)$ is *non-periodic.*

Eqn. (3.26) is known as *synthesis equation* and eqn. (3.27) is known as *analysis equation.*

The Fourier transform $X(j\omega)$ for a continuous-time signal $x(t)$ exists if the following conditions (referred to as *Dirichlet conditions*) are satisfied.

i) $x(t)$ is absolutely integrable.

i.e.,

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty$$

ii) $x(t)$ have a finite number of maxima and minima within any finite interval.

iii) $x(t)$ have a finite number of discontinuities within any finite interval.

The above conditions are sufficient but not necessary conditions.

3.7.1 Properties of FT

In this section, we will discuss the different properties of Fourier Transform. They are,

(a) Linearity

(b) Time shift

(c) Frequency shift

(d) Scaling

(e) Time differentiation

(f) Frequency differentiation

(g) Integration

(h) Convolution

(i) Modulation

(j) Parseval's theorem

(k) Symmetry

(a) Linearity:

$$\text{If} \quad x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\text{and} \quad y(t) \xleftrightarrow{FT} Y(j\omega)$$

$$\text{then} \quad z(t) = ax(t) + by(t) \xleftrightarrow{FT} Z(j\omega) = aX(j\omega) + bY(j\omega)$$

Proof. We have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt$$

$$\therefore\ Z(j\omega) = \int_{-\infty}^{\infty} z(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} [ax(t) + by(t)]e^{-j\omega t}dt$$

$$= a\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt + b\int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt$$

$$Z(j\omega) = aX(j\omega) + bY(j\omega)$$

Hence the proof.

(b) Time Shift:

$$\text{If} \quad x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\text{then} \quad y(t) = x(t - t_o) \xleftrightarrow{FT} Y(j\omega) = e^{-j\omega t_o}X(j\omega)$$

Proof. We have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$\therefore\ Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(t-t_o)e^{-j\omega t}dt$$

Put $t - t_o = a$, then $dt = da$

$$Y(j\omega) = \int_{-\infty}^{\infty} x(a)e^{-j\omega(a+t_o)}da$$

$$= e^{-j\omega t_o}\int_{-\infty}^{\infty} x(a)e^{-j\omega a}da$$

$$Y(j\omega) = e^{-j\omega t_o}X(j\omega)$$

Hence the proof.

(c) **Frequency Shift:**

$$\text{If} \quad x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\text{then} \quad y(t) = e^{j\beta t}x(t) \xleftrightarrow{FT} Y(j\omega) = X(j(\omega-\beta))$$

Proof. We have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$\therefore\ Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} e^{j\beta t}x(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\beta)t}dt$$

$$\therefore\ Y(j\omega) = X(j(\omega-\beta))$$

Hence the proof.

(d) Scaling:

$$\text{If} \quad x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\text{then} \quad y(t) = x(at) \xleftrightarrow{FT} Y(j\omega) = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

Proof. We have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\therefore \; Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

$$Y(j\omega) = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

If 'a' is positive i.e., $a > 0$,

$$\text{Put} \quad at = \tau,$$

$$dt = \frac{1}{a} d\tau$$

$$\therefore \; Y(j\omega) = \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j\frac{\omega}{a}\tau} d\tau$$

$$Y(j\omega) = \frac{1}{a} X\left(\frac{j\omega}{a}\right) \tag{3.28}$$

If 'a' is negative i.e., $a < 0$

$$\text{Put} \quad -at = \tau,$$

$$dt = -\frac{1}{a} d\tau$$

$$\therefore \; Y(j\omega) = -\frac{1}{a} \int_{\infty}^{-\infty} x(-\tau) e^{-j\frac{\omega}{a}(-\tau)} d\tau$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(-\tau) e^{-j\frac{\omega}{a}(-\tau)} d\tau$$

$$\therefore \; Y(j\omega) = \frac{1}{a} X\left(\frac{j\omega}{a}\right) \tag{3.29}$$

From eqn. (3.28) and eqn. (3.29) we get,

$$Y(j\omega) = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

Hence the proof.

(e) **Time differentiation:**

$$\text{If} \quad x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\text{then} \quad \frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(j\omega)$$

Proof. We have,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \tag{3.30}$$

Differentiating both the sides with respect to 't' we get,

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega\right]$$

Changing the order of differentiation and integration we get,

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\left(\frac{\partial}{\partial t} e^{j\omega t}\right) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, j\omega . e^{j\omega t} d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega X(j\omega)) e^{j\omega t} d\omega \tag{3.31}$$

Comparing eqn. 3.31 with eqn. 3.30 we get,

$$\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(j\omega)$$

Hence the proof.

(f) Frequency differentiation:

$$\text{If} \quad x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\text{then} \quad -jtx(t) \xleftrightarrow{FT} \frac{dX(j\omega)}{d\omega}$$

Proof. We have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad (3.32)$$

Differentiating both the sides with respect to 'ω' we get,

$$\frac{dX(j\omega)}{d\omega} = \frac{d}{d\omega}\left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]$$

Interchanging the order of differentiation and integration we get,

$$\frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{\infty} x(t)\left(\frac{\partial}{\partial\omega}e^{-j\omega t}\right)dt$$

$$= \int_{-\infty}^{\infty} x(t)\,(-jt)e^{-j\omega t}dt$$

$$\frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{\infty} \left[-jtx(t)\right]e^{-j\omega t}dt \qquad (3.33)$$

Comparing eqn. (3.33) with eqn. (3.32) we get,

$$-jtx(t) \xleftrightarrow{FT} \frac{dX(j\omega)}{d\omega}$$

Hence the proof.

(g) Integration:

$$\text{If} \quad x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\text{then} \quad \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{X(j\omega)}{j\omega} + \pi X(j0)\delta(\omega).$$

Proof. Refer to example 3.94.

(h) Convolution:

If $\quad x(t) \xleftrightarrow{FT} X(j\omega)$

and $\quad y(t) \xleftrightarrow{FT} Y(j\omega)$

then $\quad z(t) = x(t) * y(t) \xleftrightarrow{FT} Z(j\omega) = X(j\omega)Y(j\omega)$,

Proof. We have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}\,dt$$

$$\therefore\ Z(j\omega) = \int_{-\infty}^{\infty} z(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} [x(t) * y(t)]e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau\right]e^{-j\omega t}\,dt$$

Changing the order of integrations we get,

$$Z(j\omega) = \int_{-\infty}^{\infty} x(\tau)\left(\int_{-\infty}^{\infty} y(t-\tau)e^{-j\omega t}dt\right)d\tau$$

Put $\quad t-\tau = a, \quad$ then $\quad dt = da,$

$$Z(j\omega) = \int_{-\infty}^{\infty} x(\tau)\int_{-\infty}^{\infty} y(a)e^{-j\omega(a+\tau)}\,da\,d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \int_{-\infty}^{\infty} y(a)e^{-j\omega a}\,da$$

$$\therefore\ Z(j\omega) = X(j\omega)Y(j\omega)$$

Therefore convolution in time domain is equivalent to multiplication in frequency domain.

(i) Modulation:

$$\text{If} \quad x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\text{and} \quad y(t) \xleftrightarrow{FT} Y(j\omega)$$

$$\text{then} \quad z(t) = x(t)y(t) \xleftrightarrow{FT} Z(j\omega) = \frac{1}{2\pi}[X(j\omega) * Y(j\omega)]$$

Proof. We have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt$$

$$\therefore \; Z(j\omega) = \int_{-\infty}^{\infty} z(t)e^{-j\omega t}dt$$

$$Z(j\omega) = \int_{-\infty}^{\infty} [x(t)y(t)]e^{-j\omega t}dt \qquad (3.3$$

We have,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\beta)e^{j\beta t}d\beta \qquad (3.3$$

Substituting eqn. 3.35 in eqn. 3.34 we get,

$$Z(j\omega) = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\beta)e^{j\beta t}d\beta\right]y(t)e^{-j\omega t}dt$$

Changing the order of integrations we get,

$$Z(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\beta)\int_{-\infty}^{\infty} y(t)e^{-j(\omega-\beta)t}dtd\beta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\beta)Y(j(\omega-\beta))d\beta$$

$$\therefore Z(j\omega) = \frac{1}{2\pi}[X(j\omega) * Y(j\omega)]$$

Therefore multiplication in time domain is equivalent to convolution in frequency domain.

(j) **Parseval's theorem:**

If $x(t) \xleftrightarrow{FT} X(j\omega)$

then
$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \quad (3.36)$$

In eqn. (3.36), $|X(j\omega)|^2$ is known as *energy density spectrum* of the signal $x(t)$.
We know that the Left Hand Side of eqn. (3.36) is the energy of the signal $x(t)$.

Proof. We have,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} x(t)x^*(t)dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}d\omega\right]dt$$

Changing the order of integrations we get,

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)X(j\omega)d\omega$$

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

$$\therefore \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Hence the proof.

(k) **Duality:**

If $x(t) \xleftrightarrow{FT} X(j\omega)$

then $X(jt) \xleftrightarrow{FT} 2\pi x(-\omega)$

Proof. We have,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Interchanging 't' and 'ω' we get,

$$x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jt)e^{j\omega t}dt$$

Replacing 'ω' by '$-\omega$' we get,

$$x(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jt)e^{-j\omega t}dt$$

$$\therefore\ 2\pi x(-\omega) = \int_{-\infty}^{\infty} X(jt)e^{-j\omega t}dt \qquad (3.37)$$

Comparing eqn. 3.37 with,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

we get,

$$X(jt) \xleftrightarrow{\text{FT}} 2\pi x(-\omega)$$

Hence the proof.

(I) Symmetry: If $x(t) \xleftrightarrow{\text{FT}} X(j\omega)$

then (i) If $x(t)$ is real, then $X^*(j\omega) = X(-j\omega)$

(ii) If $x(t)$ is real and even, then $\text{Img}\{X(j\omega)\} = 0$ [i.e., $X(j\omega)$ is purely real]

(iii) If $x(t)$ is real and odd, then $\text{Re}\{X(j\omega)\} = 0$ [i.e., $X(j\omega)$ is purely imaginary]

Proof. If $x(t)$ is real, $x(t) = x^*(t)$

Let $x(t) = x_e(t) + x_o(t) \xleftrightarrow{\text{FT}} X(j\omega) = X_R(j\omega) + jX_I(j\omega) \qquad (3.38)$

We have, $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad (3.39)$$

Taking complex conjugate on both the sides,

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{j(-\omega)t}d\omega \qquad (3.40)$$

Comparing eqn. (3.40) with eqn. (3.39) we get,

$$x(t) \xleftrightarrow{FT} X^*(-j\omega) \qquad [\because x(t) = x^*(t)] \qquad (3.41)$$

From eqn. (3.39) and eqn. (3.40) we get,

$$X(j\omega) = X^*(-j\omega)$$

Taking complex conjugate on both the sides,

$$X^*(j\omega) = X(-j\omega)$$

From eqn. (3.38) we get,

$$x(-t) = x_e(-t) + x_o(-t)$$

$$= x_e(t) - x_o(t)$$

$$\therefore\ x(-t) = x_e(t) - x_o(t) \xleftrightarrow{FT} X(-j\omega) = X^*(j\omega)$$

$$= X_R(j\omega) - jX_I(j\omega) \qquad (3.42)$$

Adding eqn. (3.38) and eqn. (3.42) we get,

$$2x_e(t) \xleftrightarrow{FT} 2X_R(j\omega)$$

$$\therefore\ x_e(t) \xleftrightarrow{FT} X_R(j\omega)$$

Therefore, FT of real and even signal is purely real.

Subtracting eqn. (3.42) from eqn. (3.38) we get,

$$2x_o(t) \xleftrightarrow{FT} 2jX_I(j\omega)$$

$$\therefore\ x_o(t) \xleftrightarrow{FT} jX_I(j\omega)$$

Therefore, FT of real and odd signal is purely imaginary.

EXAMPLES

Example 3.69 Obtain the Fourier transform of the signal,

$$x(t) = e^{-at}u(t) \quad ; a > 0$$

Draw its magnitude and phase spectra.

Solution. We have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$X(j\omega) = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$= \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_0^{\infty}$$

$$X(j\omega) = \frac{1}{a+j\omega} \quad ; \quad a > 0$$

$$\therefore \; |X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

$$\measuredangle X(j\omega) = -\tan^{-1}(\omega/a)$$

The magnitude and phase spectra are shown in Fig. E3.69.1 and E3.69.2 respectively.

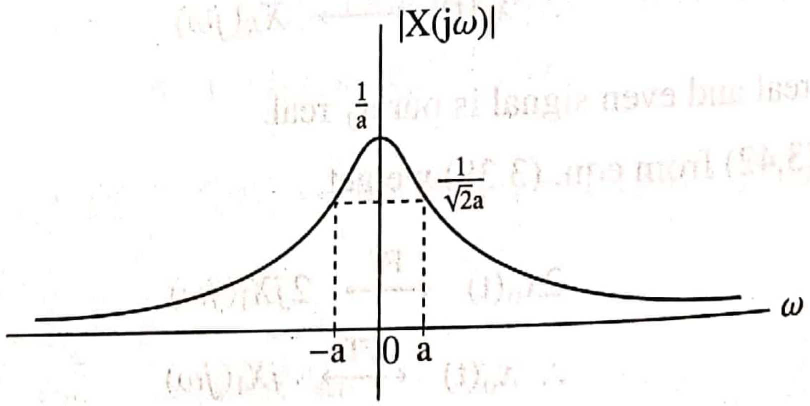


Fig. E3.69.1.

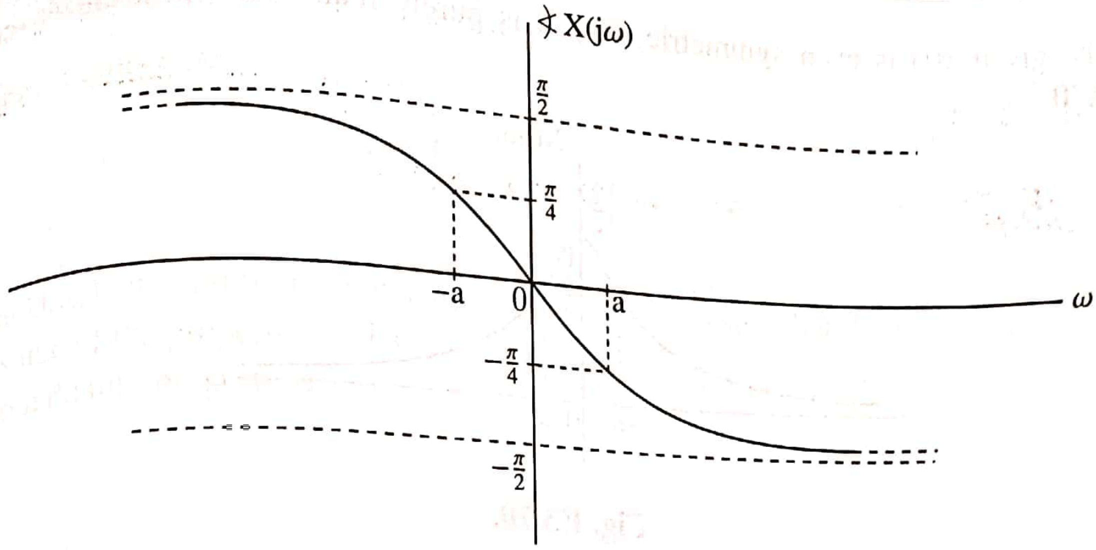


Fig. E3.69.2.

Example 3.70 Find the Fourier Transform of,

$$x(t) = e^{-a|t|} \qquad ; a > 0$$

Draw its spectrum.

Solution. We have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}dt + \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$X(j\omega) = \frac{2a}{a^2+\omega^2}$$

Since the given $x(t)$ is even symmetric, $X(j\omega)$ is purely real. The spectrum is shown in Fig. E3.70.

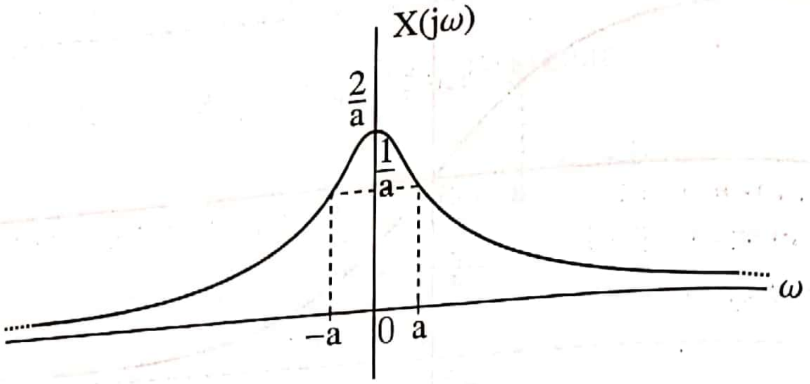


Fig. E3.70.

Example 3.71 Determine the Fourier transform of the unit impulse function.

$$\text{i.e.,} \qquad x(t) = \delta(t)$$

Draw its spectrum.

Solution. We have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\,dt$$

$$= e^{-j\omega t}\Big|_{t=0} \qquad [\because \text{ sifting property}]$$

$$X(j\omega) = 1$$

The spectrum is shown in Fig. E3.71.1.

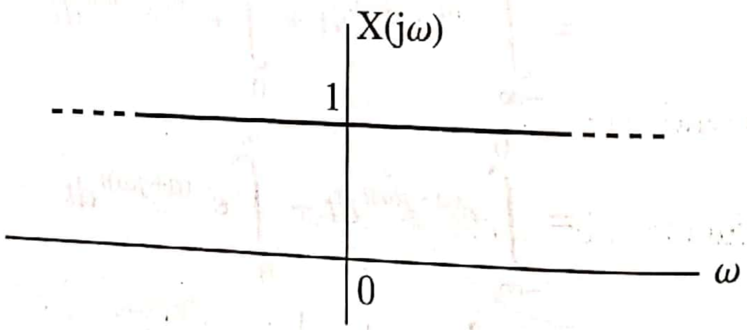


Fig. E3.71.1.

Example 3.72 Find the FT of $x(t) = 1$.

Solution. In this case,

$$\int_{-\infty}^{\infty} |x(t)|dt = \int_{-\infty}^{\infty} dt \to \infty$$

i.e., the Dirichlet condition is not satisfied. But we can show that the FT of $x(t) = 1$ exists by using some of the properties of FT.

From duality property we have,

$$\text{If} \qquad x(t) \overset{FT}{\longleftrightarrow} X(j\omega)$$

$$\text{then} \qquad X(jt) \overset{FT}{\longleftrightarrow} 2\pi x(-\omega) \qquad \text{(Duality property)}$$

We have,

$$\delta(t) \overset{FT}{\longleftrightarrow} 1$$

$\therefore$ Using duality property we have,

$$1 \overset{FT}{\longleftrightarrow} 2\pi\delta(-\omega)$$

But $\delta(-\omega)$ is an even function. $\therefore\ \delta(-\omega) = \delta(\omega)$

$$\therefore\ x(t) = 1 \overset{FT}{\longleftrightarrow} 2\pi\delta(\omega)$$

Example 3.73 Find the Fourier transform of the signum function.

$$\text{i.e.,} \qquad x(t) = \text{sgn}(t)$$

Draw the magnitude and phase spectra.

Solution. The signum function is defined as,

$$\text{sgn}(t) = 1 \quad ; t > 0$$
$$= -1 \quad ; t < 0$$

The plot of signum function is shown in Fig. E3.73 below.

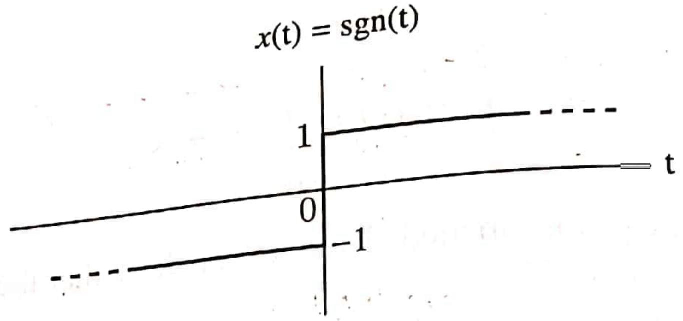


Fig. E3.73.

Differentiating $x(t) = \text{sgn}(t)$ with respect to time we get,

$$\frac{dx(t)}{dt} = 2\delta(t)$$

Taking FT on both the sides using time differentiation property we get,

$$j\omega X(j\omega) = 2$$

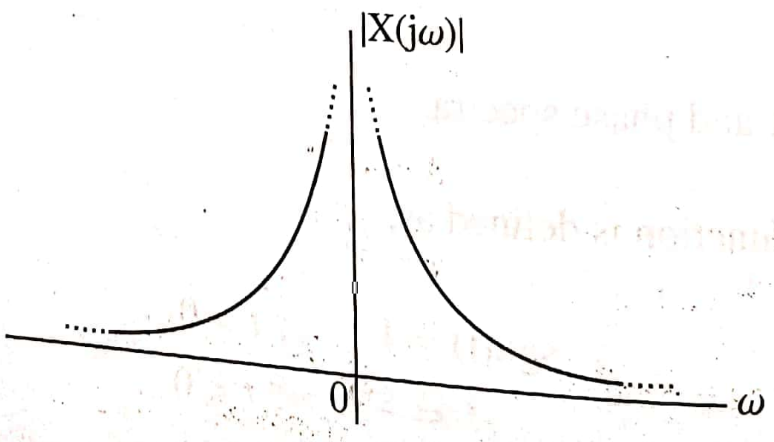
$$\therefore\ X(j\omega) = \frac{2}{j\omega}$$

$$\therefore\ |X(j\omega)| = \frac{2}{\omega}$$

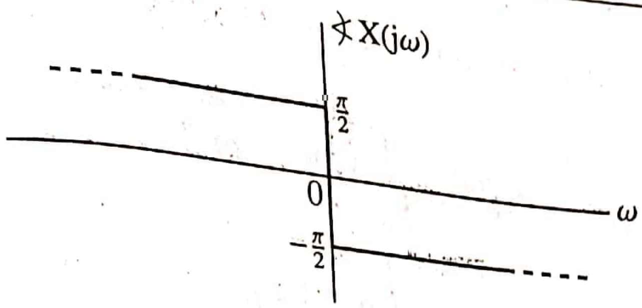
$$\measuredangle X(j\omega) = -\frac{\pi}{2} \quad ;\ \omega > 0$$

$$= \frac{\pi}{2} \quad ;\ \omega < 0$$

The magnitude and phase spectra are shown in Fig. E3.73.1 (a) & (b) respectively.



(a)



(b)

Fig. E3.73.1.

Example 3.74 Find the Fourier transform of unit step function.

Solution. Given:

$$x(t) = u(t)$$

The signum function can be expressed in terms of $u(t)$ as,

$$\text{sgn}(t) = 2u(t) - 1$$

$$\therefore \; u(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t)$$

Taking FT on both the sides we get,

$$F\{u(t)\} = \left(\frac{1}{2}\right)2\pi\delta(\omega) + \frac{1}{2}\left(\frac{2}{j\omega}\right)$$

$$= \pi\delta(\omega) + \frac{1}{j\omega}$$

Example 3.75 For the rectangular pulse shown in Fig. E3.75, draw the spectrum.

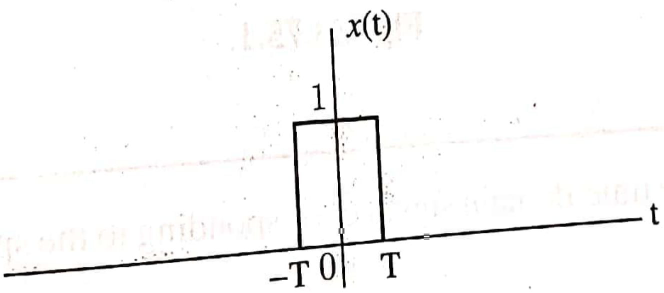


Fig. E3.75.

Solution. From Fig. E3.75 we have,

$$x(t) = 1 \quad ; \; -T < t < T$$
$$= 0 \quad ; \; \text{otherwise.}$$

We have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$= \int_{-T}^{T} 1.e^{-j\omega t}dt$$

$$= \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-T}^{T}$$

$$= \frac{1}{-j\omega}\left[e^{-j\omega T} - e^{j\omega T}\right]$$

$$X(j\omega) = \frac{2\sin\omega T}{\omega}$$

When $\omega = 0$; $X(j\omega) = 2T$ [Using *L*-Hospital's Rule]

The spectrum is shown in Fig. E3.75.1 below.

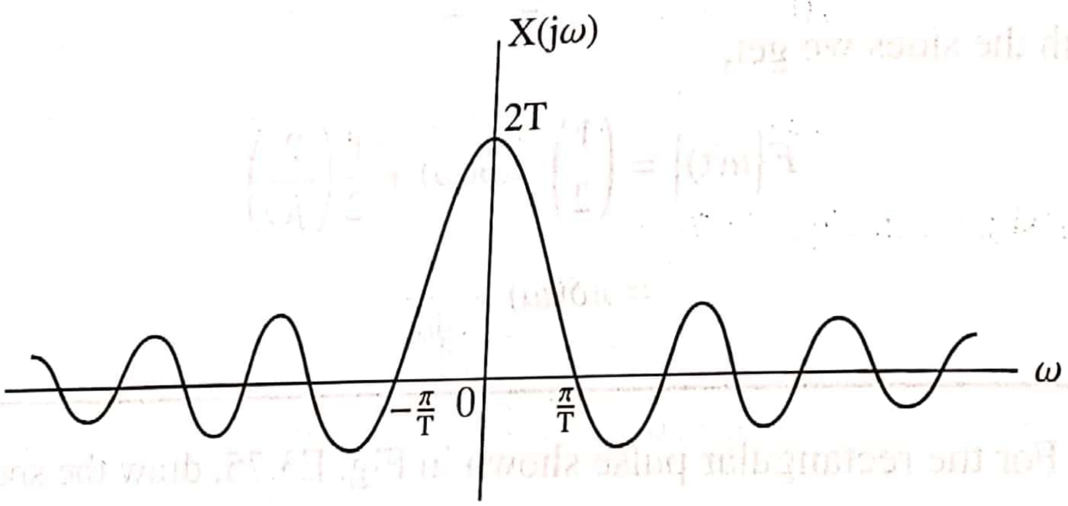


Fig. E3.75.1.

...lution. From Fig. E3.76 we have,

$$X(j\omega) = 1 \quad ; -W < \omega < W$$
$$= 1 \quad ; \text{ otherwise}$$

...e know that,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} 1.e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-W}^{W}$$

$$= \frac{1}{2\pi jt}\left(e^{jWT} - e^{-jWt}\right)$$

$$x(t) = \frac{\sin(Wt)}{\pi t}$$

$$x(0) = \frac{W}{\pi} \quad \text{[using } L\text{-Hospital's Rule]}$$

...he signal $x(t)$ is shown in Fig. E3.76.1 below.

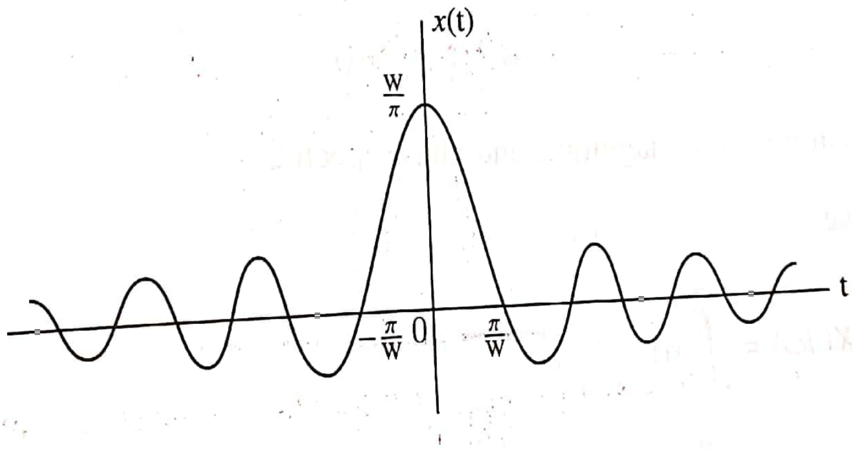


Fig. E3.76.1.

Example 3.77 Evaluate the Fourier transform for the signal,

$$x(t) = e^{-3t}u(t-1)$$

Find the expression for magnitude and phase spectra.

Solution. Given:

$$x(t) = e^{-3t}u(t-1)$$

We have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$\therefore\ X(j\omega) = \int_{1}^{\infty} e^{-3t}e^{-j\omega t}dt$$

$$= \int_{1}^{\infty} e^{-(3+j\omega)t}dt$$

$$X(j\omega) = \frac{e^{-(3+j\omega)}}{3+j\omega}$$

$$\therefore\ |X(j\omega)| = \frac{e^{-3}}{\sqrt{9+\omega^2}}$$

$$\text{and } \measuredangle X(j\omega) = -\omega - \tan^{-1}(\omega/3)$$

$$\therefore\ |X(j\omega)| = \frac{1}{\sqrt{(4-\omega^2)^2 + (4\omega)^2}} \text{ and } \angle X(j\omega) = -\tan^{-1}\left(\frac{4\omega}{4-\omega^2}\right)$$

Example 3.79 For the signal $x(t)$ shown below in Fig. E3.79, find the Fourier transform.

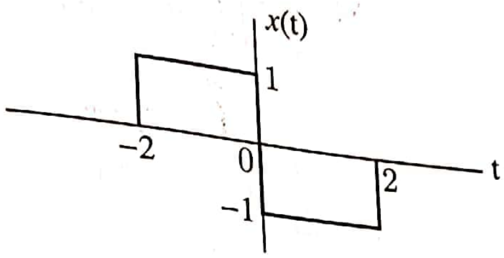


Fig. E3.79.

Solution. From Fig. E3.79 we have,

$$x(t) = \begin{cases} 1 & ;\ -2 < t < 0 \\ -1 & ;\ 0 < t < 2 \\ 0 & ;\ \text{otherwise} \end{cases}$$

We have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$= \int_{-2}^{0} 1 \cdot e^{-j\omega t}\,dt + \int_{0}^{2} (-1) \cdot e^{-j\omega t}\,dt$$

$$= \int_{-2}^{0} e^{-j\omega t}\,dt - \int_{0}^{2} e^{-j\omega t}\,dt$$

$$\therefore\quad X(j\omega) = \frac{2[\cos(2\omega) - 1]}{j\omega}$$

Example 3.80 Determine the time-domain signal corresponding to the following Fourier transform.

$$X(j\omega) = e^{-2\omega}u(\omega)$$

Solution. We have,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$\therefore \quad x(t) = \frac{1}{2\pi}\int_{0}^{\infty} e^{-2\omega}e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} e^{-(2-jt)\omega}d\omega$$

$$x(t) = \frac{1}{2\pi(2-jt)}$$

Example 3.81 Determine the time-domain signal corresponding to the spectra shown in Fig. E3.81 (a) & (b) respectively.

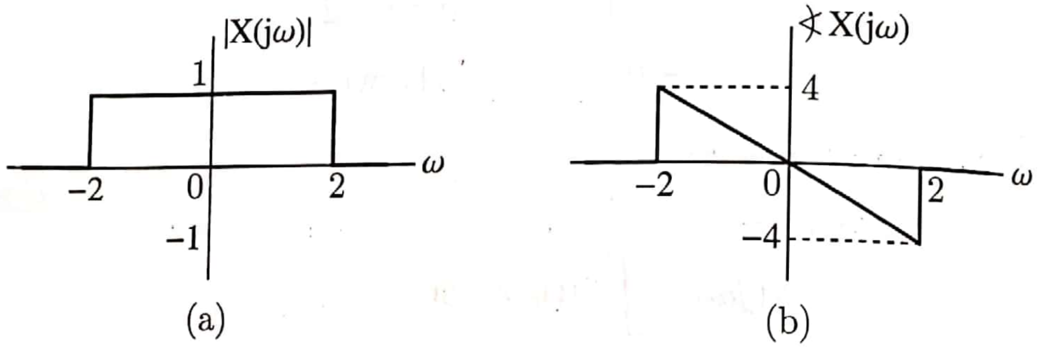


Fig. E3.81.

Solution. From Fig. E3.81 we have,

$$X(j\omega) = 1 \cdot e^{-j2\omega} \quad ; \; -2 < \omega < 2$$

$$= 0 \quad ; \text{ otherwise}$$

$$\therefore \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-2}^{2} 1 \cdot e^{-j2\omega}e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-2}^{2} e^{j\omega(t-2)}d\omega$$

$$\therefore \quad x(t) = \frac{\sin(2(t-2))}{\pi(t-2)} \quad ; t \neq 2$$

$$= \frac{2}{\pi} \quad ; t = 2 \quad \text{[Using } L\text{-Hospital's Rule]}$$

Example 3.82 Determine the time-domain signal corresponding to the spectrum shown in Fig. E3.82 (a) & (b) respectively.

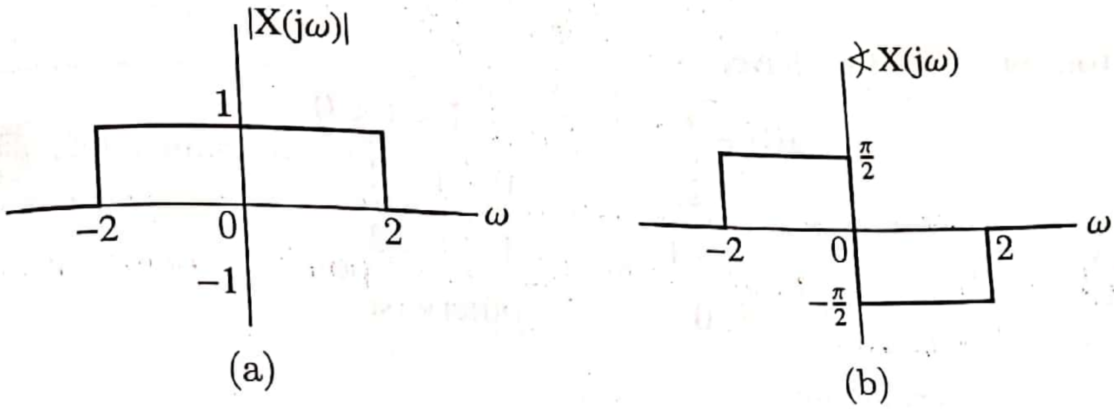


Fig. E3.82.

Solution. From Fig. E3.82 we have,

$$X(j\omega) = 1 \cdot e^{j\pi/2} = j \quad ; -2 < \omega < 0$$

$$= 1 \cdot e^{-j\pi/2} = -j \quad ; 0 < \omega < 2$$

$$= 0 \quad ; \text{otherwise}$$

We know that,

$$\therefore \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-2}^{0} j \cdot e^{j\omega t}d\omega - \int_{0}^{2} j \cdot e^{j\omega t}d\omega\right]$$

$$\therefore \quad x(t) = \frac{1-\cos(2t)}{\pi t} \quad ; t \neq 0$$

$$= 0 \quad ; t = 0$$

Example 3.83 Compute the Fourier Transform for the signal $x(t)$ shown in Fig. E3.83.

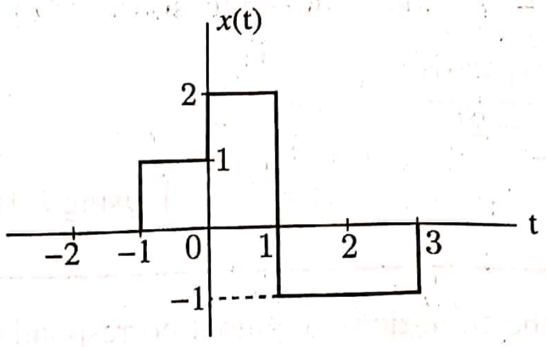


Fig. E3.83.

Solution. From Fig. E3.83 we have,

$$x(t) = \begin{cases} 1 & ; -1 < t < 0 \\ 2 & ; 0 < t < 1 \\ -1 & ; 1 < t < 3 \\ 0 & ; \text{otherwise} \end{cases}$$

We have,

$$\therefore \quad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$= \int_{-1}^{0} 1 \cdot e^{-j\omega t}\,dt + \int_{0}^{1} 2 \cdot e^{-j\omega t}\,dt - \int_{1}^{3} 1 \cdot e^{-j\omega t}\,dt$$

$$X(j\omega) = \frac{1}{j\omega}(1 + e^{j\omega} - 3e^{-j\omega} + e^{-j3\omega})$$

Example 3.84 Compute the Fourier transform of the signal,

$$x(t) = \begin{cases} 1 + \cos(\pi t) & ; |t| \le 1 \\ 0 & ; |t| > 1 \end{cases}$$

Solution. We have,

$$\therefore \quad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$= \int_{-1}^{1} [1 + \cos(\pi t)]\, e^{-j\omega t}\,dt$$

$$= \int_{-1}^{1} 1 \cdot e^{-j\omega t}\,dt + \int_{-1}^{1} \cos(\pi t) e^{-j\omega t}\,dt$$

$$= \int_{-1}^{1} e^{-j\omega t}\,dt + \int_{-1}^{1} \left(\frac{e^{j\pi t} + e^{-j\pi t}}{2}\right) e^{-j\omega t}\,dt$$

$$= \int_{-1}^{1} e^{-j\omega t}\,dt + \frac{1}{2}\int_{-1}^{1} e^{j(\pi-\omega)t}\,dt + \frac{1}{2}\int_{-1}^{1} e^{-j(\pi+\omega)t}\,dt$$

$$X(j\omega) = \frac{2\sin(\omega)}{\omega} + \frac{\sin(\omega)}{\pi-\omega} - \frac{\sin(\omega)}{\pi+\omega}$$

Example 3.85 Determine the time-domain signal corresponding to the Fourier transform,

$$X(j\omega) = \cos\left(\frac{\omega}{2}\right) + j\sin\left(\frac{\omega}{2}\right) \quad ; |\omega| < \pi$$

$$= 0 \quad ; \text{otherwise}$$

Solution. We have,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\,d\omega$$

$$\therefore \quad x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\cos\left(\frac{\omega}{2}\right) + j\sin\left(\frac{\omega}{2}\right)\right] e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega/2} e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(\frac{1}{2}+t)}\,d\omega$$

$$\therefore \quad x(t) = \frac{\cos(\pi t)}{\pi(t + {}^1\!/_2)} \quad ; t \neq -\frac{1}{2}$$

$$= 1 \quad ; t = -\frac{1}{2} \qquad \text{[Using } L\text{-Hospital's Rule]}$$

Example 3.86 Using partial fraction expansion, determine the inverse Fourier transform of,

$$X(j\omega) = \frac{5j\omega + 12}{(j\omega)^2 + 5j\omega + 6}$$

Solution. Given :

$$X(j\omega) = \frac{5j\omega + 12}{(j\omega)^2 + 5j\omega + 6}$$

$$= \frac{5j\omega + 12}{(j\omega + 3)(j\omega + 2)}$$

By partial fraction expansion we get,

$$X(j\omega) = \frac{3}{(j\omega + 3)} + \frac{2}{(j\omega + 2)}$$

Taking inverse Fourier transform we get,

$$x(t) = 3e^{-3t}u(t) + 2e^{-2t}u(t) \qquad \left[\because\ e^{-at}u(t) \overset{FT}{\longleftrightarrow} \frac{1}{a + j\omega}\right]$$

$$\therefore \quad x(t) = (3e^{-3t} + 2e^{-2t})u(t)$$

Example 3.87 Find the inverse Fourier transform of,

$$X(j\omega) = \frac{-j\omega}{(j\omega)^2 + 3j\omega + 2}$$

Solution. Given :

$$X(j\omega) = \frac{-j\omega}{(j\omega)^2 + 3j\omega + 2}$$

$$= \frac{-j\omega}{(j\omega + 1)(j\omega + 2)}$$

By partial fraction expansion we get,

$$X(j\omega) = \frac{1}{(j\omega + 1)} + \frac{(-2)}{(j\omega + 2)}$$

Taking inverse Fourier transform we get,

$$x(t) = e^{-t}u(t) - 2e^{-2t}u(t)$$

$$\therefore \quad x(t) = (e^{-t} - 2e^{-2t})u(t)$$

Example 3.88 Find the inverse Fourier transform of the following using appropriate properties.

$$X(j\omega) = \frac{j\omega}{(2 + j\omega)^2}$$

Solution. Given :

$$X(j\omega) = \frac{j\omega}{(2 + j\omega)^2}$$

We know that,

$$e^{-2t}u(t) \overset{FT}{\longleftrightarrow} \frac{1}{2 + j\omega}$$

Using frequency differentiation property we get,

$$-j\,t\,e^{-2t}u(t) \overset{FT}{\longleftrightarrow} \frac{d}{d\omega}\left[\frac{1}{2 + j\omega}\right]$$

i.e., $$-j\,t\,e^{-2t}u(t) \overset{FT}{\longleftrightarrow} \frac{-j}{(2 + j\omega)^2}$$

$$\therefore \quad t \cdot e^{-2t}u(t) \overset{FT}{\longleftrightarrow} \frac{1}{(2 + j\omega)^2}$$

Using time differentiation property we get,

$$\therefore \quad \frac{d}{dt}\left[t \cdot e^{-2t}u(t)\right] \overset{FT}{\longleftrightarrow} j\omega\frac{1}{(2 + j\omega)^2}$$

$$\therefore \quad x(t) = \frac{d}{dt}[t \cdot e^{-2t}u(t)]$$

$$= (-2t \cdot e^{-2t} + e^{-2t})u(t)$$

$$\therefore \quad x(t) = (1 - 2t)e^{-2t}u(t)$$

Solution. The plot of $\frac{dx(t)}{dt}$ is shown in Fig. E3.89.1 below.

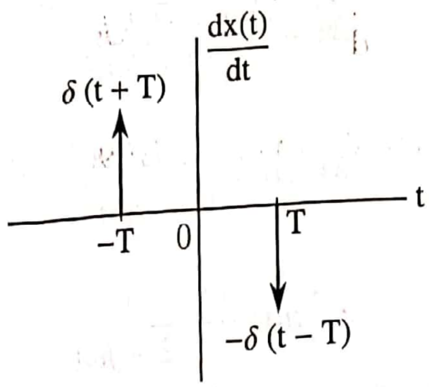


Fig. E3.89.1.

We have,
$$\frac{dx(t)}{dt} = \delta(t+T) - \delta(t-T)$$

Taking Fourier Transform using time differentiation property we get,

$$j\omega X(j\omega) = e^{j\omega T} - e^{-j\omega T}$$

$$X(j\omega) = \frac{e^{j\omega T} - e^{-j\omega T}}{j\omega}$$

$$= \frac{2\sin(\omega T)}{\omega}$$

Example 3.90 Find the Fourier Transform of the signal $x(t)$ shown in Fig. E3.90 below using time differentiation property.

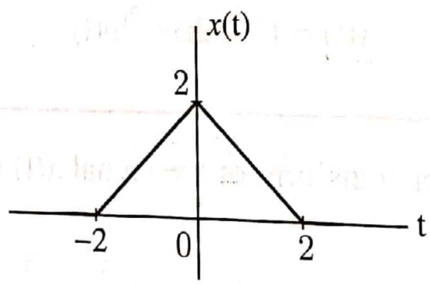


Fig. E3.90.

Solution. From time differentiation property of FT we have,

$$\text{If} \quad x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\text{then} \quad \frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega\, X(j\omega)$$

Once again applying time differentiation property we get,

$$\frac{d^2x(t)}{dt^2} \overset{FT}{\longleftrightarrow} (j\omega)^2 X(j\omega)$$

The plots of $\frac{dx(t)}{dt}$ and $\frac{d^2x(t)}{dt^2}$ are shown in Fig. E3.90.1 (a) & (b) below.

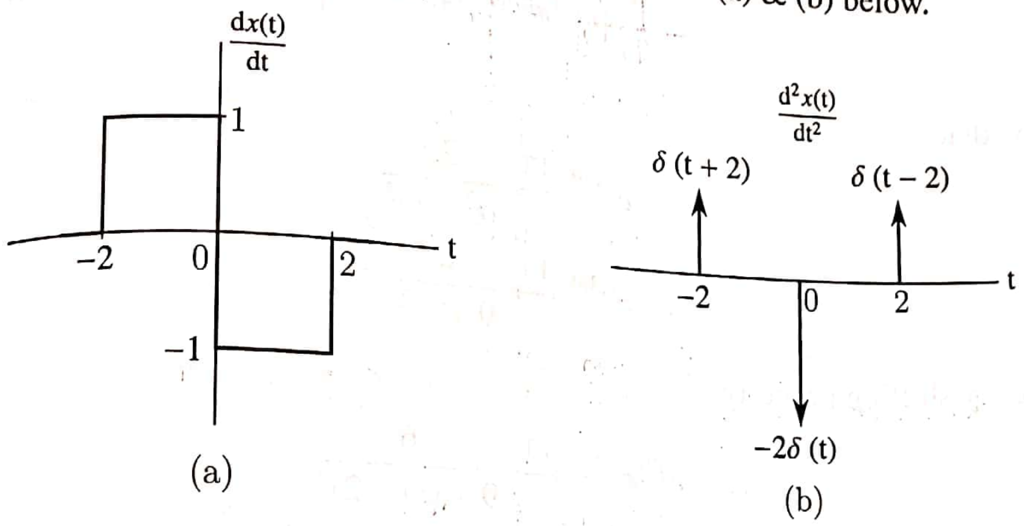


Fig. E3.90.1.

$$\therefore \quad \frac{d^2x(t)}{dt^2} = \delta(t+2) - 2\delta(t) + \delta(t-2)$$

Taking FT on both the sides using time differentiation property we get,

$$(j\omega)^2 X(j\omega) = e^{j2\omega} - 2 + e^{-j2\omega}$$

$$\therefore \quad X(j\omega) = \frac{2 - e^{j2\omega} - e^{-j2\omega}}{\omega^2}$$

$$= \frac{2[1 - \cos(2\omega)]}{\omega^2}$$

$$\therefore \quad X(j\omega) = \left(\frac{2\sin(\omega)}{\omega}\right)^2$$

Example 3.91 Find the Fourier transform of the signal,

$$x(t) = e^{-3|t|}\sin(2t)$$

using appropriate properties.

Solution. Given :

$$x(t) = e^{-3|t|}\sin(2t)$$
$$= \frac{e^{j2t} - e^{-j2t}}{2j} \cdot e^{-3|t|}$$
$$= \frac{1}{2j}\left[e^{j2t}e^{-3|t|} - e^{-j2t}e^{3|t|}\right]$$

We know that,

$$e^{-a|t|} \overset{FT}{\longleftrightarrow} \frac{2a}{a^2+\omega^2}$$
$$\therefore \quad e^{-3|t|} \overset{FT}{\longleftrightarrow} \frac{6}{9+\omega^2}$$

Using frequency shifting property,

$$\therefore \quad e^{j2t}e^{-3|t|} \overset{FT}{\longleftrightarrow} \frac{6}{9+(\omega-2)^2}$$

Similarly,

$$e^{-j2t}e^{-3|t|} \overset{FT}{\longleftrightarrow} \frac{6}{9+(\omega+2)^2}$$

$$\therefore \quad x(t) \overset{FT}{\longleftrightarrow} \frac{1}{2j}\left[\frac{6}{9+(\omega-2)^2} - \frac{6}{9+(\omega+2)^2}\right]$$
$$= \frac{3}{j}\left[\frac{1}{9+(\omega-2)^2} - \frac{1}{9+(\omega+2)^2}\right]$$

Example 3.92 Find the Fourier transform of,

$$x(t) = \sum_{k=0}^{\infty} \alpha^k \delta(t - kT) \qquad ; |\alpha| < 1$$

Solution. We have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$
$$= \int_{-\infty}^{\infty}\sum_{k=0}^{\infty} \alpha^k \delta(t-kT)e^{-j\omega t}\,dt$$

Changing the order of integration and summation,

$$X(j\omega) = \sum_{k=0}^{\infty} \alpha^k \int_{-\infty}^{\infty} e^{-j\omega t} \delta(t - kT)\, dt$$

Using sifting property of unit impulse function we get,

$$X(j\omega) = \sum_{k=0}^{\infty} \alpha^k e^{-j\omega t}\Big|_{t=kT}$$

$$= \sum_{k=0}^{\infty} \alpha^k e^{-j\omega kT}$$

$$= \sum_{k=0}^{\infty} (\alpha e^{-j\omega T})^k$$

$$\therefore \quad X(j\omega) = \frac{1}{1 - \alpha e^{-j\omega T}} \qquad \left[\because \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \quad ; |\alpha| < 1\right]$$

Example 3.93 Find the Fourier transform of the following signal using appropriate properties.

$$x(t) = \sin(\pi t) e^{-2t} u(t)$$

Solution. Given :

$$x(t) = \sin(\pi t) e^{-2t} u(t)$$

$$= \left[\frac{e^{j\pi t} - e^{-j\pi t}}{2j}\right] e^{-2t} u(t)$$

$$= \frac{e^{-2t} \cdot e^{j\pi t} u(t)}{2j} - \frac{e^{-2t} \cdot e^{-j\pi t} u(t)}{2j}$$

We know that,

$$e^{-2t} u(t) \overset{FT}{\longleftrightarrow} \frac{1}{2 + j\omega}$$

Using frequency shifting property we get,

$$e^{j\pi t} e^{-2t} u(t) \overset{FT}{\longleftrightarrow} \frac{1}{2 + j(\omega - \pi)}$$

Using linearity property we get,

$$\frac{1}{2j}e^{j\pi t}e^{-2t}u(t) \xleftrightarrow{FT} \frac{1}{2j} \cdot \frac{1}{2+j(\omega-\pi)}$$

Similarly,

$$\frac{1}{2j}e^{-j\pi t}e^{-2t}u(t) \xleftrightarrow{FT} \frac{1}{2j} \cdot \frac{1}{2+j(\omega+\pi)}$$

$$\therefore \quad x(t) = \sin(\pi t)e^{-2t}u(t) \xleftrightarrow{FT} \frac{1}{j2}\left[\frac{1}{2+j(\omega-\pi)} - \frac{1}{2+j(\omega+\pi)}\right]$$

$$\therefore \quad X(j\omega) = \frac{1}{j2}\left[\frac{1}{2+j(\omega-\pi)} - \frac{1}{2+j(\omega+\pi)}\right]$$

Example 3.94 Prove that,

$$\text{If} \quad x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\text{then} \quad \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{X(j\omega)}{j\omega} + \pi X(j0)\delta(\omega)$$

Proof. We know that,

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau \quad \text{(E3.}$$

Take $x_1(t) = x(t)$ and $x_2(t) = u(t)$.
Substituting these in eqn. (E3.94.1) we get,

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau \quad \text{(E3.}$$

But,

$$u(t-\tau) = 1 \quad ; t-\tau \geq 0 \text{ i.e., } \tau \leq t$$
$$= 0 \quad ; t-\tau < 0 \text{ i.e., } \tau > t$$

$\therefore$ eqn. (E3.94.2) becomes,

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)u(t-\tau)d\tau + \int_{t}^{-\infty} \cancelto{0}{x(\tau)u(t-\tau)}\, d\tau$$

$$\therefore \quad x(t) * u(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

i.e., $$\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t)$$

Taking Fourier transform on both sides we get,

$$F\left[\int_{-\infty}^{t} x(\tau)d\tau\right] = F\,[x(t) * u(t)]$$

Using convolution property we get,

$$F\left[\int_{-\infty}^{t} x(\tau)d\tau\right] = X(j\omega)\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] \qquad \left\{\because \; F[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}\right\}$$

$$= \frac{X(j\omega)}{j\omega} + \pi X(j\omega)\delta(\omega)$$

$$F\left[\int_{-\infty}^{t} x(\tau)d\tau\right] = \frac{X(j\omega)}{j\omega} + \pi X(j0)\delta(\omega) \qquad [\because \text{ sampling property}]$$

$$\therefore \quad \int_{-\infty}^{t} x(\tau)d\tau \overset{FT}{\longleftrightarrow} \frac{X(j\omega)}{j\omega} + \pi X(j0)\delta(\omega)$$

$\therefore$ Using duality property we have,

$$X(jt) = \frac{2}{t^2+1} \overset{FT}{\longleftrightarrow} 2\pi x(-\omega) = 2\pi e^{-|-\omega|} = 2\pi e^{-|\omega|}$$

$$\therefore \quad \frac{2}{t^2+1} \overset{FT}{\longleftrightarrow} 2\pi e^{-|\omega|}$$

Example 3.96 Find the Fourier transform of the signal $x(t)$ using appropriate properties.

$$x(t) = \frac{d}{dt}\left[te^{-2t}\sin(t)u(t)\right]$$

Solution. Given :

$$x(t) = \frac{d}{dt}[te^{-2t}\sin(t)u(t)]$$

$$= \frac{d}{dt}\left[te^{-2t}\left(\frac{e^{jt}-e^{-jt}}{2j}\right)u(t)\right]$$

$$= \frac{1}{2j}\frac{d}{dt}\left[te^{jt}e^{-2t}u(t) - te^{-jt}e^{-2t}u(t)\right]$$

We have,

$$e^{-2t}u(t) \overset{FT}{\longleftrightarrow} \frac{1}{2+j\omega}$$

Using frequency differentiation property we get,

$$te^{-2t}u(t) \overset{FT}{\longleftrightarrow} j\frac{d}{d\omega}\left[\frac{1}{2+j\omega}\right]$$

$$= \frac{1}{(2+j\omega)^2}$$

Using frequency shift property we get,

$$te^{jt}e^{-2t}u(t) \overset{FT}{\longleftrightarrow} \frac{1}{[2+j(\omega-1)]^2}$$

Similarly,

$$te^{-jt}e^{-2t}u(t) \overset{FT}{\longleftrightarrow} \frac{1}{[2+j(\omega+1)]^2}$$

$$\therefore \quad \left[te^{jt}e^{-2t}u(t) - te^{-jt}e^{-2t}u(t)\right] \overset{FT}{\longleftrightarrow} \left[\frac{1}{[2+j(\omega-1)]^2} - \frac{1}{[2+j(\omega+1)]^2}\right]$$

Using time differentiation property we get,

$$\frac{d}{dt}\left[te^{-2t}\sin(t)u(t)\right] \xleftrightarrow{FT} j\omega\cdot\frac{1}{2j}\left[\frac{1}{[2+j(\omega-1)]^2}+\frac{1}{[2+j(\omega+1)]^2}\right]$$

$$= \frac{\omega}{2}\left[\frac{1}{[2+j(\omega-1)]^2}-\frac{1}{[2+j(\omega+1)]^2}\right]$$

Example 3.97 Find the FT of the signal given by,

$$x(t) = \int_{-\infty}^{t}\frac{\sin(\pi\tau)}{\pi\tau}d\tau$$

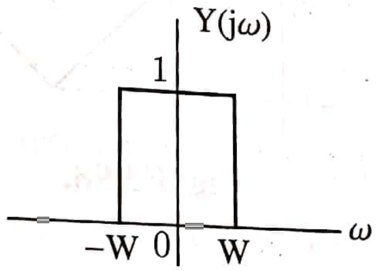
Solution. From integration property we have,

$$\text{If} \quad x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\text{then} \quad \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{X(j\omega)}{j\omega} + \pi X(j0)\delta(\omega) \qquad \text{(E3.97.1)}$$

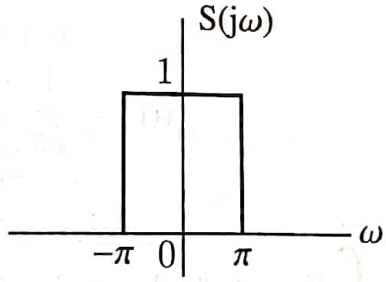
We know that,

$$y(t) = \frac{\sin(Wt)}{\pi t} \xleftrightarrow{FT}$$



(Refer to example 3.76)

$$\therefore \quad s(t) = \frac{\sin(\pi t)}{\pi t} \xleftrightarrow{FT}$$



Using eqn. (E3.97.1) we get,

$$x(t) = \int_{-\infty}^{t} s(\tau)d\tau = \int_{-\infty}^{t}\frac{\sin(\pi\tau)}{\pi\tau}d\tau \xleftrightarrow{FT} X(j\omega) = \pi \quad ; \ \omega = 0$$

$$= \frac{1}{j\omega} \quad ; \ -\pi < \omega < \pi$$

$$= \quad ; \ \text{otherwise}$$

Example 3.98 For the signal $x(t)$ shown in Fig. E3.98 below, evaluate the following quantities without explicitly computing $X(j\omega)$.

(a) $\int_{-\infty}^{\infty} X(j\omega)d\omega$

(b) $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$

(c) $\int_{-\infty}^{\infty} X(j\omega)e^{j2\omega}d\omega$

(d) $\measuredangle X(j\omega)$

(e) $X(j0)$

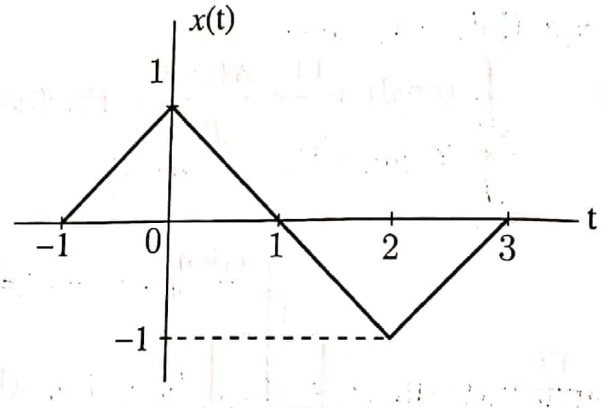


Fig. E3.98.

Solution. (a) We have,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$\therefore \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = 2\pi x(t) \qquad \text{(E3.98.1)}$$

Substituting t = 0 in eqn. (E3.98.1) we get,

$$\therefore \int_{-\infty}^{\infty} X(j\omega)d\omega = 2\pi x(0)$$

$$= 2\pi \cdot 1$$

$$= 2\pi$$

3.8 Applications of Fourier Representations

So far we discussed about the Fourier representations for different classes of signals, i.e., DTFS for periodic discrete-time signals, FS for periodic continuous-time signals, DTFT for non-periodic discrete-time signals and FT for non-periodic continuous-time signals. These Fourier representations are very much useful in the analysis of interaction between signals and systems. Also it helps in the numerical evaluation of signal properties or system behaviour. With the aid of these representations the analysis become easy when we deal (i) with a mixture of periodic and non-periodic signals, (ii) with a system that involve both continuous-time and discrete-time signals. For these reasons, we should establish the relationships between the representations of the different classes of signals.

In this section, we will obtain the relationship between frequency response descriptions of LTI system and the time-domain representations of systems like impulse response representations and differential/difference equation representations. We will establish the relationships among the representations of different classes of signals.

(b) From Parseval's theorem we have,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

$$\therefore \quad \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= 2\pi\left[\int_{-1}^{0}(t+1)^2\, dt + \int_{0}^{2}(1-t)^2\, dt + \int_{2}^{3}(t-3)^2\, dt\right]$$

$$= \frac{8\pi}{3}$$

(c) Substituting t = 2, in eqn. (E3.98.1) we get,

$$\int_{-\infty}^{\infty} X(j\omega)e^{j2\omega}\, d\omega = 2\pi x(2)$$

$$= 2\pi(-1) = -2\pi$$

(d) Here $x(t)$ is an odd signal $x_o(t)$ shifted to the right by 1 unit.

$$\text{i.e.,} \qquad x(t) = x_o(t-1)$$

Taking FT on both the sides we get,

$$X(j\omega) = X_o(j\omega)\cdot e^{-j\omega} \tag{E3.98.2}$$

We know that argument of an odd signal is $\frac{\pi}{2}$.

$\therefore$ eqn. (E3.98.2) can be written as,

$$X(j\omega) = |X_o(j\omega)|e^{j\frac{\pi}{2}}\cdot e^{-j\omega}$$

$$= |X_o(j\omega)|e^{j\left(\frac{\pi}{2}-\omega\right)}$$

$$\therefore \quad \measuredangle X(j\omega) = \frac{\pi}{2} - \omega$$

(e) We have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$

Substituting $\omega = 0$ we get,

$$X(j0) = \int_{-\infty}^{\infty} x(t)\, dt$$

$$= 0 \qquad [\because\ x(t) \text{ is shifted odd signal}]$$

3.8 Applications of Fourier Representations

So far we discussed about the Fourier representations for different classes of signals, i.e., DTFS for periodic discrete-time signals, FS for periodic continuous-time signals, DTFT for non-periodic discrete-time signals and FT for non-periodic continuous-time signals. These Fourier representations are very much useful in the analysis of interaction between signals and systems. Also it helps in the numerical evaluation of signal properties or system behaviour. With the aid of these representations the analysis become easy when we deal (i) with a mixture of periodic and non-periodic signals, (ii) with a system that involve both continuous-time and discrete-time signals. For these reasons, we should establish the relationships between the representations of the different classes of signals.

In this section, we will obtain the relationship between frequency response descriptions of LTI system and the time-domain representations of systems like impulse response representations and differential/difference equation representations. We will establish the relationships among the representations of different classes of signals.

3.9 Frequency Response of LTI Systems

In this section, we will discuss the relationships between the time-domain representations and frequency domain representations of LTI systems. Analysing an LTI system in frequency domain is easy due to many factors. One among these is the convolution in the time-domain transforms to multiplication in the frequency domain, i.e., the output of a system is obtained simply by multiplying the Fourier representation of the input with the system frequency response. The frequency response is the amplitude and phase change the system imparts to a complex sinusoid.

3.9.1 Impulse Response

In this section, we will discuss the relationship between the frequency response and the impulse response of an LTI system. The impulse response and frequency response of a continuous-time and discrete-time LTI systems forms a FT and DTFT pairs respectively.

i.e., $h(t) \xleftrightarrow{\text{FT}} H(j\omega)$

and $h(n) \xleftrightarrow{\text{DTFT}} H(e^{j\Omega})$

For a continuous-time system having impulse response $h(t)$ and input $x(t)$, the output $y(t)$ is given by,

$$y(t) = x(t) * h(t)$$

Taking FT on both the sides (using convolution property) we get,

$$Y(j\omega) = X(j\omega)H(j\omega)$$

$$\therefore \quad H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \tag{3.43}$$

In eqn. (3.43), $H(j\omega)$ is known as *frequency response* of the system. $|H(j\omega)|$ and $\measuredangle H(j\omega)$ are called *magnitude* and *phase response* respectively.

For a discrete-time LTI system having impulse response $h(n)$ and input $x(n)$, the output $y(n)$ is given by,

$$y(n) = x(n) * h(n)$$

Taking DTFT on both the sides (using convolution property) we get,

$$Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega})$$

$$\therefore \quad H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} \tag{3.44}$$

In eqn. (3.44), $H(e^{j\Omega})$ is known as *frequency response* of the system. $|H(e^{j\Omega})|$ and $\measuredangle H(e^{j\Omega})$ are called *magnitude* and *phase response* respectively.

EXAMPLES

Example 3.99 Find the frequency response of a continuous-time LTI system represented by the impulse response,

$$h(t) = e^{-|t|}$$

Solution. The frequency response of a continuous-time LTI system is given by the FT of $h(t)$

$$\therefore \quad H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{0} e^{t}e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-t}e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{0} e^{(1-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(1+j\omega)t}\, dt$$

$$= \frac{1}{1-j\omega} + \frac{1}{1+j\omega}$$

$$H(j\omega) = \frac{2}{1+\omega^2}$$

Example 3.100 Obtain the frequency response of a continuous-time LTI system whose impulse response $h(t)$ is as shown in Fig. E3.100.

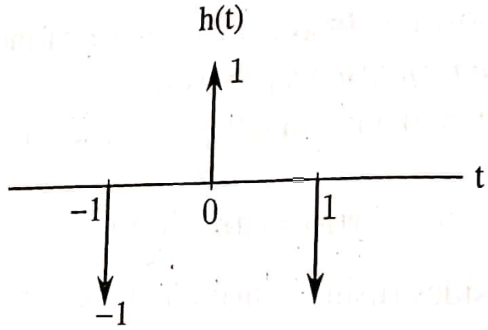


Fig. E3.100.

Solution. From Fig. E3.100 we have,

$$h(t) = -\delta(t+1) + \delta(t) - \delta(t-1)$$

Frequency response is given by,

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} [-\delta(t+1) + \delta(t) - \delta(t-1)]\, e^{-j\omega t}\, dt$$

$$= -e^{j\omega} + 1 - e^{-j\omega} \quad \text{[using sifting property]}$$

$$H(j\omega) = 1 - 2\cos(\omega)$$

Example 3.101 The impulse response of a continuous-time LTI system is given by,

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

Find the frequency response and plot the magnitude and phase response.

Solution. We have,

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt$$

$$= \frac{1}{RC}\int_{0}^{\infty} e^{-t/RC}e^{-j\omega t}\,dt$$

$$= \frac{1}{RC}\int_{0}^{\infty} e^{-\left(j\omega+\frac{1}{RC}\right)t}$$

$$H(j\omega) = \frac{1/RC}{j\omega + 1/RC}$$

∴ The magnitude response is,

$$|H(j\omega)| = \frac{1/RC}{\sqrt{\omega^2 + (1/RC)^2}}$$

The magnitude response $|H(j\omega)|$ is shown in Fig. E3.101.1

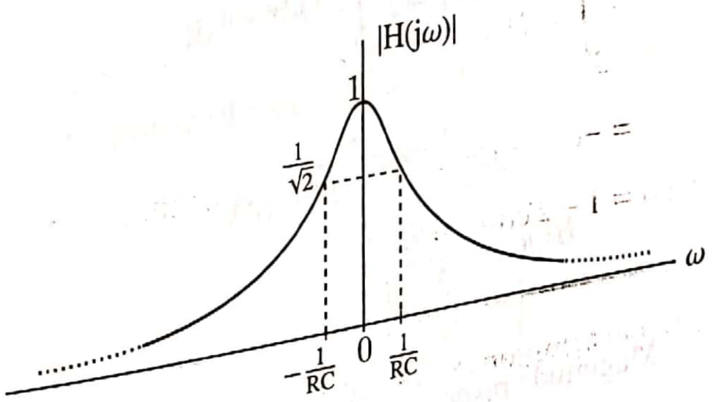


Fig. E3.101.1.

The phase response $\measuredangle H(j\omega) = -\tan^{-1}(\omega RC)$ and is shown in Fig. E3.101.2.

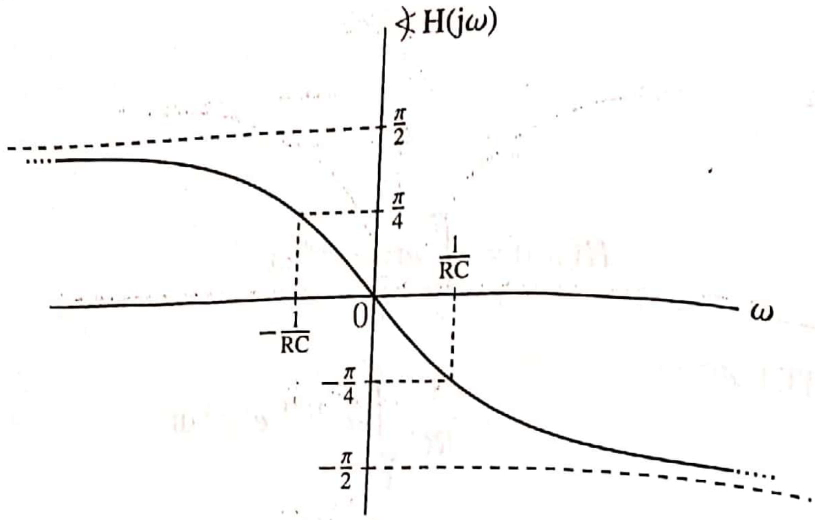


Fig. E3.101.2.

Example 3.102 Draw the frequency response of the system described by the impulse response,

$$h(t) = \delta(t) - 2e^{-2t}u(t)$$

Solution. We have,

$$\text{Frequency response} \quad H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\,dt - \int_{0}^{\infty} 2\,e^{-2t}e^{-j\omega t}\,dt$$

$$= 1 - 2\int_{0}^{\infty} e^{-(2+j\omega)t}\,dt$$

$$= 1 - \frac{2}{2+j\omega}$$

$$H(j\omega) = \frac{j\omega}{2+j\omega}$$

$$\therefore \text{ Magnitude response } |H(j\omega)| = \frac{\omega}{\sqrt{4+\omega^2}}$$

$$\text{\& Phase response } \measuredangle H(j\omega) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega}{2}\right)$$

The magnitude and phase spectra are shown in Fig. E3.101.1 and Fig. E3.102.2 respectively.

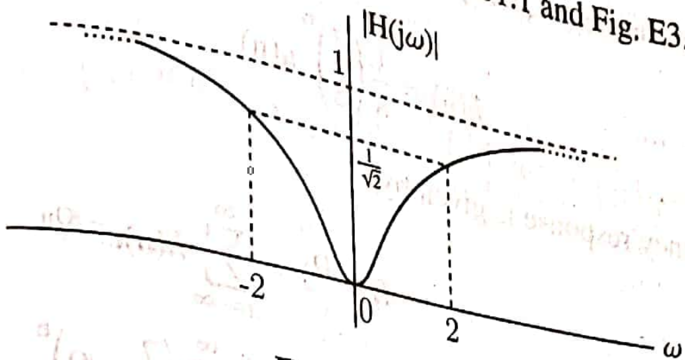


Fig. E3.102.1.

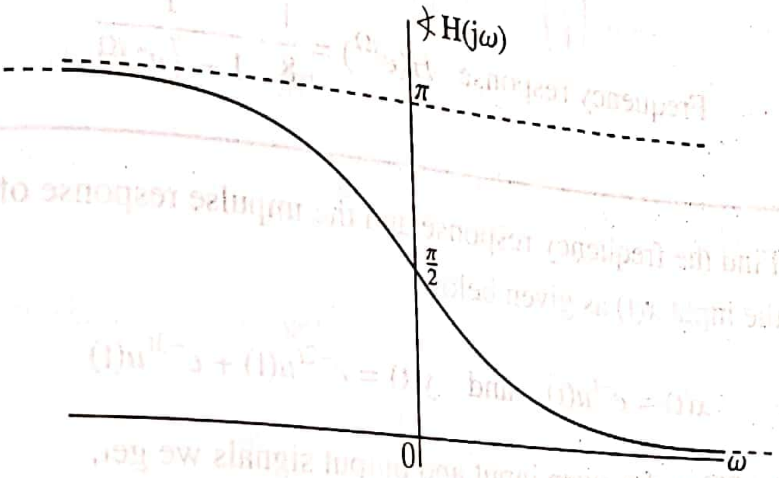


Fig. E3.102.2.

Example 3.103. Obtain the frequency response of a discrete-time LTI system represented by the impulse response,

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

Solution. The frequency response of a discrete-time LTI system is given by the DTFT of $h(n)$.

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}e^{-j\Omega}\right)^n$$

Frequency response $H(e^{j\Omega}) = \dfrac{1}{1-\frac{1}{2}e^{-j\Omega}}$

Example 3.104 Obtain the frequency response of a discrete-time LTI system with impulse response,

$$h(n) = \frac{1}{8}\left(\frac{7}{8}\right)^n u(n)$$

Solution. The frequency response is given by,

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\Omega n}$$

$$= \frac{1}{8}\sum_{n=0}^{\infty}\left(\frac{7}{8}e^{-j\Omega}\right)^n$$

Frequency response $H(e^{j\Omega}) = \dfrac{1}{8}\cdot\dfrac{1}{1-\frac{7}{8}e^{-j\Omega}}$

Example 3.105 Find the frequency response and the impulse response of the system having the output $y(t)$ for the input $x(t)$ as given below.

$$x(t) = e^{-t}u(t) \quad \text{and} \quad y(t) = e^{-2t}u(t) + e^{-3t}u(t)$$

Solution. Taking the FT of the given input and output signals we get,

$$X(j\omega) = \frac{1}{1+j\omega}$$

and $$Y(j\omega) = \frac{1}{2+j\omega} + \frac{1}{3+j\omega}$$

$\therefore$ The frequency response $H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)} = \dfrac{1+j\omega}{2+j\omega} + \dfrac{1+j\omega}{3+j\omega}$

Using partial fraction expansion,

$$H(j\omega) = 2 - \frac{1}{2+j\omega} - \frac{2}{3+j\omega}$$

Taking inverse FT we get the impulse response,

$$h(t) = 2\delta(t) - e^{-2t}u(t) - 2e^{-3t}u(t)$$

$$\therefore \quad h(t) = 2\delta(t) - e^{-2t}\left[1 + 2e^{-t}\right]u(t)$$

Example 3.106. Obtain the frequency response and the impulse response of the system having the output $y(n)$ for the input $x(n)$ as given below.

$$x(n) = \left(\frac{1}{2}\right)^n u(n) \quad ; \quad y(n) = \frac{1}{4}\left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{4}\right)^n u(n)$$

Solution. Given :

$$x(n) = \left(\frac{1}{2}\right)^n u(n) \tag{E3.106.1}$$

and
$$y(n) = \frac{1}{4}\left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{4}\right)^n u(n) \tag{E3.106.2}$$

Taking DTFT of eqn. (E3.106.1) we get,

$$X(e^{j\Omega}) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}}$$

Taking DTFT of eqn. (E3.106.2) we get,

$$Y(e^{j\Omega}) = \frac{\frac{1}{4}}{1-\frac{1}{2}e^{-j\Omega}} + \frac{1}{1-\frac{1}{4}e^{-j\Omega}}$$

∴ The frequency response,

$$H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})}$$

$$= \frac{1}{4} + \frac{1-\frac{1}{2}e^{-j\Omega}}{1-\frac{1}{4}e^{-j\Omega}}$$

$$H(e^{j\Omega}) = \frac{1}{4} + \frac{1}{1-\frac{1}{4}e^{-j\Omega}} - \frac{\frac{1}{2}e^{-j\Omega}}{1-\frac{1}{4}e^{-j\Omega}}$$

Taking inverse DTFT we get,

Impulse response $h(n) = \frac{1}{4}\delta(n) + \left(\frac{1}{4}\right)^n u(n) - \frac{1}{2}\left(\frac{1}{4}\right)^{n-1} u(n-1).$

3.10 Solution of Differential / Difference Equations

The frequency response is the amplitude and the phase change the system imparts to a complex sinusoid. This indicates that the frequency response is for the system's steady state response to a sinusoidal. In contrast to differential/difference equation descriptions for a system, the frequency response description cannot represent initial conditions i.e., it can only describe a system in a steady state condition.

The differential equation representation for a continuous-time LTI system is of the form,

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Taking FT on both the sides using time differentiation property we get,

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$\therefore \quad H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum\limits_{k=0}^{M} b_k (j\omega)^k}{\sum\limits_{k=0}^{N} a_k (j\omega)^k}$$

where $H(j\omega)$ is the *frequency response* of the system. Thus the frequency response of a continuous-time LTI system described by a linear constant co-efficient differential equation is a ratio of two polynomials in '$j\omega$'. Alternatively, if we have frequency response of a continuous-time LTI system as a ratio of polynomials in '$j\omega$', we can determine the corresponding differential equation.

Similarly, the difference equation representation for a discrete-time LTI system is of the form,

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

Taking DTFT on both the sides using time-shift property we get,

$$\sum_{k=0}^{N} a_k (e^{-j\Omega})^k Y(e^{j\Omega}) = \sum_{k=0}^{M} b_k (e^{-j\Omega})^k X(e^{j\Omega})$$

$$\therefore \quad H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = \frac{\sum\limits_{k=0}^{M} b_k (e^{-j\Omega})^k}{\sum\limits_{k=0}^{N} a_k (e^{-j\Omega})^k}$$

Thus, in a discrete-time system, the frequency response is a ratio of polynomials in '$e^{-j\Omega}$'. Alternatively, if we have frequency response of a discrete-time LTI system as a ratio of polynomials in '$e^{-j\Omega}$', we can obtain the corresponding difference equation.

EXAMPLES

Example 3.107 Find the frequency response and the impulse response of the system described by the differential equation,

$$\frac{dy(t)}{dt} + 8y(t) = x(t)$$

Solution. Given :
$$\frac{dy(t)}{dt} + 8y(t) = x(t)$$

Taking FT on both sides we get,

$$j\omega\, Y(j\omega) + 8Y(j\omega) = X(j\omega)$$

$$Y(j\omega)[8 + j\omega] = X(j\omega)$$

$\therefore$ The frequency response $H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)} = \dfrac{1}{8 + j\omega}$.

Taking inverse FT, we get the impulse response,

$$h(t) = e^{-8t}u(t)$$

Example 3.108 Repeat Example (3.107) for,

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = -\frac{dx(t)}{dt}$$

Solution. Given :
$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = -\frac{dx(t)}{dt}.$$

Taking FT on both the sides we get,

$$(j\omega)^2 Y(j\omega) + 5(j\omega)Y(j\omega) + 6Y(j\omega) = -j\omega\, X(j\omega)$$

$\therefore$ The frequency response,

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{-j\omega}{(j\omega)^2 + 5(j\omega) + 6}$$

$$-j\omega$$

By partial fraction expansion we get,

$$H(j\omega) = \frac{2}{2 + j\omega} - \frac{3}{3 + j\omega}$$

Taking inverse FT we get the impulse response,

$$h(t) = 2e^{-2t}u(t) - 3e^{-3t}u(t)$$

$$\therefore \quad h(t) = (2e^{-2t} - 3e^{-3t})u(t)$$

Example 3.109 Obtain the frequency response and the impulse response of the system described by the difference equation,

$$y(n) + \frac{1}{2}y(n-1) = x(n) - 2x(n-1)$$

Solution. Given : $\quad y(n) + \frac{1}{2}y(n-1) = x(n) - 2x(n-1).$

Taking DTFT on both the sides we get,

$$Y(e^{j\Omega}) + \frac{1}{2}e^{-j\Omega}Y(e^{j\Omega}) = X(e^{j\Omega}) - 2e^{-j\Omega}X(e^{j\Omega})$$

$\therefore$ The frequency response,

$$H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = \frac{1 - 2e^{-j\Omega}}{1 + \frac{1}{2}e^{-j\Omega}}$$

$$H(e^{j\Omega}) = \frac{1}{1 + \frac{1}{2}e^{-j\Omega}} - \frac{2e^{-j\Omega}}{1 + \frac{1}{2}e^{-j\Omega}}$$

Taking inverse DTFT we get,

$$\text{Impulse response } h(n) = \left(\frac{-1}{2}\right)^n u(n) - 2\left(\frac{-1}{2}\right)^{n-1} u(n-1)$$

Example 3.110 Find the differential equation description for the system having impulse response,

$$h(t) = \frac{1}{a}e^{-t/a}u(t)$$

Solution. Given :

$$h(t) = \frac{1}{a}e^{-t/a}u(t)$$

Taking FT on the both the sides, we get the frequency response,

$$H(j\omega) = \frac{1}{a}\cdot\frac{1}{(1/a + j\omega)}$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{1 + ja\omega}$$

$$\therefore \quad aj\omega\, Y(j\omega) + Y(j\omega) = X(j\omega)$$

Taking inverse FT we get,

$$a\frac{dy(t)}{dt} + y(t) = x(t)$$

Example 3.111 Obtain the difference equation description for the system having impulse response,

$$h(n) = \delta(n) + 2\left(\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^n u(n)$$

Solution. Given :

$$h(n) = \delta(n) + 2\left(\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^n u(n)$$

Taking DTFT on both the sides we get,

$$H(e^{j\Omega}) = 1 + \frac{2}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{1}{1 + \frac{1}{2}e^{-j\Omega}}$$

$$H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = \frac{4 + \frac{1}{2}e^{-j\Omega} - \frac{1}{4}e^{-j2\Omega}}{1 - \frac{1}{4}e^{-j2\Omega}}$$

$$\left[\frac{-1}{4}e^{-j2\Omega} + 1\right]Y(e^{j\Omega}) = \left[\frac{-1}{4}e^{-j2\Omega} + \frac{1}{2}e^{-j\Omega} + 4\right]X(e^{j\Omega})$$

Taking inverse DTFT we get,

$$\frac{-1}{4}y(n-2) + y(n) = \frac{-1}{4}x(n-2) + \frac{1}{2}x(n-1) + 4x(n)$$

Example 3.112 Find the differential equation that represents the system with the frequency response,

$$H(j\omega) = \frac{2 + 3j\omega - 3(j\omega)^2}{1 + 2j\omega}$$

Solution. Given :

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{2 + 3j\omega - 3(j\omega)^2}{1 + 2j\omega}$$

$$\therefore \quad Y(j\omega)[1 + 2j\omega] = X(j\omega)[2 + 3j\omega - 3(j\omega)^2]$$

Taking inverse FT (using time differentiation property)

$$y(t) + 2\frac{dy(t)}{dt} = 2x(t) + 3\frac{dx(t)}{dt} - 3\frac{d^2x(t)}{dt^2}$$

$$\therefore \quad \frac{dy(t)}{dt} + \frac{1}{2}y(t) = -\frac{3}{2}\frac{d^2x(t)}{dt^2} + \frac{3}{2}\frac{dx(t)}{dt} + x(t)$$

Example 3.113 Obtain the difference equation for the system with the frequency response,

$$H(e^{j\Omega}) = 1 + \frac{e^{-j\Omega}}{\left(1 + \frac{1}{2}e^{-j\Omega}\right)\left(1 + \frac{1}{4}e^{-j\Omega}\right)}$$

Solution. Given :

$$H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = 1 + \frac{e^{-j\Omega}}{\left(1 - \frac{1}{2}e^{-j\Omega}\right)\left(1 + \frac{1}{4}e^{-j\Omega}\right)}$$

$$= \frac{1 + \frac{3}{4}e^{-j\Omega} - \frac{1}{8}e^{-j2\Omega}}{1 - \frac{1}{4}e^{-j\Omega} - \frac{1}{8}e^{-j2\Omega}}$$

$$\therefore \quad Y(e^{j\Omega})\left[1 - \frac{1}{4}e^{-j\Omega} - \frac{1}{8}e^{-j2\Omega}\right] = X(e^{j\Omega})\left[1 + \frac{3}{4}e^{-j\Omega} - \frac{1}{8}e^{-j2\Omega}\right]$$

Taking inverse DTFT on both the sides, using time shift property,

$$y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = x(n) + \frac{3}{4}x(n-1) - \frac{1}{8}x(n-2)$$

3.11 Fourier Transform Representation for Periodic Signals

We discussed that FS and DTFS are the Fourier representations for periodic continuous-time and periodic discrete-time signals respectively. Even though the FT and DTFT does not converge for the periodic signals, it could be found by incorporating impulses in the representations. This concept can be used to analyse examples involving a mixture of periodic and non-periodic signals. In this section, we discuss the relationship between (i) FT and FS (ii) DTFT and DTFS representations.

3.11.1 Relationship between FT and FS :

We studied that a periodic continuous-time signal $x(t)$ with fundamental frequency ω_o can be expressed as,

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_o t} \tag{3.45}$$

where $X(k)$ are FS coefficients.

We know that,

$$\delta(t) \overset{FT}{\longleftrightarrow} 1$$

Using duality property we get,

$$1 \overset{FT}{\longleftrightarrow} 2\pi\,\delta(-\omega)$$

Since '1' is real and even signal we get,

$$1 \overset{FT}{\longleftrightarrow} 2\pi\,\delta(\omega)$$

Using frequency shift property we have,

$$e^{jk\omega_o t} \overset{FT}{\longleftrightarrow} 2\pi\,\delta(\omega - k\omega_o) \tag{3.46}$$

We have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \tag{3.47}$$

Substituting eqn. (3.45) in eqn. (3.47) we get,

$$X(j\omega) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_o t}e^{-j\omega t}\,dt$$

Changing the order of integration and summation,

$$X(j\omega) = \sum_{k=-\infty}^{\infty} X(k) \underbrace{\int_{-\infty}^{\infty} e^{jk\omega_o t} e^{-j\omega t}\, dt}_{\text{F.T. of } e^{jk\omega_o t}} \qquad (3.48)$$

Substituting eqn. (3.46) in eqn. (3.48) we get,

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X(k)\delta(\omega - k\omega_o)$$

Therefore, the FT of a periodic signal $x(t)$ is a series of impulses separated by the fundamental frequency ω_o and the strength of k^{th} impulse is $2\pi\, X(k)$.

For eg., consider that Fig. 3.3 shows the FS representation of a continuous-time signal $x(t)$ with fundamental frequency ω_o.

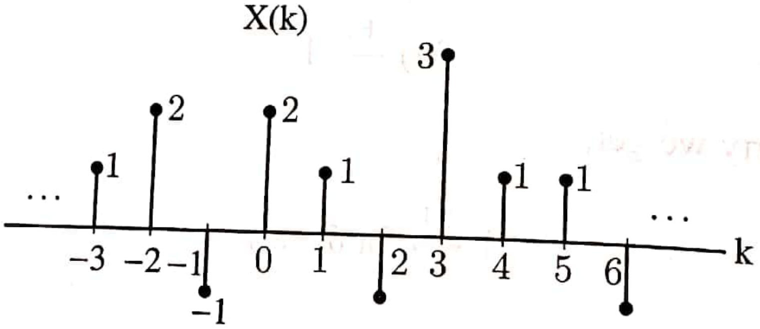


Fig. 3.3

Then the FT representation for the same signal $x(t)$ is shown in Fig. 3.4 below.

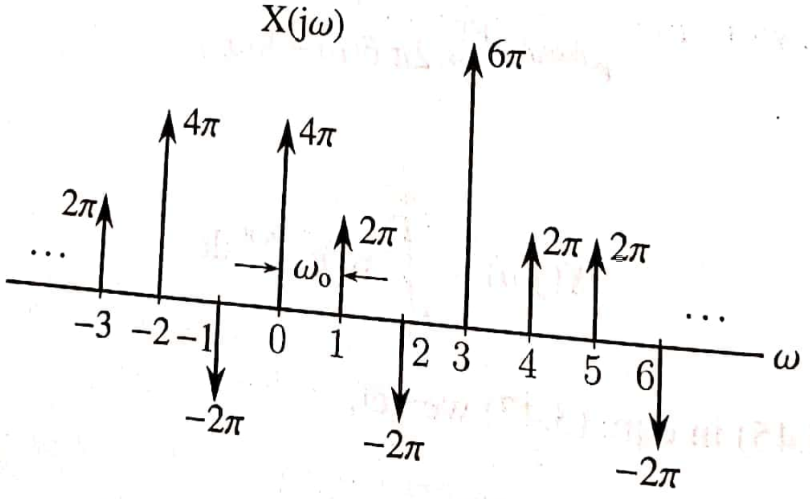


Fig. 3.4

EXAMPLES

Example 3.114 Find the FT representation for the periodic signal $x(t) = \cos(\omega_o t)$ and draw the spectrum.

Solution. Given :

$$x(t) = \cos(\omega_o t)$$

$$x(t) = \frac{1}{2}e^{j\omega_o t} + \frac{1}{2}e^{-j\omega_o t} \quad \text{(E3.114.1)}$$

We have,

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_o t} \quad \text{(E3.114.2)}$$

Comparing eqn. (E3.114.1) and eqn. (E3.114.2) we get,

$$X(1) = \frac{1}{2}$$

$$X(-1) = \frac{1}{2}$$

$$X(k) = 0 \quad ; k \neq \pm 1$$

The FT of $x(t)$ is given by,

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X(k)\delta(\omega - k\omega_o)$$

$$= 2\pi\left[X(1)\delta(\omega - \omega_o) + X(-1)\delta(\omega + \omega_o)\right]$$

$$X(j\omega) = \pi\,\delta(\omega - \omega_o) + \pi\,\delta(\omega + \omega_o)$$

The spectrum is shown in Fig. E3.114.1 below.

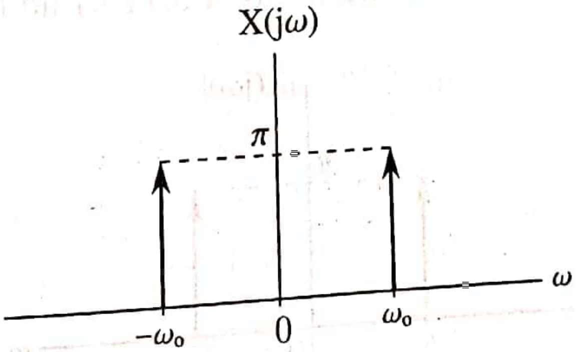


Fig. E3.114.1.

Example 3.115 Obtain the FT representation for the periodic signal $x(t)$ and draw the spectrum where $x(t) = \sin(\omega_o t)$

Solution. Given :

$$x(t) = \sin(\omega_o t)$$

$$x(t) = \frac{1}{2j}e^{j\omega_o t} - \frac{1}{2j}e^{-j\omega_o t} \tag{E3.115.1}$$

We have,

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_o t} \tag{E3.115.2}$$

Comparing eqn. (E3.115.1) and eqn. (E3.115.2) we get,

$$X(1) = \frac{1}{2j} = -j\frac{1}{2}$$

$$X(-1) = -\frac{1}{2j} = j\frac{1}{2}$$

$$X(k) = 0 \quad ; \text{k} \neq \pm 1$$

The FT of $x(t)$ is given by,

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X(k)\delta(\omega - k\omega_o)$$

$$= 2\pi\left[X(1)\delta(\omega - \omega_o) + X(-1)\delta(\omega + \omega_o)\right]$$

$$X(j\omega) = -j\pi\,\delta(\omega - \omega_o) + j\pi\,\delta(\omega + \omega_o)$$

The magnitude and phase spectra are shown in Fig. E3.115.1 and Fig. E3.115.2 respectively.

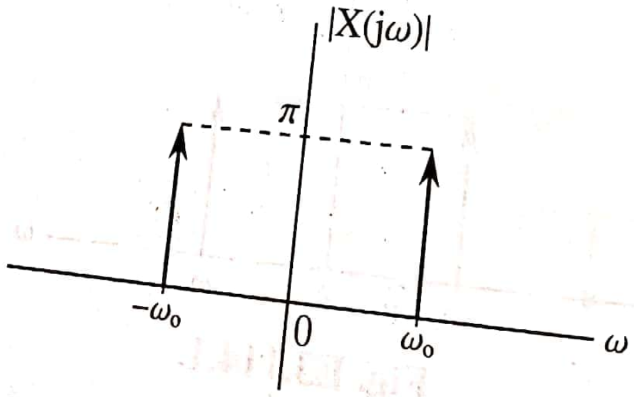


Fig. E3.115.1.

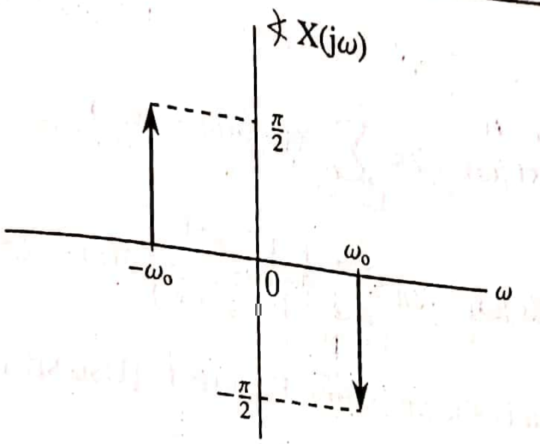


Fig. E3.115.2.

Example 3.116 For the signal $x(t)$ shown in Fig. E3.116; determine the FT and draw the magnitude spectrum.

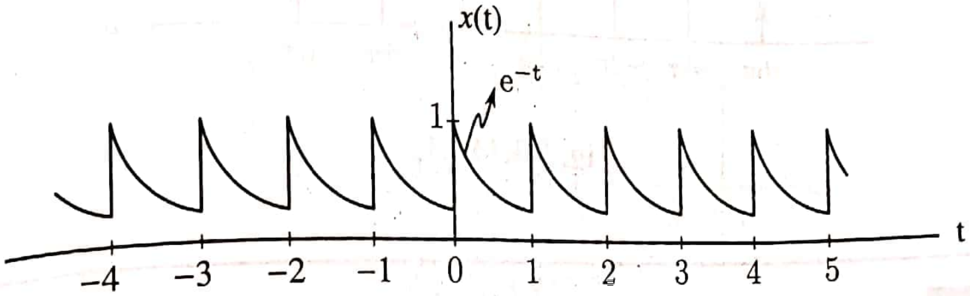


Fig. E3.116.

Solution. By observing Fig. E3.116 we have,

$$T = 1 \quad \text{and} \quad \omega_o = \frac{2\pi}{T} = 2\pi$$

We have,

$$X(k) = \frac{1}{T}\int_{\langle T \rangle} x(t)e^{-jk\omega_o t}\,dt$$

$$= \int_0^1 e^{-t}e^{-jk(2\pi)t}\,dt$$

$$= \int_0^1 e^{-(1+j2\pi k)t}\,dt$$

$$X(k) = \frac{1-e^{-1}}{1+j2\pi k}$$

The FT is,

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X(k)\delta(\omega - k\omega_o)$$

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} \left\{\frac{1-e^{-1}}{1+j2\pi k}\right\} \delta(\omega - 2\pi k)$$

The magnitude spectrum is shown in Fig. E3.116.1. [Use sifting property]

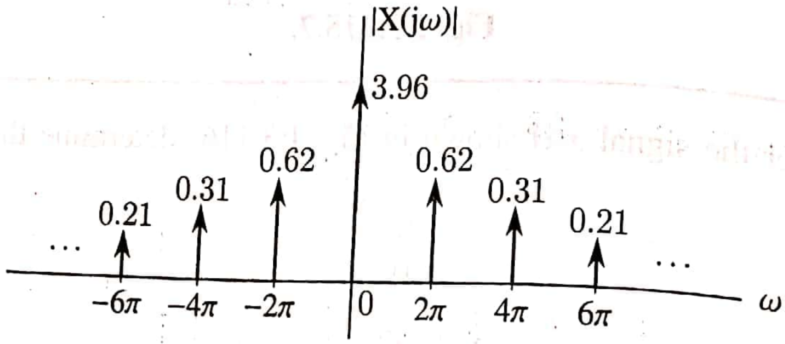


Fig. E3.116.1.

Example 3.117 Find the Fourier transform of the periodic impulse train,

$$\delta_{T_o}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_o)$$

and draw the spectrum.

Solution. The sketch of the given signal is shown in Fig. E3.117.1

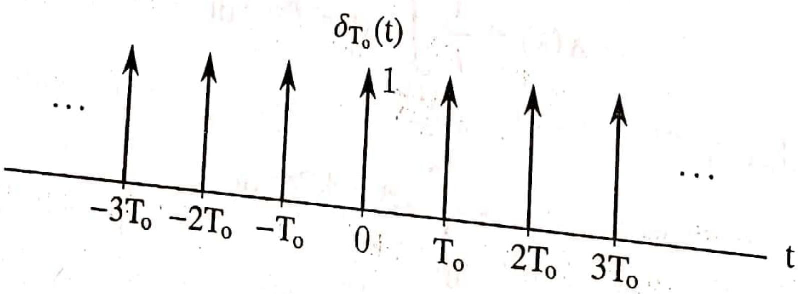


Fig. E3.117.1.

It is *Dirac comb.*

Referring to Example 3.27 we have,

$$X(k) = \frac{1}{T_o} \quad ; \quad \omega_o = \frac{2\pi}{T_o}$$

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X(k)\delta(\omega - k\omega_o)$$

$$= 2\pi \sum_{k=-\infty}^{\infty} \frac{1}{T_o}\delta(\omega - k\omega_o)$$

$$= \frac{2\pi}{T_o} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_o)$$

The spectrum is shown in Fig. E3.117.2 below.

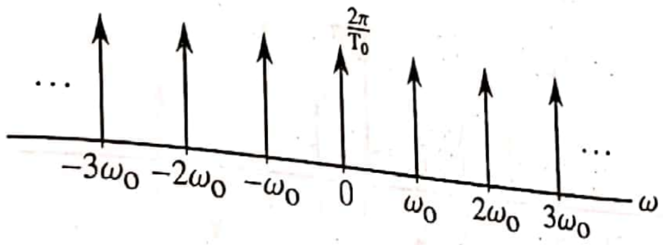


Fig. E3.117.2.

3.11.2 Relationship between DTFT and DTFS :

We studied that a periodic discrete-time signal $x(n)$ with fundamental frequency Ω_o can be expressed as,

$$x(n) = \sum_{k=\langle N\rangle} X(k)e^{jk\Omega_o n}$$

where $X(k)$ are DTFS coefficients.

Then the DTFT of this signal $x(n)$ is given by,

$$X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} X(k)\delta(\Omega - k\Omega_o) \qquad (3.49)$$

Therefore, the DTFT of a periodic signal $x(n)$ is a series of impulses separated by the fundamental frequency Ω_o and the strength of k^{th} impulse is $2\pi X(k)$.

For eg., consider that a discrete-time periodic sequence $x(n)$ with fundamental frequency Ω_o has DTFS (for one cycle $N = 10$) as shown in Fig. 3.5 below.

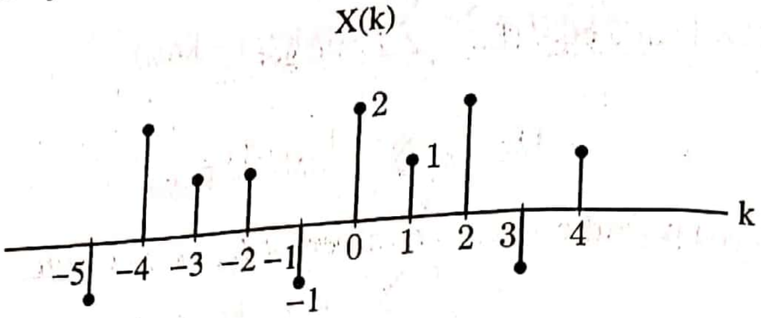


Fig. 3.5.

The DTFT representation of $x(n)$ is as shown in Fig. 3.6 below.

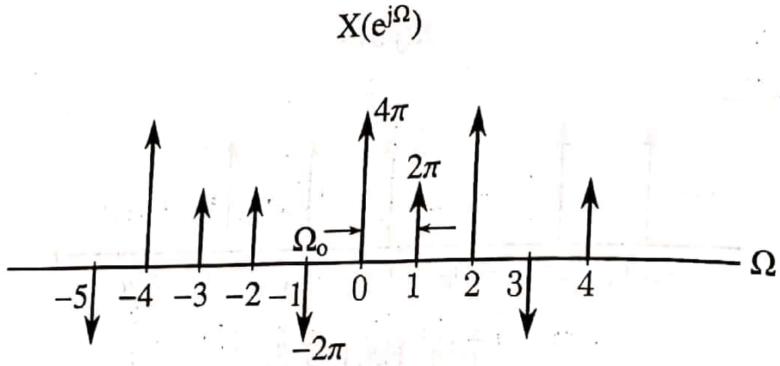


Fig. 3.6.

EXAMPLES

Example 3.118 Find the DTFT representation for the periodic signal,

$$x(n) = \cos\left(\tfrac{\pi}{3}n\right)$$

Also draw the spectrum.

Solution. Given :

$$x(n) = \cos\left(\tfrac{\pi}{3}n\right)$$

$$\therefore \quad \Omega_o = \frac{\pi}{3} = \frac{2\pi}{6}$$

$\therefore$ Fundamental period $N = 6$

$$x(n) = \cos\left(\frac{\pi}{3}n\right)$$

$$x(n) = \frac{1}{2}e^{j\frac{\pi}{3}n} + \frac{1}{2}e^{-j\frac{\pi}{3}n} \qquad \text{(E3.118.1)}$$

Also we have,

$$x(n) = \sum_{k=\langle N \rangle} X(k) e^{jk\Omega_o n} \qquad \text{(E3.118.2)}$$

Comparing eqn. (E3.118.1) and eqn. (E3.118.2) we get

$$X(1) = \frac{1}{2} \quad \text{and} \quad X(-1) = \frac{1}{2}$$

Since DTFS X(k) forms a periodic sequence of period N, we can write,

$$\ldots = X(-11) = X(-5) = X(1) = X(7) = X(13) = \ldots = \frac{1}{2}$$

$$\ldots = X(-7) = X(-1) = X(5) = X(11) = X(17) = \ldots = \frac{1}{2}$$

and other X(k)s are zero.

$\therefore$ The DTFT of x(n) is,

$$X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} X(k)\delta(\Omega - k\Omega_o)$$

$$\therefore \quad X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} X(k)\delta\left(\Omega - k\frac{\pi}{3}\right)$$

The spectrum is shown in Fig. E3.118.1 below.

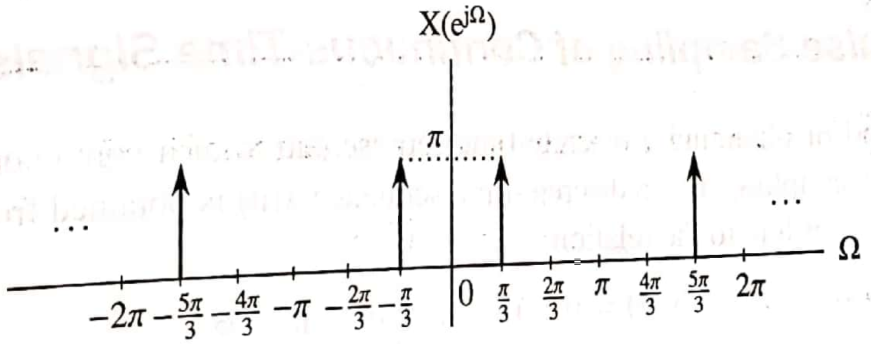


Fig. E3.118.1.

Example 3.119 Find the DTFT of the periodic signal,

$$x(n) = \sin\left(\frac{4\pi}{21}n\right) + \cos\left(\frac{10\pi}{21}n\right) + 1$$

Sketch the magnitude spectrum.

Solution. Referring to Example 3.2 we have,

$$X(0) = 1$$

$$X(-2) = -\frac{1}{2j} \quad ; \; X(2) = \frac{1}{2j}$$

$$X(-5) = \frac{1}{2} \quad ; \; X(5) = \frac{1}{2}$$

$$\& \qquad \Omega_o = \frac{2\pi}{21} \quad \therefore \; N = 21$$

We have,

$$X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} X(k)\delta\left(\Omega - \left(\frac{2\pi}{21}\right)k\right)$$

The magnitude spectrum (for one cycle) is shown in Fig. E3.119.1 below.

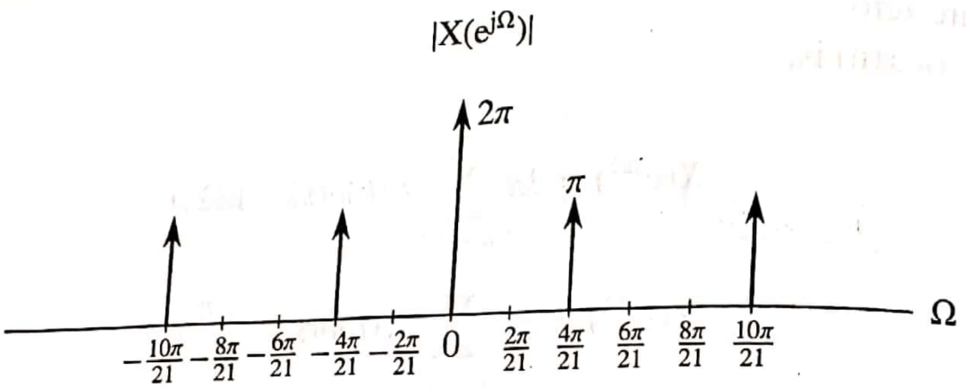


Fig. E3.119.1.

3.12 Impulse Sampling of Continuous-Time Signals

A typical method of obtaining a discrete-time representation of a continuous-time signal is through periodic sampling. i.e., a discrete-time sequence $x(n)$ is obtained from a continuous-time signal $x(t)$ according to the relation,

$$x_s(t) = x(n\tau) \qquad ; \; -\infty < n < \infty$$

where 'τ' is the sampling period and $\omega_s = \dfrac{2\pi}{\tau}$ is the sampling frequency in rad/sec. The pictorial representation of sampling is shown in Fig. 3.7 below.

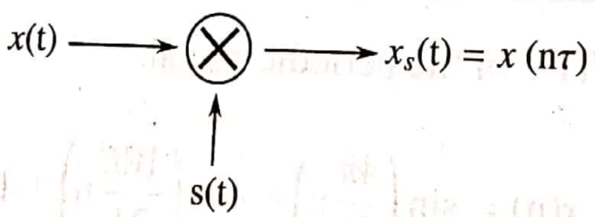


Fig. 3.7.

where $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\tau)$ = periodic impulse train with period τ.

From Fig. 3.7 we have,

$$x_s(t) = x(t)s(t)$$

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - n\tau) \tag{3.50}$$

By sampling property of the impulse function we get,

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(n\tau)\delta(t - n\tau) \tag{3.51}$$

The FT of a periodic impulse train $s(t)$ is,

$$S(j\omega) = \frac{2\pi}{\tau} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \tag{3.52}$$

Taking the FT of eqn. (3.50) using modulation property we get,

$$X_s(j\omega) = \frac{1}{2\pi} X(j\omega) * S(j\omega) \tag{3.53}$$

Substituting eqn. (3.52) in eqn. (3.53) we get,

$$X_s(j\omega) = \frac{1}{2\pi} X(j\omega) * \left\{ \frac{2\pi}{\tau} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \right\}$$

$$\therefore \quad X_s(j\omega) = \frac{1}{\tau} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \tag{3.54}$$

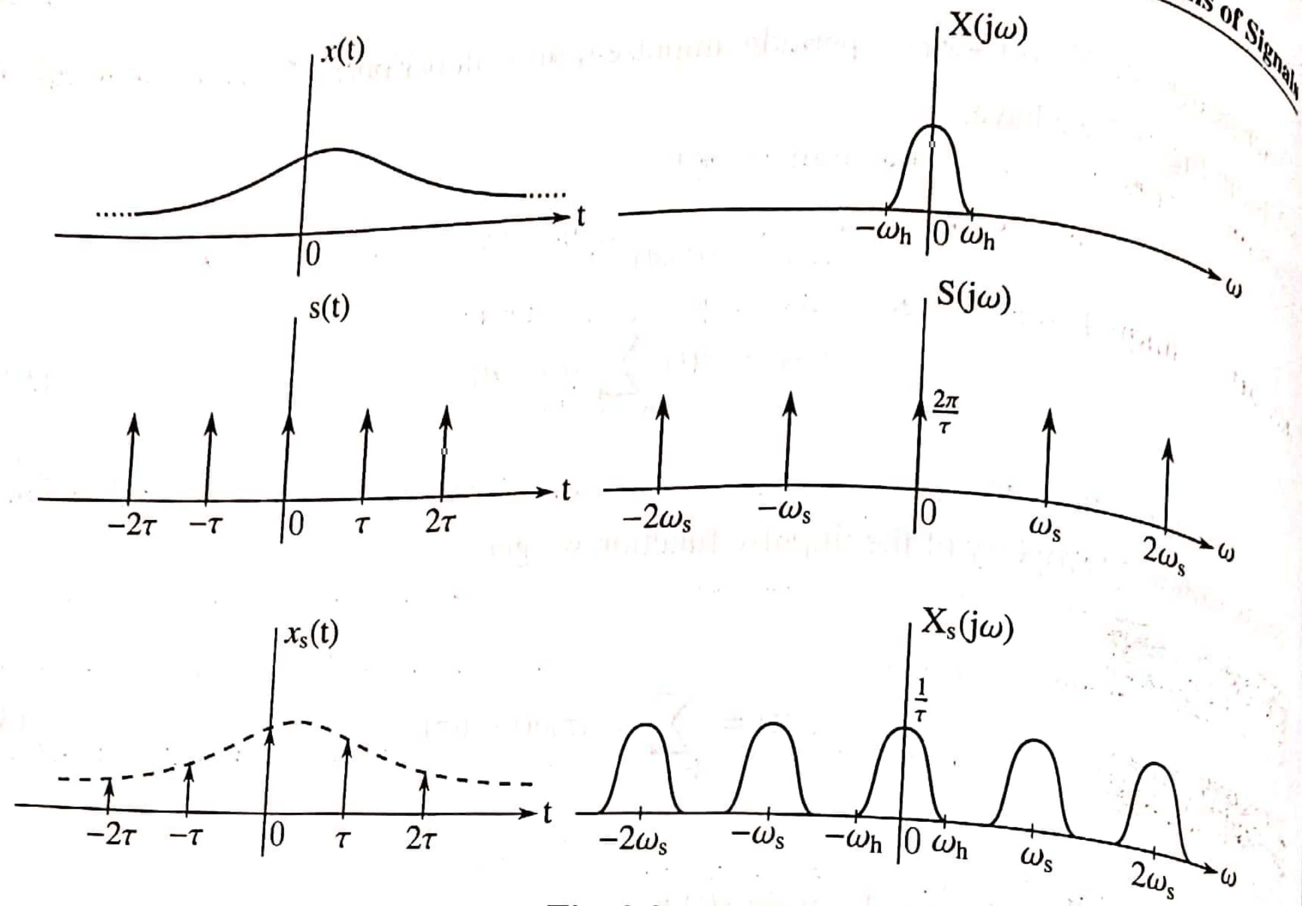


Fig. 3.8.

Eqn. (3.53) provides the relationship between the Fourier transforms of $x_s(t)$ and $x(t)$. We have the FT of $x_s(t)$ is periodic repetition of FT of $x(t)$. The entire explanation is represented graphically as shown in Fig. 3.8.

From Fig. 3.8 it is evident that when,

$$\omega_s - \omega_h > \omega_h$$

$$\text{i.e.,} \qquad \omega_s > 2\omega_h$$

the replicas of $X_s(j\omega)$ do not overlap each other. Consequently, $x(t)$ can be recovered from $x_s(t)$ by passing it through a low pass filter.

If $\omega_s \not> 2\omega_h$, the replicas of $X_s(j\omega)$ overlap each other as shown in Fig. 3.9 below.

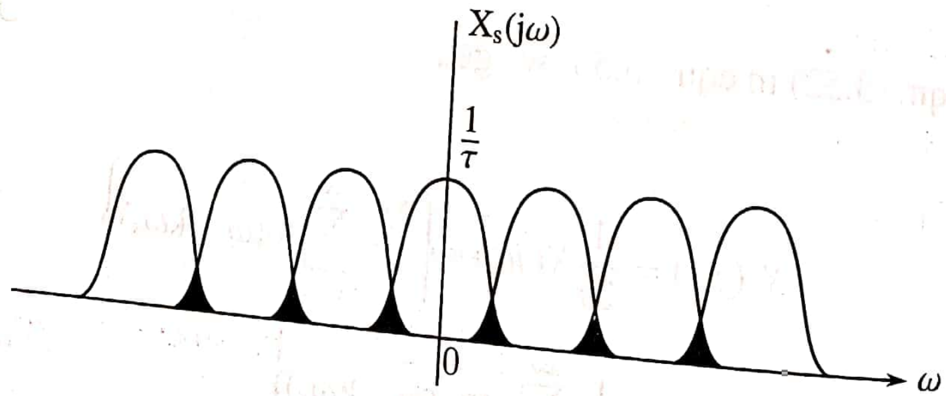


Fig. 3.9.

Therefore, the signal $x(t)$ cannot be recovered by low pass filtering. This type of distortion is know *aliasing distortion* or simply *aliasing*.

Sampling theorem **:** If $x(t)$ is bandlimited with,

$$X(j\omega) = 0 \quad \text{for} \quad |\omega| \geq \omega_h \quad \text{(i.e., highest frequency of } x(t) = \omega_h)$$

then $x(t)$ is uniquely determined by its samples taken at rate,

$$\omega_s = \frac{2\pi}{\tau} \geq 2\omega_h$$

The minimum sampling frequency $2\omega_h$ is called *Nyquist rate* and the actual sampling frequency ω_S is known as *Nyquist frequency.*

EXAMPLES

Example 3.120 A signal $x(t) = \cos(5\pi t) + 0.5\cos(10\pi t)$ is ideally sampled with sampling period τ_s.

Find the minimum sampling frequency (i.e., Nyquist rate)

Solution. Given :

$$x(t) = \cos(5\pi t) + 0.5\cos(10\pi t)$$

The given signal consists 2 frequency components.

$$\omega_1 = 5\pi \text{ rad/sec.} \qquad ; f_1 = 2.5 \text{ Hz}$$

$$\omega_2 = 10\pi \text{ rad/sec.} = \omega_h \qquad ; f_2 = 5 \text{ Hz} = f_h$$

$\therefore$ The highest frequency is $\omega_h = \omega_2 = 10\pi$.

$\therefore$ Minimum sampling frequency $= 2\omega_h = 20\pi$ rad/sec *or* $2f_h = 10$ Hz.

Example 3.121 Specify the Nyquist rate for each of the following signals.

(i) $x_1(t) = \text{sinc}(200t)$

(ii) $x_2(t) = \text{sinc}^2(200t)$

Solution. (i) Given :

$$x_1(t) = \text{sinc}(200t) = \frac{\sin(200\pi t)}{(200\pi t)} \qquad \left[\because \text{sinc}(\theta) = \frac{\sin(\pi\theta)}{(\pi\theta)}\right]$$

Nyquist rate $= 2 \times 200\pi = 400\pi$ rad/sec or 200 Hz

(ii) Given :

$$x_2(t) = \text{sinc}^2(200t)$$
$$= \left[\frac{\sin(200\pi t)}{200\pi t}\right]^2$$
$$= \frac{1}{(200\pi t)^2}\left[\frac{1}{2} - \frac{1}{2}\cos(400\pi t)\right]$$

$\therefore \quad \omega = 400\pi$

Nyquist rate $= 2 \times 400\pi = 800\pi$ rad/sec or 400 Hz.

Example 3.122 Determine the Nyquist rate corresponding to the following signals.

(i) $x(t) = \cos(150\pi t)\sin(100\pi t)$

(ii) $x(t) = \cos^3(200\pi t)$

(iii) $x(t) = \text{sinc}(200t) + \text{sinc}^2(200t)$

Solution. (i) Given: $x(t) = \cos(150\pi t)\sin(100\pi t)$

$$= \frac{1}{2}\sin(250\pi t) - \frac{1}{2}\sin(50\pi t)$$

The signal consists 2 frequency components.

$$\omega_1 = 250\pi \quad \text{rad / sec} \qquad ; f_1 = 125 \text{ Hz}$$
$$\omega_2 = 50\pi \quad \text{rad / sec} \qquad ; f_2 = 25 \text{ Hz}$$

$\therefore$ The highest frequency $\omega_h = \omega_1 = 250\pi$.

$\therefore$ Nyquist rate $= 2\omega_h = 500\pi$ rad / sec or $2f_h = 250$ Hz.

(ii) Given: $x(t) = \cos^3(200\pi t)$

$$= \frac{3}{4}\cos(200\pi t) + \frac{1}{4}\cos(600\pi t) \qquad \left[\because \cos^3(\theta) = \frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta)\right]$$

The highest frequency $\omega_h = 600\pi$ rad / sec or $f_h = 300$Hz.

$\therefore$ Nyquist rate $= 2\omega_h = 1200\pi$ rad / sec or $2f_h = 600$ Hz.

(iii) Given: $x(t) = \text{sinc}(200t) + \text{sinc}^2(200t)$

$$= \frac{\sin(200\pi t)}{(200\pi t)} + \left[\frac{\sin(200\pi t)}{(200\pi t)}\right]^2$$

$$= \frac{\sin(200\pi t)}{(200\pi t)} + \frac{1}{2(200\pi t)^2} - \frac{1}{2(200\pi t)^2}\cos(400\pi t)$$

The highest frequency $\omega_h = 400\pi$ rad / sec or $f_h = 200$ Hz

$\therefore$ Nyquist rate $= 2\omega_h = 800\pi$ rad / sec or $2f_h = 400$ Hz

Example 3.123 If the Nyquist rate for $x(t)$ is ω_s. Find the Nyquist rate for each of the following signals.

(i) $x(2t)$ (iv) $x(t)x(t)$

(ii) $x\left(\frac{t}{3}\right)$ (v) $x(t+1)$

(iii) $x(t) * x(t)$ (vi) $\int_{-\infty}^{t} x(\tau)d\tau$

Solution. Assume that highest frequency of $x(t) = \omega_h$ rad / sec.

(i) The signal $x(2t)$ is a compressed version of $x(t)$ by a factor of 2. i.e., the highest frequency of $x(2t) = 2\omega_h$.

$\therefore$ Nyquist rate $\omega_s = 2(2\omega_h) = 4\omega_h$

(ii) The signal $x\left(\frac{t}{3}\right)$ is an expanded version of $x(t)$ by a factor of 3. i.e., the highest frequency of $x(\frac{t}{3}) = \frac{\omega_h}{3}$.

$\therefore$ Nyquist rate $\omega_S = \frac{2}{3}\omega_h$.

(iii) The signal $x(t) * x(t)$ is convolution of $x(t)$ with itself.

Then the highest frequency remains same that of $x(t)$ i.e., ω_h.

$\therefore$ Nyquist rate $\omega_S = 2\omega_h$.

(iv) The signal $x(t)x(t)$ is multiplication of $x(t)$ with itself.

Then the highest frequency is get doubled i.e., $2\omega_h$.

$\therefore$ Nyquist rate $\omega_S = 2(2\omega_h) = 4\omega_h$.

(v) The signal $x(t+1)$ is time shifted version of $x(t)$.

Time shifting operation doesnot change the frequency of the signal.

$$\therefore \quad \text{Nyquist rate } \omega_S = 2\omega_h.$$

(vi) The integral operation on $x(t)$, i.e., $\int_{-\infty}^{t} x(\tau)dt$ doesnot change the frequency of the signal.

$$\therefore \quad \text{Nyquist rate } \omega_s = 2\omega_h.$$

Example 3.124 The output $x(t)$ of an ideal low-pass filter which has cut-off frequency $\omega_c = 1000\pi$ rad / sec is impulse sampled with the following sampling periods.

(i) $T_S = 0.5 \times 10^{-3}$

(ii) $T_S = 2 \times 10^{-3}$

(iii) $T_S = 10^{-4}$

Which of these sampling periods would guarantee that $x(t)$ can be recovered from its sampled version using an appropriate low-pass filter.

Solution. Since $x(t)$ is the output of an ideal low-pass filter ($\omega_c = 1000\pi$ rad / sec), it has highest frequency $\omega_h = 1000\,\pi$ rad / sec.

We know that $x(t)$ can be recovered from its sampled version by passing it through a low-pass filter only if,

$$\text{Sampling frequency } \omega_S \geq 2\omega_h$$

$$\therefore \quad \omega_S \geq 2(1000\pi) = 2000\pi \text{ rad / sec}$$

$$\therefore \quad f_S \geq 2(500) = 1000 \text{ Hz}$$

$$\therefore \quad T_S \leq \frac{1}{1000} = 10^{-3}.$$

$\therefore$ Only $T_S = 0.5 \times 10^{-3}$ and $T_S = 10^{-4}$ satisfy the condition.

3.13 Review Questions

(1) Mention the appropriate Fourier representation for different classes of signals.

(2) When the two signals are said to be orthogonal ?

(3) State and prove the following properties of DTFS.

(i) Linearity (ii) Time shift (iii) Frequency shift

(iv) Convolution (v) Modulation (vi) Duality.

(4) State and prove the following properties of Fourier series.

(i) Linearity (ii) Time shift (iii) Frequency shift (iv) Scaling

(v) Time differentiation (vi) Convolution (vii) Modulation.

(5) State and prove the following properties of DTFT.

(i) Linearity (ii) Time shift (iii) Frequency shift

(iv) Frequency differentiation (v) Convolution (vi) Modulation.

(6) What are Dirichlet's conditions ?

(7) State and prove the following properties of Fourier Transform.

(i) Linearity (ii) Time shift (iii) Frequency shift

(iv) Scaling (v) Time differentiation (vi) Integration.

(vii) Convolution (viii) Modulation.

(8) What is frequency response of LTI system ?

(9) Obtain the relationship between FT and FS.

(10) What is impulse sampling ?

(11) What do you mean by aliasing ?

(12) State and prove sampling theorem.

(13) What is Nyquist rate?

3.14 Problems with Answers

P 3.1. Evaluate the DTFS representation for the signal,

$$x(n) = \cos\left(\frac{2\pi}{3}n\right) + \sin\left(\frac{2\pi}{7}n\right)$$

Plot the magnitude and phase spectrum.

Ans. Refer Fig. P3.1 (a) and (b).

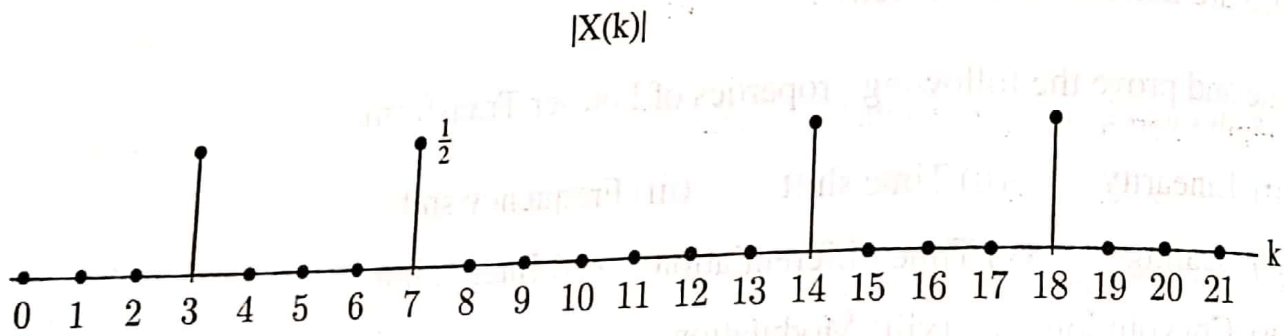
$$N = 21$$

$$X(7) = X(14) = \frac{1}{2}$$

$$X(3) = \frac{1}{2j}$$

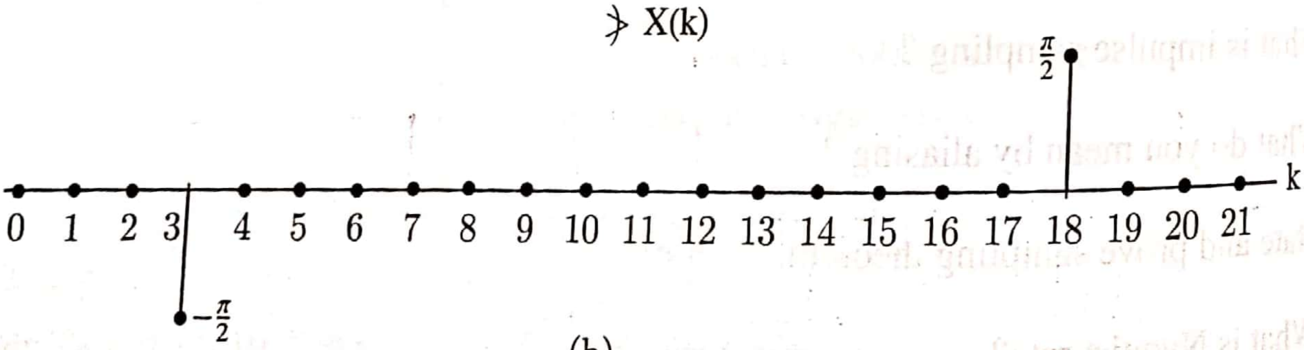
$$X(18) = -\frac{1}{2j}$$

Magnitude spectrum



(a)

Phase spectrum



(b)

Fig. P3.1.

P3.2. Obtain the DTFS of the signal shown in Fig. P3.2.

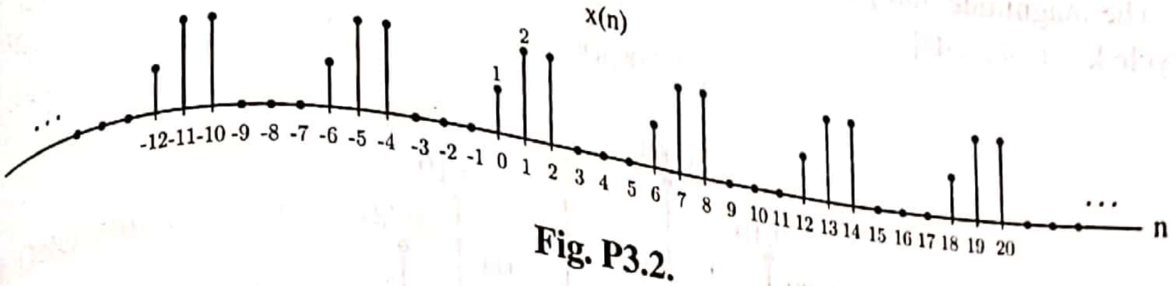


Fig. P3.2.

Ans. $X(k) = 1 + 4je^{-j\pi k/2}\sin\left(\frac{\pi k}{6}\right)$; $0 \le k \le 5$

P3.3. Find the DTFS coefficients for the signal $x(n)$ shown in Fig. P3.3.

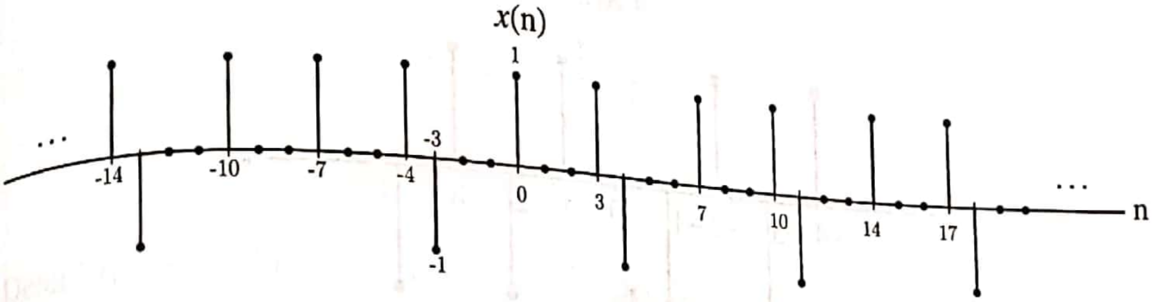


Fig. P3.3.

Ans. Fundamental period $N = 7$. $\therefore\ \Omega_o = \frac{2\pi}{7}$.

Take n, k ∈ {0, . . . 6}

$$X(k) = \frac{1}{7}\left(1 + e^{-j\frac{6\pi}{7}k} - e^{-j\frac{8\pi}{7}k}\right) \quad ; k \in (0, \ldots 6).$$

P3.4. Evaluate the DTFS representation for the signal shown in Fig. P3.4. Sketch its magnitude and phase spectrum.

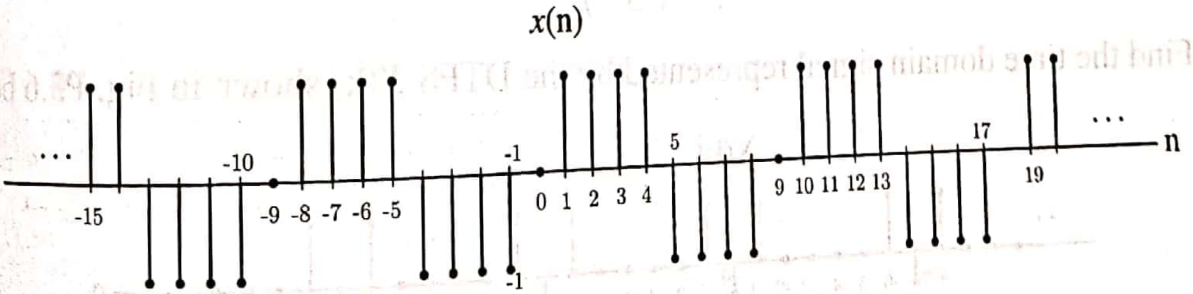


Fig. P3.4.

Ans. The magnitude and phase spectra are shown in Fig. P3.4.1 and P3.4.2 respectively [for one cycle $k \in (-4 \ldots 4)$]

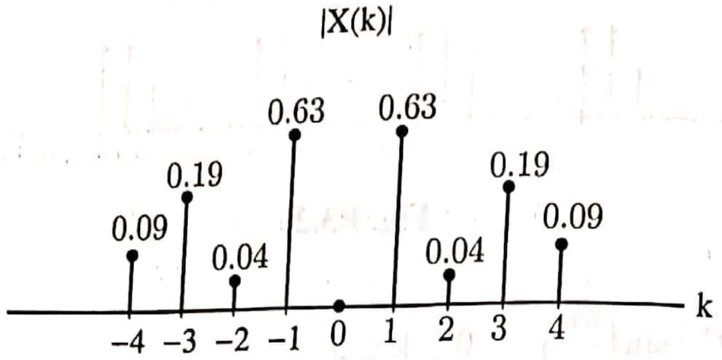


Fig. P3.4.1.

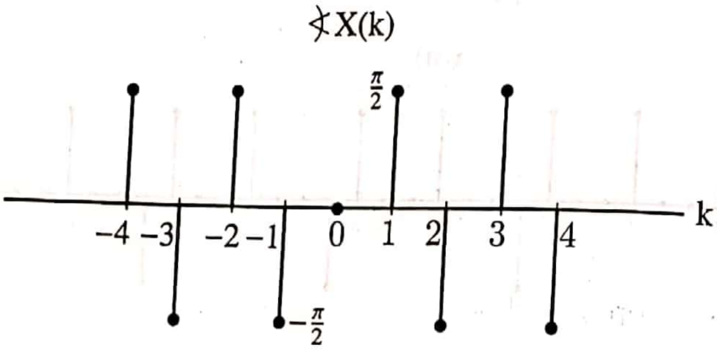


Fig. P3.4.2.

P 3.5. Determine the time domain signal represented by the DTFS coefficients,

$$X(k) = \sum_{m=-\infty}^{\infty} \delta(k-2m) - 2\delta(k+3m)$$

Ans. $N = 6 \quad ; \Omega_0 = \frac{\pi}{3}$

$$n, k \in \{-2, -1, 0, 1, 2, 3\}$$

$$x(n) = 2\cos\left(\frac{2\pi}{3}n\right) - 1 - 2(-1)^n$$

P 3.6. Find the time domain signal represented by the DTFS $X(k)$ shown in Fig. P3.6 below.

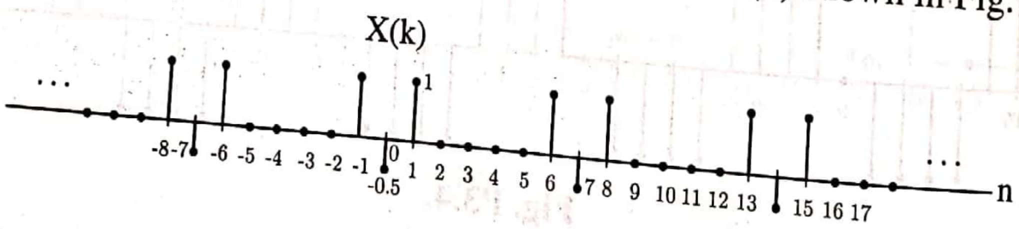


Fig. P3.6.

Ans. $N = 7 \quad ; \Omega_o = \frac{2\pi}{7}$

$$n, k \in (-3, -2, -1, 0, 1, 2, 3)$$

$$x(n) = 2\cos\left(\frac{2\pi}{7}n\right) - \frac{1}{2}$$

P 3.7. Determine the FS representation for the signal.

$$x(t) = \cos(4t) + \sin(6t)$$

Ans. $\omega_o = 2 \quad ; T = \pi$

$$X(2) = X(-2) = \frac{1}{2}$$

$$X(3) = \frac{1}{2j} \quad ; X(-3) = -\frac{1}{2j}$$

$$\& \quad X(k) = 0 \quad \text{for} \quad k \neq \pm 2 ; \pm 3$$

P 3.8. Determine the FS representation for the periodic signal,

$$x(t) = e^{-t} \quad ; -1 < t < 1$$

with period $T = 2$.

Ans. $X(k) = \frac{(-1)^k}{2(1 + jk\pi)}[e - e^{-1}] \quad ;$ for all k.

P 3.9. Evaluate the FS representation for the signal $x(t)$ with period $T = 4$ given by,

$$x(t) = \sin(\pi t) \quad ; 0 \le t \le 2$$
$$= 0 \quad ; 2 \le t \le 4$$

Ans. $T = 4 \quad ; \omega_o = \frac{\pi}{2}$

$$X(k) = \frac{2je^{-jk\frac{\pi}{2}}\sin(\frac{k\pi}{2})}{\pi(4 - k^2)} \quad ; k \neq \pm 2$$

$$X(2) = X(-2) = 0$$

P 3.10. Find the FS coefficients for the periodic signal shown in Fig. P3.10

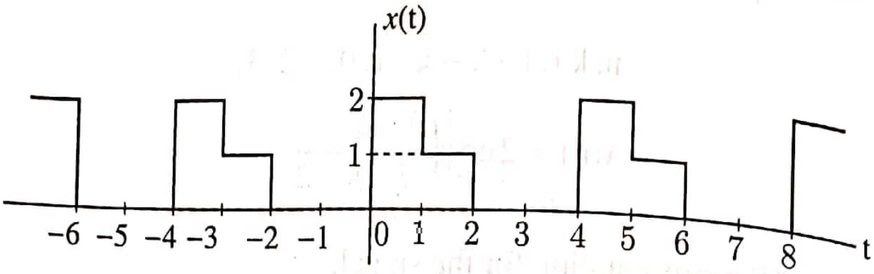


Fig. P3.10.

Ans. $T = 4 \quad ; \ \omega_o = \dfrac{\pi}{2}$

$$X(k) = \frac{e^{-jk\frac{\pi}{2}} \sin(\frac{k\pi}{2}) + e^{-jk\frac{\pi}{4}} \sin(\frac{k\pi}{4})}{k\pi} \quad ; \text{ for all 'k' except } k = 0$$

$$X(0) = \frac{3}{4}$$

P 3.11. Determine the DTFT of the signal,

$$x(n) = \left(\frac{1}{4}\right)^n u(n+2)$$

Ans. $X(e^{j\Omega}) = \dfrac{16e^{j2\Omega}}{1 - \frac{1}{4}e^{-j\Omega}}$

P 3.12. Determine the DTFT of the signal,

$$x(n) = \left(\frac{1}{2}\right)^n \{u(n+3) - u(n-2)\}$$

Ans. $X(e^{j\Omega}) = 8e^{j3\Omega}\left[\dfrac{1-(\frac{1}{2})^5 e^{-j5\Omega}}{1-\frac{1}{2}e^{-j\Omega}}\right]$

P 3.13. Compute the DTFT of the signal,

$$x(n) = \delta(4 - 2n)$$

Ans. $X(e^{j\Omega}) = e^{-j2\Omega}$

P 3.14. Find the DTFT of the signal,

$$x(n) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{4}\right)^n \delta(n - 3k)$$

Ans. $X(e^{j\Omega}) = \dfrac{1}{1 - \left(\frac{1}{4}e^{-j\Omega}\right)^3}$

P 3.15. Determine the DTFT of the signal $x(n)$ shown in Fig. P3.15

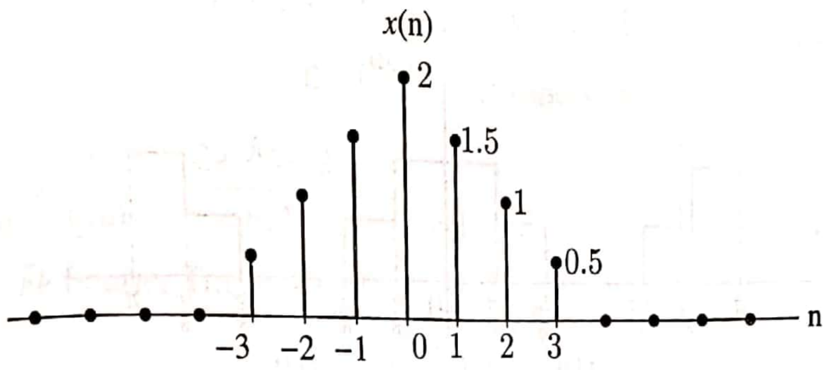


Fig. P3.15.

Ans. $X(e^{j\Omega}) = 2 + \cos(\Omega) + 2\cos(2\Omega) + \cos(3\Omega)$

P 3.16. Find the time domain signal corresponding to the DTFT,

$$X(e^{j\Omega}) = 1 - 2e^{-j3\Omega} + 4e^{j2\Omega} + 3e^{-j6\Omega}$$

Ans. $x(n) = \{4, 0, 1, 0, 0, -2, 0, 0, 3\}$
$\uparrow$

P 3.17. Find the inverse DTFT of,

$$X(e^{j\Omega}) = \cos\left(\frac{\Omega}{2}\right) + j\sin(\Omega) \qquad ; -\pi \le \Omega < \pi$$

Ans. $x(n) = \dfrac{-(-1)^n}{2\pi\left(n^2 - \frac{1}{4}\right)} + \dfrac{1}{2}\delta(n-1) - \dfrac{1}{2}\delta(n+1)$

P 3.18. Find the inverse DTFT of,

$$|X(e^{j\Omega})| = 0 \qquad ; 0 \le |\Omega| \le \frac{\pi}{3}$$

$$= 1 \qquad ; \frac{\pi}{3} \le |\Omega| \le \frac{2\pi}{3}$$

$$= 0 \qquad ; \frac{2\pi}{3} < |\Omega| \le \pi$$

$$\measuredangle X(e^{j\Omega}) = 2\Omega$$

Ans. $x(n) = \left[\dfrac{\sin\left(\frac{\pi n}{3}\right)}{\pi n}\right]\left[\cos\left(\frac{\pi n}{2}\right)\right]$

P 3.19. Determine the time domain signal corresponding to the spectrum shown in Fig. P3.19 below.

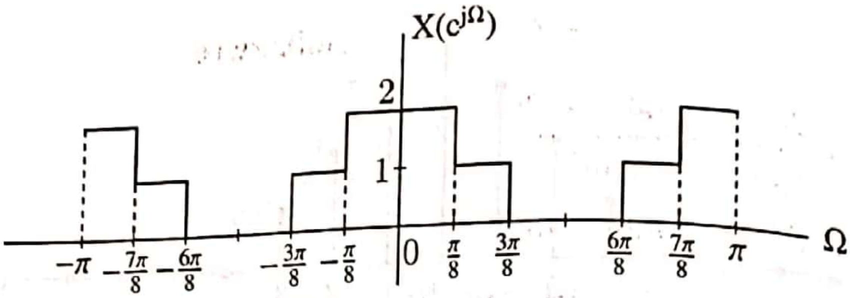


Fig. P3.19.

Ans. $x(n) = \dfrac{1}{n\pi}\left[\sin\left(\dfrac{7\pi n}{8}\right) + \sin\left(\dfrac{6\pi n}{8}\right) - \sin\left(\dfrac{3\pi n}{8}\right) - \sin\left(\dfrac{\pi n}{8}\right)\right]$

P 3.20. Let $x(n)$ be a sequence given by,

$$x(n) = \{-1, 2, \underset{\uparrow}{-3}, 2, -1\}$$

with DTFT $X(e^{j\Omega})$. Evaluate the following functions without computing $X(e^{j\Omega})$.

(a) $X(e^{j0})$ (b) $X(e^{j\pi})$ (c) $\int_{-\pi}^{\pi} X(e^{j\Omega})d\Omega$ (d) $\int_{-\pi}^{\pi} |X(e^{j\Omega})|^2 d\Omega$

Ans. (a) -1 (b) -9 (c) -6π (d) 38π

P 3.21. Determine the inverse DTFT of each of the following DTFTs.

(a) $X(e^{j\Omega}) = 1 + 2\cos(\Omega) + 3\cos^2(\Omega)$

(b) $Y(e^{j\Omega}) = j\left[3 + 4\cos(\Omega) + 2\cos^2(\Omega)\right]\sin(\Omega)$

Ans. (a) $x(n) = \{0.75, 1, \underset{\uparrow}{2.5}, 1, 0.75\}$

(b) $y(n) = \{0.25, 1, 1.75, \underset{\uparrow}{0}, -1.75, -1, -0.25\}$

P 3.22. Find the Fourier Transform of the signal,

$$x(t) = e^{2+t}u(-t+1)$$

Ans. $X(j\omega) = \dfrac{e^3 e^{-j\omega}}{1 - j\omega}$

P 3.23. Find the Fourier Transform of the signal,

$$x(t) = 1 - t^2 \quad ; 0 \le t \le 1$$
$$= 0 \quad ; \text{otherwise}$$

Ans. $X(j\omega) = \dfrac{1}{j\omega} + \dfrac{2e^{-j\omega}}{(j\omega)^2} - \dfrac{2e^{-j\omega} - 2}{j\omega^3}$

P 3.24. Compute the Fourier Transform of,

$$x(t) = \left[t\, e^{-2t} \sin(4t)\right] u(t)$$

using appropriate properties.

Ans. $X(j\omega) = \dfrac{j\pi}{[2 + j(\omega + 4)]^2} - \dfrac{j\pi}{[2 + j(\omega - 4)]^2}$

P 3.25. Find the FT of the signal,

$$x(t) = \sin(t) + \cos\left(2\pi t + \frac{\pi}{4}\right)$$

Ans. $X(j\omega) = j\pi\delta(\omega + 1) - j\pi\delta(\omega - 1) + e^{j\frac{\pi}{4}}\delta(\omega - 2\pi) + e^{-j\frac{\pi}{4}}\delta(\omega + 2\pi)$

P 3.26. Find the time domain signal corresponding to,

$$X(j\omega) = \cos\left(4\omega + \frac{\pi}{3}\right)$$

Ans. $x(t) = \dfrac{e^{-j\frac{\pi}{3}}}{2}\delta(t - 4) + \dfrac{e^{j\frac{\pi}{3}}}{2}\delta(t + 4)$

P 3.27. Find the inverse FT of,

$$X(j\omega) = 2\left[\delta(\omega - 1) - \delta(\omega + 1)\right] + 3\left[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)\right]$$

Ans. $x(t) = \dfrac{j2}{\pi}\sin(t) + \frac{3}{\pi}\cos(2\pi t)$

P 3.28. Let $X(j\omega)$ be the FT of the signal $x(t)$ shown in Fig. P3.28. Evaluate the following without explicitly evaluating $X(j\omega)$.

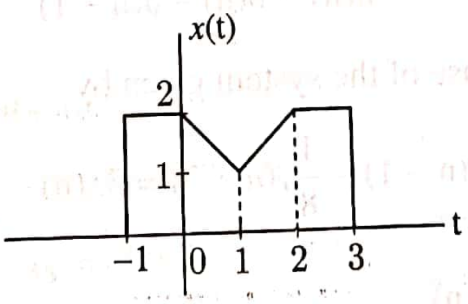


Fig. P3.28.

(a) $X(j0)$

(b) $\measuredangle X(j\omega)$

(c) $\int_{-\infty}^{\infty} X(j\omega)d\omega$

(d) $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$

Ans. (a) 7 (b) $-\omega$ (c) 4π (d) 26π

P 3.29. Obtain the frequency response of a system having impulse response,

$$h(n) = (-1)^n \quad ; |n| \leq 10$$
$$= 0 \quad ; \text{otherwise}$$

Ans. $H(e^{j\Omega}) = \dfrac{\cos\left(\frac{21}{2}\Omega\right)}{\cos\left(\frac{\Omega}{2}\right)}$

P 3.30. Obtain the frequency response of a continuous time LTI system with impulse response.

$$h(t) = 4e^{-2t}\cos(12t)u(t)$$

Ans. $H(j\omega) = 2\left[\frac{1}{2+j(\omega+12)} + \frac{1}{2+j(\omega-12)}\right]$

P 3.31. Obtain the frequency response and the impulse response of the system having the output $y(n)$ for the input $x(n)$ as given below.

$$x(n) = \left(\frac{1}{4}\right)^n u(n) \quad \text{and} \quad y(n) = \left(\frac{1}{4}\right)^n u(n) - \left(\frac{1}{4}\right)^{n-1} u(n-1)$$

Ans.

$$H(e^{j\Omega}) = 1 - e^{-j\Omega}$$

$$h(n) = \delta(n) - \delta(n-1)$$

P 3.32. Find the impulse response of the system given by,

$$y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = 3x(n) - \frac{3}{4}x(n-1)$$

Ans. $h(n) = \left[\left(\frac{1}{2}\right)^n + 2\left(-\frac{1}{4}\right)^n\right]u(n)$

P 3.33. Find the differential equation representation of a system with impulse response,

$$h(t) = (2e^{-2t} - 3e^{-3t})u(t)$$

Ans. $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = -\frac{dx(t)}{dt}$

P 3.34. Find the differential equation description for the system having impulse response,

$$h(t) = 2e^{-2t}u(t) - 2te^{-2t}u(t)$$

Ans. $\frac{d^2y(t)}{dt^2} + \frac{4\,dy(t)}{dt} + 4y(t) = 2\frac{dx(t)}{dt} + 2x(t)$

P 3.35. Find the differential equation description for the system with the frequency response,

$$H(j\omega) = \frac{2 + 3j\omega - 3(j\omega)^2}{1 + j2\omega}$$

Ans. $2\frac{dy(t)}{dt} + y(t) = -3\frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + 2x(t)$

P 3.36. A signal $x(t) = \cos(10\pi t) + 3\cos(20\pi t)$ is ideally sampled with sampling period τ_s. Find the Nyquist rate.

Ans. Nyquist rate $= 2\omega_h = 40\pi$ rad/sec or 20 Hz

P 3.37. Specify the Nyquist rate for each of the following signals.

(i) $x_1(t) = \text{sinc}(500t)$

(ii) $x_2(t) = \text{sinc}^2(500t)$

Ans. (i) 1000π rad/sec or 500 Hz

(ii) 2000π rad/sec or 1000 Hz

P 3.38. Determine the Nyquist rate for each of the following signals

(i) $x(t) = \sin(200\pi t)$

(ii) $x(t) = \sin^2(200\pi t)$

(iii) $x(t) = 1 + \cos(200\pi t) + \sin(400\pi t)$

Ans. (i) $\omega_S = 400\pi$ rad / sec or $f_S = 200\text{Hz}$

(ii) $\omega_S = 800\pi$ rad / sec or $f_S = 400\text{Hz}$

(iii) $\omega_S = 800\pi$ rad / sec or $f_S = 400\text{Hz}$