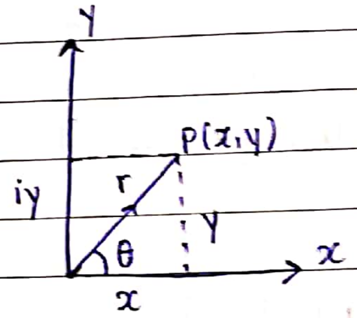
UNIT-III Complex Analysis

Functions of a Complex Variable :-

Complex number $z = x+iy$; $i = \sqrt{-1}$

If (r, θ) be the polar coordinates of P -



$z = r\cos\theta + ir\sin\theta$

$= r[\cos\theta + i\sin\theta]$

$\boxed{z = re^{i\theta}}$ ⇒ represents complex number in Polar form

If $z = x+iy$ is a complex number then $\bar{z} = x-iy$ is a complex conjugate.

NOTE :- $|z-a| = r$ represents a circle with centre at a & radius r.

Function of a complex variable -

If x & y are real variables then $z = x+iy$ is called a complex variable. If corresponding to each value of a complex variable z in a given region R there correspond one or more values of another complex variable w. $[u+iv]$
{i/p is a comp. variable $x+iy$ ⇒ o/p is also comp. ie $u+iv$}
w is called function of complex variable z & is denoted as $w = f(z) = u+iv$.

Derivative of f(z) :-

Let $w = f(z)$ be a single valued funct^n of variable $z = x+iy$ then derivative of $w = f(z)$ is

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$ provided the

limit exists and has same value for all the diff. ways in which δz approaches to zero.

Analytic Functions-

A function f(z) which is a single value and possesses a unique derivative at all points of region R then f(z) is called an analytic function in that region.

Remark: i) An analytic $funct^n$ is also called regular $funct^n$ or holomorphic $funct^n$.

ii) $Funct^n$ which is analytic everywhere in the complex plane is known as an entire function.

iii) A point at which ~~an analytic~~ $funct^n$ fails to be analytic or fails to posses a unique derivative is called singular point of function.

Eg: $f(z) = \dfrac{z}{z-a}$; {$funct^n$ fails to be analytic at $z=a$ ∴ $z=a$ is a singular pt}

Cauchy-Riemann [C-R] equations in Cartesian form-

(6 Marks) Theorem:-

"Necessary $condit^n$ that single valued $funct^n$ f(z). $f(z) = U(x,y) + i\,v(x,y)$ be analytic at any pt. $z = x+iy$ is that there exists four continous first order partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ which are continuous & satisfy equations $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$, these are

$(U_x = V_y)$ $(V_x = -U_y)$

called as CR equations."

⇒ Proof:-

Let f(z) be analytic at a point z ∴ by definition of analytic $funct^n$

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \text{ exists \& is unique.}$$

In cartesian form :-

$$f(z) = u(x,y) + i\,v(x,y)$$

let $\delta x, \delta y, \delta z$ be the increments in x, y, z respectively

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)] - [u(x,y) + iv(x,y)]}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) - u(x,y)] + i[v(x+\delta x, y+\delta y) - v(x,y)]}{\delta z}$$

$$= \lim_{\delta z \to 0} \left\{ \frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta z} + i \frac{[v(x+\delta x, y+\delta y) - v(x,y)]}{\delta z} \right.$$

sinze $z = x+iy$ $\therefore \delta z \Rightarrow \delta x + i\delta y$

Since δz approaches 0 in two ways

Case<i>: Along real axis { ie along $\delta x \to 0$ }

ie $y = 0 \Rightarrow \delta y = 0$

$\Rightarrow \delta z = \delta x$

Equation ① becomes-

$$f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x}$$

$$+ i \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{——} ②$$

Case <ii> : Along imaginary axis:

ie $x = 0 \Rightarrow \delta x = 0$

$\Rightarrow \delta z = i\delta y$

Equation ① becomes-

$$f'(z) = \lim_{i\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + \cancel{i} \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{\cancel{i}\delta y}$$

$$f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{——} ③$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{——} ③$$

Existence of $f'(z)$ thus guarantees the existence of the four first order partial derivatives whichare continuous in eq^n ② & ③

equating ② & ③ :-

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real part:-

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$$

Equating imaginary part:-

$$\boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

REMARK: The real & imaginary parts of an analytic functⁿ are called conjugate funct'ns.

$\therefore$ if $f(z) = u(x,y) + iv(x,y)$ is an analytic functⁿ then $u(x,y)$ & $v(x,y)$ are conjugate functions and relation b/w two conjugate functions is given by CR equations.

Cauchy- Riemann [C-R] equations in polar form:-

"If a functⁿ $f(z)$ is analytic at a point z then there exists 4 continuous first order partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ & satisfy equation called C-R equatⁿˢ

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$ "

Note:-
$f'(z) = f'(re^{i\theta})$
$= e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$

$\Rightarrow$ Proof:-

let (r,θ) be polar coordinates of pt z whose cartesian coordinates are (x,y) then

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$z = x+iy = r\cos\theta + i\sin\theta$$

$$z = re^{i\theta}$$

$$u+iv = f(z) = f(re^{i\theta}) \text{ — } ①$$

diff. ① partially wrt r keeping θ constant

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})\, e^{i\theta} \text{ — } ②$$

diff ① partially wrt θ keeping r constant

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = rie^{i\theta} f'(re^{i\theta}) \text{ — } ③$$

From ② & ③

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$$

equating imaginary parts

$$i\frac{\partial v}{\partial \theta} = ir\frac{\partial u}{\partial r}$$

$$\boxed{\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}}$$

equating real parts -

$$\boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$$

$\Rightarrow \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ & $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ are CR equations.

LP 3) Prove that given funct^{ns} are analytic & find their derivative.

ii) ~~sinhz +~~ e^z

$f(z) = $ ~~sinhz~~ e^z

$= e^{x+iy}$

$= e^x . e^{iy}$

$= e^x[\cos y + i\sin y]$

$f(z) = e^x\cos y + ie^x\sin y = u + iv$

$\Rightarrow u = e^x\cos y \quad v = e^x\sin y$

$\frac{\partial u}{\partial x} = e^x\cos y \quad \frac{\partial v}{\partial x} = e^x\sin y$

$\frac{\partial u}{\partial y} = -\sin y e^x \quad \frac{\partial v}{\partial y} = \cos y e^x$

since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, CR eq^{ns} are satisfied

$\therefore f(z) = e^z$ ~~are~~ is a analytic (/holomorphic/regular) funct^n.

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$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= e^x \cos y + i e^x \sin y$$
$$= e^x(\cos y + i\sin y)$$
$$= e^x . e^{iy}$$
$$= e^{x+iy}$$
$$\boxed{f'(z) = e^z}$$

vi) (?) $z + e^z = (x+iy)(e^x . e^{iy})$
$$= (x+iy)(e^x.(\cos y + i\sin y)) = (x+iy)[e^x\cos y + ie^x \sin y]$$
$$= xe^x\cos y + ixe^x\sin y + iye^x\cos y - ye^x\sin y$$

$$f(z) = [xe^x\cos y - ye^x\sin y] + i[xe^x\sin y + ye^x\cos y]$$
$$= u + iv$$
$$\Rightarrow u = xe^x\cos y - ye^x\sin y \ \& \ v = xe^x\sin y + ye^x\cos y$$
$$\frac{\partial u}{\partial x} = \cos y[xe^x + e^x] - e^x y\sin y = e^x[x\cos y + \cos y - y\sin y]$$
$$\frac{\partial u}{\partial y} = -xe^x\sin y - e^x[y\cos y + \sin y]$$
$$\frac{\partial v}{\partial x} = \sin y[xe^x + e^x] + y\cos y e^x$$
$$\frac{\partial v}{\partial y} = xe^x\cos y + e^x[-y\sin y + \cos y] = e^x[x\cos y + \cos y - y\sin y]$$
$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$; CR equat^{ns} are satisfied.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= e^x[x\cos y + \cos y - y\sin y] + i[\sin y(xe^x + e^x) + y\cos y e^x]$$
$$= e^x[x\cos y + \cos y - y\sin y + ix\sin y + i\sin y + iy\cos y]$$
$$= e^x[\cos y + x\cos y - y\sin y) + i(x\sin y + \sin y + y\cos y)]$$
$$f'(z) = e^x[\cos y + i\sin y + x(\cos y + i\sin y) - y(\sin y - i\cos y)$$
$$= e^x[e^{iy} + xe^{iy} - y(\sin y - i\cos y)]$$

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iii) $f(z) = \sinh z = \frac{e^{z} - e^{-z}}{2}$

$= \frac{1}{2}\{e^{x+iy} - e^{-(x+iy)}\}$

$= \frac{1}{2}\{e^{x}.\ e^{iy} - e^{-x}.\ e^{-iy}\}$

$= \frac{1}{2}\{e^{x}[\cos y + i\sin y] - e^{-x}[\cos y - i\sin y]\}$

$= \frac{1}{2}\{(\cos y + i\sin y)(e^{x} - e^{-x})\}$

$= \frac{(e^{x} - e^{-x})}{2}\cos y + i\frac{(e^{x} - e^{-x})}{2}\sin y$

(X) $f(z) = \sinh x\cos y + i\sinh x\sin y$

$\Rightarrow u = \sinh x\cos y$

$v = \sinh x\sin y$

$\frac{\partial u}{\partial x} = \cos y\cosh x \qquad \frac{\partial u}{\partial y} = -\sinh x\sin y$

$\frac{\partial v}{\partial x} = \cosh x\sin y \qquad \frac{\partial v}{\partial y} = \sinh x\cos y$

$= \frac{1}{2}\{e^{x}[\cos y + i\sin y) - e^{-x}(\cos x - i\sin y)]$

$= \frac{1}{2}\{(e^{x} - e^{-x})\cos y + i(e^{x} + e^{-x})\sin y\}$

$= \sinh x\cos y + \cosh x\sin y$

$= u + iv$

$u = \sinh x\cos y \qquad v = \cosh x\sin y$

$\frac{\partial u}{\partial x} = \cosh x\cos y \qquad \frac{\partial v}{\partial x} = \sinh x\sin y$

$\frac{\partial u}{\partial y} = -\sinh y\sin y \qquad \frac{\partial v}{\partial y} = \cosh x\cos y$

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$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

CR equat^{ns} hold good

$\therefore f(z) = \sinh z$ is Analytic funct^{n}.

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \cosh x \cos y + i \sinh x \sin y$

$= \frac{(e^{x} + e^{-x})}{2} \cos\left(\frac{e^{iy} + e^{-iy}}{2}\right) + i\left(\frac{e^{x} - e^{-x}}{2}\right)\left(\frac{e^{iy} - e^{-iy}}{2}\right)$

put $x = z$ & $y = 0$

$\boxed{f'(z) = \cosh x}$

iv) $f(z) = \log z$ $\{= \log(x+iy)$ can't use$\}$ $\therefore$ convert to polar.

In polar form $z = re^{i\theta}$

$f(z) = \log(re^{i\theta}) = \log r + \log e^{i\theta} = \log r + i\theta$

$= u + iv$

$u = \log r$ $\quad \theta = v$

$\frac{\partial u}{\partial r} = \frac{1}{r}$ $\quad \frac{\partial u}{\partial \theta} = 0$

$\frac{\partial v}{\partial r} = 0$ $\quad \frac{\partial v}{\partial \theta} = 1$

since $\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} \Rightarrow \frac{\partial u}{\partial r}$ & $\frac{\partial v}{\partial \theta} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

CR eq^{ns} are satisfied $\therefore$ $\log z$ is analytic.

$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$

$= e^{-i\theta}\left[\frac{1}{r} + i0\right] = \frac{e^{-i\theta}}{r} = \frac{1}{re^{i\theta}} = \frac{1}{z}$ //.

v) $f(z) = z^n$

using polar form $\rightarrow z = re^{i\theta}$

$f(z) = (re^{i\theta})^n = r^n e^{in\theta}$

$= r^n [\cos n\theta + i \sin n\theta]$

$u = r^n \cos n\theta$ and $v = r^n \sin n\theta$

$\frac{\partial u}{\partial r} = n r^{n-1} \cos n\theta \qquad \frac{\partial v}{\partial r} = n r^{n-1} \sin n\theta$

$\frac{\partial u}{\partial \theta} = -r^n n \sin n\theta \qquad \frac{\partial u}{\partial \theta} = r^n n \cos n\theta$

$\frac{\partial u}{\partial r} = n r^{n-1} \cos n\theta = \frac{1}{r} \frac{\partial v}{\partial \theta}$

$\frac{\partial v}{\partial r} = n r^{n-1} \sin n\theta = \frac{-1}{r} \frac{\partial u}{\partial \theta}$

$f'(z) = e^{-i\theta} \left[\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right]$

$= e^{-i\theta} [n r^{n-1} \cos n\theta + i n r^{n-1} \sin \theta]$

$= e^{-i\theta} [n r^{n-1} (e^{in\theta})]$

$= n r^{n-1} e^{in\theta - i\theta}$

$= n r^{n-1} e^{i\theta(n-1)}$

$= n r^{n-1} e^{i\theta(n-1)}$

$= n (re^{i\theta})^{n-1}$

$= n z^{n-1}$ //.

Harmonic Functions-

A function ϕ is said to be harmonic if it satisfies laplace eq^n. ~~th~~ $\cdot \nabla^2 \phi = 0$

In cartesian form $\phi(x,y)$ is harmonic if

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

In polar form $\phi(r, \theta)$ is harmonic if

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0$$

REMARK: • The real & imaginary parts of an analytic $funct^n$ are harmonic.

• BUT The converse is not true i.e we can example for u & v satisfying laplace eq^n but not satisfying CR equations.

$$\text{Analytic } funct^n \longrightarrow \text{Harmonic } funct^n$$
$$\text{Analytic } funct^n \not\longleftarrow \text{Harmonic } funct^n$$

Definition:

let $u(x,y)$ be harmonic function then a $funct^n$ $v(x,y)$ is said to be harmonic conjugate of $u(x,y)$ if it satisfies following conditions-

i> $v(x,y)$ is harmonic

ii> CR eq^{ns} holds good.

Statement of theorem:

"If $f(z) = u+iv$ is an analytic function in a domain D, then v is harmonic conjugate of u conversely if v is harmonic conjugate of u in domain D then $f(z) = u+iv$ is analytic $funct^n$ in domain D."

Construct^n of analytic $funct^n$ $f(z)$ given it's real / img. parts by Milne - Thomson's Method:

Procedure:-

(I) Cartesian Form:

Step 1: Given u & v as a function of x & y.

Step 2: Find partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$

Step 3: Consider $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

Case<i> : If u is given then use CR equation-

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\therefore \boxed{f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}} \quad \text{—— (1)}$$

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Case ii) If V is given
then use CR equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \quad \text{--- (2)}$$

Step 4: ~~No~~ Substitute expressions for partial derivatives in RHS of (1) & (2) & then put $x = z$ & $y = 0$ to obtain $f'(z)$ as a function of z only.

Step 5: Integrate wrt z to get $f(z)$.

(II) Polar Form:

Step 1: Given u & v as funct^n of r & θ.

Step 2: Find partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial v}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial \theta}$

Step 3: Consider $f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$

Case (i) If u is given: then using CR eq^n

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\therefore f'(z) = \left(\frac{\partial u}{\partial r} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right)e^{-i\theta} \quad \text{--- (1)}$$

Case (ii) If v is given

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\therefore f'(z) = e^{-i\theta}\left[\frac{1}{r}\frac{\partial v}{\partial \theta} + i\frac{\partial v}{\partial r}\right] \quad \text{--- (2)}$$

Step 4: Substitute expressions for partial derivat^n in RHS of (1) & (2) & then put $r = z$ & $\theta = 0$ to obtain $f'(z)$ as funct^n of z only.

Step 5: Integrate wrt z to get $f(z)$.

LPH) Detrmine analytic funct^n whose real part is.

i) ~~Q~~ $x\cos 2y - y\sin 2y$ {real part}

given $u = x\cos 2y - y\sin 2y$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

use CReq^n $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$\frac{\partial u}{\partial x} = \cos 2y$ $\quad \frac{\partial u}{\partial y} = -2x\sin 2y - 2y\cos 2y - \sin 2y$

$\therefore f'(z) = \cos 2y + i(2x\sin 2y + 2y\cos 2y + \sin 2y)$

~~$f'(z) = \cos 2y +$~~ put $x = z, y = 0$

$f'(z) = \cancel{\cos z}\ 1 + i[2z(0) + 0] = 1$

$f'(z) = 1$

Integrate wrt z

$$f(z) = z + c$$

ii) $\cos x \cosh y$

given $u = \cos x \cosh y$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$\frac{\partial u}{\partial x} = -\sin x \cosh y$ $\quad \frac{\partial u}{\partial y} = \cos x \sinh y$

$f'(z) = -\sin x\cosh y + i\cos x \sinh y$

put $x = z$ & $y = 0$

$f'(z) = -\sin z - i\cos z\,(0)$ $\quad$ {$\cosh y = 1$, $\sinh y = 0$}

$= -\sin z$

Integrate wrt z

$$f(z) = \cos z + c$$

iii) $e^x \cos y$

given $u = e^x \cos y$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$f'(z) = e^x \cos y - i e^x \sin y$$

put $x = z$ & $y = 0$

$$f'(z) = e^z$$

Integrate:

$$f(z) = e^z + C$$

LP5) Determine analytic funct^n whose img. part is–

i) $\log(x^2+y^2) + (x-2y)$

ii) $e^x\{(x^2-y^2)\cos y - 2xy \sin y\}$

iii) $e^x(x \sin y + y \cos y)$

i) Given $v = \log(x^2+y^2) + (x-2y)$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

CR eq^n $\frac{\partial v}{\partial y} = +\frac{\partial u}{\partial x}$

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \quad \text{—(1)}$$

$$\frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2} + 1 \qquad \frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} - 2$$

$$f'(z) = \frac{2y}{x^2+y^2} - 2 + i\left(\frac{2x}{x^2+y^2} + 1\right)$$

put $x = z$ & $y = 0$

$$f'(z) = -2 + i\left[\frac{2z}{z^2} + 1\right]$$

integrate wrt:

$$f(z) = -2z + i[2\log_e z + z] + c$$

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ii) $v = e^x \{(x^2-y^2)\cos y - 2xy \sin y\}$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$

$\frac{\partial v}{\partial y} = e^x\{[(-2y)\cos y - \sin y(x^2-y^2)] - 2\sin y(x) - 2xy\cos y$

$\frac{\partial v}{\partial x} = e^x\{2x\cos y - 2y\sin y\} + \{(x^2-y^2)\cos y - 2xy\sin y\}e^x$

$\therefore f'(z) = e^x[-2y\cos y - \sin y(x^2-y^2) - 2x\sin y - 2xy\cos y]$
$+ ie^x[2x\cos y - 2y\sin y - 2xy\sin y + (x^2-y^2)\cos y]$

put $x = z$ & $y = 0$

$f'(z) = e^z[0] + ie^x[2z+z^2]$

$f'(z) = ie^z(2z+z^2)$

integrate wrt z

$f(z) = \cancel{i[e^z(2z^2 + \frac{z^3}{8}}$

$f(z) = i\,[(2z+z^2)(e^z) - (2+2z)e^z + (2)e^z] + C$

$= i\,[2ze^z + z^2e^z - 2e^z - 2ze^z + 2e^z] + C$

$= i\,[\cancel{2ze^z} - \cancel{2ze^z} + z^2e^z] + C$

$= i\,(z^2e^z) + C$ //.

iii) $e^x(x\sin y + y\cos y) = v$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

iii) $e^x \cos y$

given $u = e^x \cos y$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$f'(z) = e^x \cos y - ie^x \sin y$$

put $x = z$ & $y = 0$

$$f'(z) = e^z$$

Integrate:

$$\underline{f(z) = e^z + C}$$

LP5) Determine analytic funct^n whose img. part is–

i) $\log(x^2+y^2) + (x-2y)$

ii) $e^x\{(x^2-y^2)\cos y - 2xy \sin y\}$

iii) $e^x(x\sin y + y\cos y)$

i) Given $v = \log(x^2+y^2) + (x-2y)$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

CR eq^n $\frac{\partial v}{\partial y} = +\frac{\partial u}{\partial x}$

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \quad -(1)$$

$$\frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2} + 1 \qquad \frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} - 2$$

$$f'(z) = \frac{2y}{x^2+y^2} - 2 + i\left(\frac{2x}{x^2+y^2} + 1\right)$$

put $x = z$ & $y = 0$

$$f'(z) = -2 + i\left[\frac{2z}{z^2} + 1\right]$$

integrate w.r.t:

$$f(z) = -2z + i[2\log_e z + z] + c$$

ii) $v = e^x\{(x^2-y^2)\cos y - 2xy\sin y\}$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$

$\frac{\partial v}{\partial y} = e^x\{[(-2y)\cos y - \sin y(x^2-y^2)] - 2\sin y(x) - 2xy\cos y\}$

$\frac{\partial v}{\partial x} = e^x\{2x\cos y - 2y\sin y\} + \{(x^2-y^2)\cos y - 2xy\sin y\}e^x$

$\therefore f'(z) = e^x[-2y\cos y - \sin y(x^2-y^2) - 2x\sin y - 2xy\cos y]$
$+ ie^x[2x\cos y - 2y\sin y - 2xy\sin y + (x^2-y^2)\cos y]$

put $x=z$ & $y=0$

$f'(z) = e^z[0] + ie^x[2z+z^2]$

$f'(z) = ie^z(2z+z^2)$

integrate wrt z

$\cancel{f(z) = i\left[e^z\left(2z^2 + \frac{z^3}{3}\right)\right.}$

$f(z) = i\,[(2z+z^2)(e^z) - (2+2z)e^z + (2)e^z] + c$

$= i\,[2ze^z + z^2e^z - \cancel{2e^z} - 2ze^z + \cancel{2e^z}] + c$

$= i\,[\cancel{2ze^z} - \cancel{2ze^z} + z^2e^z] + c$

$= i\,(z^2e^z) + c$ //.

iii) $e^x(x\sin y + y\cos y) = v$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

LP9) If $f(z)$ is a regular funct^n of z prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

— let $f(z) = u+iv$ is a regular funct^n

then $|f(z)| = \sqrt{u^2+v^2}$

$$|f(z)|^2 = u^2+v^2 = \phi \text{ (say)} \quad \text{—— (1)}$$

diff (1) partially wrt x:

$$\frac{\partial\phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} \quad \text{—— (2)}$$

diff (2) wrt x

$$\frac{\partial^2\phi}{\partial x^2} = 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2 + 2v\frac{\partial^2 v}{\partial x^2} + 2\left(\frac{\partial v}{\partial x}\right)^2 \text{—— (3)}$$

III^{ly} diff (1) partially wrt y twice:-

$$\frac{\partial\phi}{\partial y} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} \quad \text{—— (4)}$$

$$\frac{\partial^2\phi}{\partial y^2} = 2u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial y}\right)^2 + 2v\frac{\partial^2 v}{\partial y^2} + 2\left(\frac{\partial v}{\partial y}\right)^2 \text{—— (5)}$$

Adding (3) & (5)

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2u\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] + 2\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right.$$
$$\left. + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] + 2v\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] \quad \text{(6)}$$

Since $f(z) = u+iv$ is a regular funct^n of z, u & v satisfy CR eq^{ns} & laplace eq^{ns}.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

and $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ & $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$

putting in eq^n (6)

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$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 + 0 + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] + 2\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$LHS = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \quad — (7)$$

Now: $f(z) = u + iv$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

$$4|f'(z)|^2 = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \quad — (8)$$

LHS = RHS.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 4|f'(z)|^2$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)| = 4|f'(z)|^2 //.$$ proved

LP10) If z is ~~harmonic~~ holomorphic or analytic funct^n of z. prove that $\left[\frac{\partial}{\partial x}|f(z)|\right]^2 + \left[\frac{\partial}{\partial y}|f(z)|\right]^2 = |f'(z)|^2$

proof: let $f(z) = u + iv$ is analytical funct^n.

$$|f(z)| = \sqrt{u^2+v^2}$$

$$|f(z)|^2 = u^2+v^2 = \phi^2 \text{ (say)} — (1)$$

diff (1) wrt x partially

$$\cancel{2}\phi\frac{\partial \phi}{\partial x} = \cancel{2}u\frac{\partial u}{\partial x} + \cancel{2}v\frac{\partial v}{\partial x} \quad — (2)$$

diff (1) wrt y partially

$$\cancel{2}\phi\frac{\partial \phi}{\partial y} = \cancel{2}u\frac{\partial u}{\partial y} + \cancel{2}v\frac{\partial v}{\partial y} \quad — (3)$$

squaring & adding (2) & (3)

$$\phi^2\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] = u^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$2uv\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} + v^2\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + 2uv\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} \quad — (4)$$

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since $f(z)$ is analytic CR eq^{ns} are satisfied.

ie $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

eq^n (4) reduces to

$$\phi^2\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] =$$

$$= u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + \cancel{2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}}$$

$$+ u^2\left[\left(\frac{-\partial v}{\partial x}\right)^2\right] + v^2\left(\frac{\partial u}{\partial x}\right)^2 - \cancel{2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}}$$

$$= \cancel{u^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]}$$

$$= (u^2+v^2)\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2(u^2+v^2) = \phi^2$$

$$= (u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = \phi^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$\Rightarrow \cancel{\phi^2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] = \cancel{\phi^2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$\text{LHS} = \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] = \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

we have $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \quad \text{--- (6)}$$

from (5) & (6)

$$\frac{\partial}{\partial x}|f(z)|^2 + \frac{\partial}{\partial y}|f(z)|^2 = |f'(z)|^2 \;//.$$