Formation of P.D.E by eliminating arbitary fn's :-

Suppose z is a fn of two arbitary fn's, we have to find partial derivatives after the second order and use the necessary partial derivatives of 2^{nd} order to form the P.D.E by eliminating arbitary fns.

Ex:- Form the P.D.E by eliminating the arbitory fns for $z = f(x^2+y^2)$

→ Diff the given fn w.r.t x and y partially

$$\frac{\partial z}{\partial x} = p = f'(x^2+y^2)\,2x \longrightarrow (1)$$

$$\frac{\partial z}{\partial y} = q = f'(x^2+y^2)\,2y \longrightarrow (2)$$

Dividing (1) & (2)

$$\frac{p}{q} = \frac{x}{y}$$

$$py = qx$$

$$py - qx = 0.$$

2) $z = f(x+ct) + \phi(x-ct)$.

$$\frac{\partial z}{\partial x} = \phi'(x-ct) + f'(x+ct) = p$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct) \longrightarrow (1)$$

$$\frac{\partial z}{\partial t} = f'(x+ct)\cdot c - \phi'(x-ct)\cdot c$$

$$\frac{\partial^2 z}{\partial t^2} = f''(x+ct)c^2 + \phi''(x-ct)\cdot c^2 \longrightarrow (2)$$

Using ① we get

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left[\frac{\partial^2 z}{\partial x^2} \right]$$

3) form a P.D.E by eliminating arbitrary fn from $z = x^2 + 2f\left(\frac{1}{y} + \log x\right)$

$$\frac{\partial z}{\partial x} = p = 2x + \frac{2}{x} f\left(\frac{1}{y} + \log x\right) \longrightarrow ②$$

$$\frac{\partial^2 z}{\partial x^2} = 2 + \left(-\frac{2}{x^2}\right) f\left(\frac{1}{y} + \log x\right) + \frac{2}{x^2} f\left(\frac{1}{y} + \log x\right)$$

$$\frac{\partial z}{\partial y} = q = 2f\left(\frac{1}{y} + \log x\right)\left(\frac{-1}{y^2}\right)$$

eliminate f' from ② and ③

$$2f'\left(\frac{1}{y} + \log x\right) = -qy^2$$

① becomes

$$p = 2x - \frac{qy^2}{x}$$

$$p - 2x = -\frac{qy^2}{x}$$

$$px - 2x^2 = -qy^2$$

$$px + qy^2 = 2x^2$$

4) $z = yf(x) + x\phi(y) \longrightarrow ①$

$$\frac{\partial z}{\partial x} = yf'(x) + \phi(y) = p \longrightarrow ②$$

$$\frac{\partial z}{\partial y} = f(x) + x\phi'(y) = q \longrightarrow ③$$

$$\frac{\partial^2 z}{\partial x^2} = y f''(x) + 0 = r \longrightarrow (4)$$

$$\frac{\partial^2 z}{\partial y^2} = x \phi''(y) = t \longrightarrow (5)$$

$$\frac{\partial^2 z}{\partial x \partial y} = s = f'(x) + \phi'(y) \longrightarrow (6)$$

from (2) $f'(x) = \frac{p - \phi(y)}{y}$

from (3) $\phi'(y) = \frac{q - f(x)}{x}$

Use this in (6)

$$\frac{\partial^2 z}{\partial x \partial y} = s = \frac{p - \phi(y)}{y} + \frac{q - f(x)}{x}$$

$$s = \frac{px - x\phi(y) + yq - yf(x)}{xy}$$

$$s \cdot xy = px - x\phi(y) + yq - yf(x)$$

$$s \cdot xy = px + yq - [x\phi(y) + yf(x)]$$

Using (1) $s \cdot xy = px + q \cdot y - z$

$$s \cdot xy + z = px + qy.$$

Note: To form P.D.E of the form $\phi(u, v) = 0$ where u and v are fn's of x, y, z (z being a fn of x and y) we proceed as follows by given data $\phi(u, v) = 0 \longrightarrow (1)$

Diff (1) partially w.r.t x and y & applying chain rule we get

$$\frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial x} = 0 \longrightarrow (2)$$

$$\frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial y} = 0 \longrightarrow (3)$$

Transforming the 2nd term in both (2) & (3) on the RHS & dividing, we obtain the required P.D

Ex:- Form P.D.E by eliminating arbitrary fns $\phi(xy + z^2, x + y + z) = 0$.

→ we have $\phi(u, v) = 0 \longrightarrow (1)$

where $u = xy + z^2$ and $v = x + y + z$

Diff partially u w.r.t x & y

$$\frac{\partial u}{\partial x} = y + 2z \cdot \frac{\partial z}{\partial x} \qquad \frac{\partial u}{\partial y} = x + 2z \cdot \frac{\partial z}{\partial y}$$

$$\frac{\partial v}{\partial x} = 1 + \frac{\partial z}{\partial x} = 1 + p \qquad \frac{\partial v}{\partial y} = 1 + \frac{\partial z}{\partial y} = 1 + q$$

Using the above eqns and applying the chain rule we get

$$\frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial x} \longrightarrow (2)$$

$$\frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial y} = -\frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial y} \longrightarrow (3)$$

Dividing (2) by (3) we get

$$\frac{\cancel{\partial y}}{\cancel{\partial x}} = \frac{\partial y}{\partial x}$$

or $$\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial v/\partial x}{\partial v/\partial y}$$

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$$= \frac{y + 2zp}{x + 2zq} = \frac{1+p}{1+q}$$

$$(1+p)(x+2zq) = (1+q)(y+2zp)$$

$$x + 2zp + px + \cancel{2pqz} = y + 2zp + qy + \cancel{2pqz}$$

$$p(x - 2z) - q(y - 2z) + (x - y) = 0.$$

2) form P.D.E by eliminating arbitrary fn f from $f(x^2 + 2yz,\ y^2 + 2zx) = 0$.

$f(u, v) = 0$

$u = x^2 + 2yz \qquad v = y^2 + 2zx.$

$$\frac{\partial u}{\partial x} = 2x + 2y\frac{\partial z}{\partial x} + 2z\frac{\partial y}{\partial x}.$$

$$\frac{\partial u}{\partial y} = 2x \cdot \frac{\partial x}{\partial y} + 2z$$

$$\frac{\partial v}{\partial x} = 2z$$

solve it.

$$\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial v/\partial x}{\partial v/\partial y}$$

$$\frac{2x + 2yp}{2z + 2yq} = \frac{2z + 2xp}{2y + 2xq}$$

$$(2x + 2yp)(2y + 2xq) = (2z + 2xp)(2z + 2yq)$$

$$p(y^2 - xz) + q(x^2 - yz) = z^2 - xy$$

3) $f(x^2 + y^2 + z^2,\ x + y + z) = 0.$

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$u = x^2 + y^2 + z^2$ $\qquad$ $v = x + y + z$

$\frac{\partial u}{\partial x} = 2x + 2z \frac{\partial z}{\partial x}$ $\qquad$ $\frac{\partial v}{\partial x} = 1 + p$.

$\frac{\partial u}{\partial y} = 2y + 2z \frac{\partial z}{\partial y}$ $\qquad$ $\frac{\partial v}{\partial y} = 1 + q$

$$\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial v/\partial x}{\partial v/\partial y}$$

$$\frac{\cancel{2}x + 2zp}{\cancel{2}y + 2zq} = \frac{1+p}{1+q}$$

$$\frac{x+zp}{y+zq} = \frac{1+p}{1+q}$$

$(1+q)(x+zp) = (1+p)(y+zq)$

$x + zp + xq + p\cancel{z}q = y + qz + py + pq\cancel{z}$

$p(z-y) + q(x-z) + x - y = 0$

ii) $F(xy + z^2, x + y + z) = 0$

$u = xy + z^2$ $\qquad$ $v = x + y + z$

$\frac{\partial u}{\partial x} = y + 2z \cdot \frac{\partial z}{\partial x}$ $\qquad$ $\frac{\partial v}{\partial x} = 1 + p$

$\frac{\partial u}{\partial y} = x + 2z \cdot \frac{\partial z}{\partial y}$ $\qquad$ $\frac{\partial v}{\partial y} = 1 + q$.

$$\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial v/\partial x}{\partial v/\partial y}$$

$$\frac{y + 2z \cdot p}{x + 2z \cdot q} = \frac{1+p}{1+q}$$

$$y + 2zp + qy + \cancel{2zpq} = x + 2zq + px + \cancel{2zpq}$$

$$\cancel{x - y + 2z(q - zp)\ x\ (-1)}$$

$$y - x + 2z(p-q) + qy - px = 0$$

Solution of p.d.e :

The soln $f(x, y, z, p, q) = 0 \longrightarrow$ ①, first order p.d.e which contains 2 arbitory constants is called a Complete integral.

A soln obtained from the Complete integral by assi-gning a particular values to the arbitory const-ants is called a particular integral.

I. Direct Integration.

Here we consider p.d.e which can be solved by direct integration.

In place of usual constants of integration, we use arbitory function of the variable held fix

Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x$

$\rightarrow$ We have $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \sin x$

Integrating with respect to x keeping y fix.

$$\frac{\partial z}{\partial y} = -\cos x + f(y)$$ where f is arbitory fn

Integrating w.r.t y keeping x fix

$$z = -y\cos x + F(y) + \phi(x)$$

where $F(y) = \int f(x)\,dy$ and ϕ is arbitory fn which is required soln

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Ex:- Solve $\frac{\partial^2 z}{\partial x^2} = xy$.

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = xy$$

$\int$ w.r.t x

$$\frac{\partial z}{\partial x} = \frac{x^2}{2} y + f(y)$$

$$z = \frac{x^3}{2\times 3} y + x f(y) + \phi(y)$$

$$z = \frac{x^3 y}{6} + x f(y) + \phi(y).$$

Ex:- $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0$

$$\frac{\partial^3 z}{\partial x^2 \partial y} = -18xy^2 - \sin(2x-y)$$

$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x \partial y}\right)$ $\int$ w.r.t x

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{\cancel{18}^{9} x^2}{\cancel{2}_1} y^2 + \frac{\cos(2x-y)}{2} + f(y)$$

$\int$ w.r.t x

$$\frac{\partial z}{\partial y} = -\frac{\cancel{9}^{3} x^3}{\cancel{3}_1} y^2 + \frac{\sin(2x-y)}{4} + x f(y) + \phi(y)$$

$\int$ w.r.t y

$$z = -\frac{3x^3 y^3}{3} + \frac{\cos(2x-y)}{4} + x F(y) + G(y)$$

$$z = -x^3 y^3 + \frac{\cos(2x-y)}{4} + x F(y) + G(y)$$

The result may be simplified by writing $\int f(y)\,dy = F(y)$ & $\int \phi(y)\,dy = G(y)$

where $\cancel{F}$, G, g are arbitrary constants.

Ex: Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$.

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) + z = 0$

given that $x = 0$, $z = e^y$ & $\frac{\partial z}{\partial x} = 1$

→ Assuming that z is the function of x alone, the given p.d.e will be of the form

$$\frac{d^2 z}{dx^2} + z = 0 \quad \longrightarrow (1)$$

Let $D = \frac{d}{dx}$ (using operator)

∴ (1) becomes $(D^2+1)z = 0$.

Auxillary eqn $D^2+1 = 0$.

$D = \pm i$. $\quad \begin{pmatrix} \alpha = 0 \\ \beta = 1 \end{pmatrix}$

And soln is

$$z = A\sin x + B\cos x$$

$$z = e^{\alpha x}(A\sin\beta x + B\cos\beta x)$$

where A and B are arbitory constants

But z is a function of x and y, A and can be arbitory function of y (∵ eqn is w.r.t

hence the soln of given eqn is

$$\boxed{z = f(y)\sin x + \phi(y)\cos x} \longrightarrow (2)$$

Diff (2) w.r.t x partially

$+g(x)$

$$\frac{\partial z}{\partial x} = f(y)\cos x + \phi(y)(-\sin x) \longrightarrow (3)$$

when $x = 0$, $z = e^y$ put in eqn (2)

$e^y = \phi(y)$

when $x = 0$ $\partial z/\partial y = 1$ put in eqn (3)

$1 = f(y)$

$\therefore$ particular soln

$$z = \sin x + e^{y} \cos x$$

Ex:- $\frac{\partial^2 z}{\partial x^2} = a^2 z$, $x=0$, $\frac{\partial z}{\partial x} = a \sin y$, $\frac{\partial z}{\partial y} = 0$

$\rightarrow$ Assume that z is the fn of x alone

$$(D + a^2) z = 0$$

$$D = \pm a$$

$$z = c_1 e^{ax} + c_2 e^{-ax}$$

But z is the fn of x and y, c_1 and c_2 can be arbitory fn of y

$\therefore$ Soln for given p.d.e is

$$z = f(y) e^{ax} + \phi(y) e^{-ax}$$

$$\frac{\partial z}{\partial x} = af(y) e^{ax} - a\phi(y) e^{-ax}$$

$$a \sin y = f(y) - \phi(y) \longrightarrow ①$$

$$\frac{\partial z}{\partial y} = f'(y) e^{ax} + \phi'(y) e^{-ax}$$

$$0 = f'(y) + \phi'(y) \longrightarrow ②$$

diff. ① w.r.t y

$$+ a \cos y = f'(y) - \phi'(y)$$

$$0 = f'(y) + \phi'(y)$$

$$+ a \cos y = 2f'(y)$$

$$+ \frac{a \cos y}{2} = f'(y)$$

$\int$ w.r.t y

$$k + \frac{1a}{2} \sin y = f(y)$$

$$\sin y = \frac{1}{2}\sin y - \phi(y) + k$$

$$\phi(y) = \frac{1}{2}\sin y - \sin y + k$$

$$\phi(y) = -\frac{1}{2}\sin y + k$$

$$\therefore z = \frac{\sin y}{2} e^{ax} - \frac{\sin y}{2} e^{-ax} + k.$$

$$z = \frac{1}{2}(\sin y + k)e^{ax} + \frac{1}{2}(k - \sin y)e^{-ax}$$

$$z = \sin y\left(\frac{e^{ax} - e^{-ay}}{2}\right) + k\left(\frac{e^{ax} + e^{-ax}}{2}\right)$$

$$z = \sin y \cdot \sinh ax + k \cosh ax.$$

Solution of the Lagrange's Linear p.d.e :

Given p.d.e of the form $Pp + Qq = R$ is called Lagrange's Linear p.d.e where

$$P = \partial u/\partial z \cdot \partial v/\partial y - \partial v/\partial z \cdot \partial u/\partial y$$
$$Q = \partial u/\partial y \cdot \partial v/\partial z - \partial v/\partial y \cdot \partial u/\partial z$$
$$R = \partial u/\partial y \cdot \partial v/\partial x - \partial v/\partial y \cdot \partial u/\partial x.$$

Procedure :

1) We form the eqn of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ which is known as auxillary eqns

2) Then solve the Auxillary eqns.

3) Suppose $u(x, y, z)$ $v(x, y, z)$ are the 2 relations so obtained then $\phi(u, v) = 0$ constitute the soln of the given p.d.e.

Ex:- Solve $x(y-z)p + y(z-x)q = z(x-y)$.

$\Rightarrow$ The given eqn is of the form

$Pp + Qq = R$ which is lagrange's first order p.d.e where

$P = x(y-z)$

$Q = y(z-x)$

$R = z(x-y)$.

form Auxillary eqn as $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \longrightarrow (1)$$

Using multiplier 1, 1, 1 we get each ratio =

$$\frac{1 \cdot dx + 1 \cdot dy + 1 \cdot dz}{1 \cdot (x(y-z) + 1 \cdot y(z-x) + 1 \cdot z(x-y))}$$

$$\text{ratio} = \frac{dx + dy + dz}{\cancel{xy} - \cancel{xz} + \cancel{yz} - \cancel{yx} + \cancel{zx} - \cancel{yz}}$$

$$\text{ratio} = \frac{dx + dy + dz}{0}$$

$$0 = dx + dy + dz.$$

Integrating

$$\int dx + \int dy + \int dz = c_1$$

$$x + y + z = c_1 \longrightarrow (2)$$

Again using multipliers as yz, zx, xy we get each ratio $= \frac{yz\,dx + zx\,dy + xy\,dz}{yz(xy - xz) + zx(zy - xy) + xy(zx}$

$$= \frac{yz\,dx + zx\,dy + xy\,dz}{\cancel{xy^2z} - \cancel{xz^2y} + \cancel{z^2xy} - \cancel{x^2zy} + \cancel{x^2yz} - \cancel{xy^2z}}$$

$$0 = yz\,dx + zx\,dy + xy\,dz$$

divide throughout by xyz.

$$0 = \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz$$

Integrating

$$\log C_2 = \log x + \log y + \log z$$

$$xyz = C_2 \longrightarrow (3)$$

from (2) and (3)

$$\phi(u, v) = 0$$

$$\phi(x+y+z-C_1,\ xyz - C_2) = 0$$

2] Solve $(x^2 - y^2 - z^2)p + 2xyq = 2xz$.

$\rightarrow$ The given eqn is of the form $Pp + Qq = R$

where $P = x^2 - y^2 - z^2$

$Q = 2xy$

$R = 2xz$

Auxillary eqn as $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \longrightarrow (1)$$

from 2 and 3 fraction

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{y} = \frac{dz}{z}$$

Integrating

$$\frac{y}{z} = C_1$$

Using multipliers (x, y, z) each ratio is

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{x(x^2-y^2-z^2) + y(2xy) + z(2xz)} = \text{ratio}$$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2} = \text{ratio}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x^3 + xy^2 + xz^2} = \text{ratio}.$$

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)} = \text{ratio}.$$

from eqn ① we have

$$\frac{x\,dx + y\cdot dy + z\cdot dz}{\cancel{x}(x^2 + y^2 + z^2)} = \frac{dz}{2\cancel{x}z}$$

$$\frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2 + y^2 + z^2} = \frac{dz}{z} \qquad \frac{f'(x)}{f(x)} = \log f(x)$$

Taking the integration

$$\log(x^2 + y^2 + z^2) = \log z + \log c_2$$

$$x^2 + y^2 + z^2 = z\,c_2$$

$$\phi(u, v) = 0$$

$$\Rightarrow \phi\left(\frac{u}{z} - c_1,\ \frac{x^2 + y^2 + z^2}{z} - c_2\right) = 0.$$

Ex:- Solve $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

$P = x(y^2 - z^2)$

$Q = y(z^2 - x^2)$

$R = z(x^2 - y^2)$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}$$

$$\frac{dx + dy + dz}{xy^2 - z^2x + yz^2 - yx^2 + zx^2 - zy^2}$$

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Take x, y, z as multiplier.

$$\frac{x\,dx + y\,dy + z\,dz}{\cancel{x^2y^2} - \cancel{z^2x^2} + \cancel{y^2z^2} - \cancel{y^2x^2} + \cancel{z^2x^2} - \cancel{z^2y^2}} = \text{ratio}$$

$$x\,dx + y\,dy + z\,dz = 0.$$

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$$

Take $1/x, 1/y, 1/z$ as multiplier

$$\frac{1/x\,dx + 1/y\,dy + 1/z\,dz}{\cancel{y^2} - \cancel{z^2} + \cancel{z^2} - \cancel{x^2} + \cancel{x^2} - \cancel{y^2}} = \text{ratio}.$$

$$\int 1/x\,dx + \int 1/y\,dy + \int 1/z\,dz = 0.$$

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2$$

$$\phi(u, v) = 0$$

$$\phi\left(\frac{x^2 + y^2 + z^2}{2} - c_1,\ xyz - c_2\right) = 0$$

Ex:- $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$.

$P = x^2(y-z)$

$Q = y^2(z-x)$

$R = z^2(x-y)$.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{x^2(y-z)} + \frac{dy}{y^2(z-x)} + \frac{dz}{z^2(x-y)}$$

Take $1/x, 1/y, 1/z$ as multiplier

$$\frac{1/x\,dx + 1/y\,dy + 1/z\,dz}{\cancel{xy} - \cancel{xz} + \cancel{yz} - \cancel{xy} + \cancel{zx} - \cancel{zy}} = \text{ratio}$$

$\log x + \log y + \log z = \log c_1$

$xyz = c_1$

Take $1/x^2, 1/y^2, 1/z^2$ as multiplier.

$$\frac{1/x^2\,dx + 1/y^2\,dy + 1/z^2\,dz}{\cancel{y} - \cancel{z} + \cancel{z} - \cancel{x} + \cancel{x} - \cancel{y}} = \text{ratio}$$

$$\int 1/x^2\,dx + \int 1/y^2\,dy + \int 1/z^2\,dz = \int 0$$

$$\cancel{\frac{\log x^2}{2x} + \frac{\log y^2}{2y} + \frac{\log z^2}{2z} = \log c_2}$$

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = c_2$$

$$-\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - c_2 = 0$$

$$\phi(u, v) = 0$$

$$\phi\left(xyz - c_1, -\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - c_2\right) = 0$$

Ex:- $zp - zq = z^2 + (x+y)^2$

$P = z$

$Q = -z$

$R = z^2 + (x+y)^2$

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2}$$

$$\cancel{\frac{dx + dy + dz}{z - z + z^2 + x^2 + y^2 + 2xy} = \text{ratio}}$$

$$\frac{dx}{z} = \frac{dy}{-z}$$

$$\int dx = -\int dy$$

$$x = -y + c_1$$

$$x + y = c_1$$

$$\frac{dx}{z} = \frac{dz}{z^2+(x+y)^2}$$

$$\frac{dx}{z} = \frac{dz}{z^2+c_1^2}$$

$$dx = \frac{z\,dz}{z^2+c_1^2}$$

$$\int dx = \int \frac{z\,dz}{z^2+c_1^2}$$

$$x = \frac{1}{2}\log(z^2+c_1^2) + c_2$$

$$\phi(u,v) = 0$$

$$\phi\left(x+y-c_1,\ x-\frac{1}{2}\log(z^2+c_1^2)-c_2\right) = 0.$$

Soln of p.d.e by method of seperation of variables (product method)

Most of the boundary prblms involving linear p.d.e can be solved by the following method.

It involves a soln which breaks up into a product of functions each of which contains only one of the variables.

Ex:- Solve $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y)u$ by separation method.

→ Let $u = XY$, $X = X(x)$ and $Y = Y(y)$ be the soln of given p.d.e

Substituting into the given p.d.e we have

$$\frac{\partial}{\partial x}(XY) + \frac{\partial}{\partial y}(XY) = 2(x+y)XY.$$

$$Y\frac{\partial X}{\partial x} + X\frac{\partial Y}{\partial y} = 2(x+y)X\cdot Y.$$

$$Y\cdot\frac{dX}{dx} + X\frac{dY}{dy} = 2(x+y)X\cdot Y$$

dividing both sides by XY

$$\frac{1}{X}\frac{dX}{dx} + \frac{1}{Y}\frac{dY}{dy} = 2(x+y).$$

$$\frac{1}{X}\frac{dX}{dx} - 2x = 2y - \frac{1}{Y}\frac{dY}{dy}$$

Equating both sides to a common constant k, we have $\frac{1}{X}\cdot\frac{dX}{dx} - 2x = k$; $-\frac{1}{Y}\frac{dY}{dy} + 2y = k$

$$\Rightarrow \frac{1}{X}\cdot\frac{dX}{dx} = 2x+k$$

$$\frac{1}{X}dX = (2x+k)dx.$$

Integrating both sides

$$\int\frac{1}{X}dX = \int(2x+k)dx.$$

$$\log X = \cancel{2}\frac{x^2}{\cancel{2}} + kx + c_1$$

$$X = e^{x^2+kx+c_1}$$

$$-\frac{1}{Y}dY = (k-2y)dy$$

$$\int -\frac{1}{Y}dY = \int(k-2y)dy$$

$$\int\frac{1}{Y}dY = \int(2y-k)dy$$

$$\log_e Y = y^2 - ky + c_2$$

$$Y = e^{y^2-ky+c_2}$$

$\therefore$ Required soln is $u = XY = e^{c_1+c_2}\cdot e^{x^2+kx+y^2-ky}$

Ex:- Using the method of separation of variables solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where $u(x,0) = 6e^{-3x}$.

$\rightarrow$ Let $u = XT$ where $X = X(x)$ and $T = T(t)$

substituting in above p.d.e we get

$$\frac{\partial(XT)}{\partial x} = 2\frac{\partial(XT)}{\partial t} + XT$$

$$T\frac{\partial X}{\partial x} = 2X\frac{\partial T}{\partial t} + XT$$

$$T\frac{\partial X}{\partial x} - XT = 2X\frac{\partial T}{\partial t}.$$

$$T\left(\frac{\partial X}{\partial x} - X\right) = 2X\frac{\partial T}{\partial t}$$

$$\frac{X'-X}{2X} = \frac{T'}{T}$$

Equating to the Common constant k

$$\frac{X'-X}{2X} = k$$

$$\frac{X'}{2X} - \frac{1}{2} = k.$$

$$\frac{X'}{X} = 1+2k.$$

$$\frac{dX}{dx} \cdot \frac{1}{X} = 1+2k.$$

$$\int \frac{1}{X}\, dX = \int (1+2k)\, dx$$

$$\log X = x + 2xk + \log c_1$$

$$X = c_1 e^{(1+2k)x} \longrightarrow (1)$$

$$\frac{T'}{T} = k.$$

$$\cancel{\log T = kt} \quad \frac{dT}{dt} \times \frac{1}{T} = k.$$

$$\frac{1}{T}\, dT = k\, dt$$

$$\int \frac{1}{T}\, dT = \int k\, dt.$$

$$\log T = kt + \log c_2$$

$$T = c_2 e^{kt} \longrightarrow (2)$$

From (1) and (2) the required soln is

$$u = XT$$

$$u = c_1 c_2\, e^{(1+2k)x} \cdot e^{kt} \longrightarrow (3).$$

Now by using initial condition.

$$u(x, 0) = 6e^{-3x}$$

Eqn (3) becomes :

$$c_1 c_2 \cdot e^{(1+2k)x}(1) = 6e^{-3x}.$$

$$c_1 c_2 \cdot e^{(1+2k)x} = 6e^{-3x}$$

$$c_1 c_2 = 6 \qquad e^{(1+2k)x} = e^{-3x}$$

$$c_1 c_2 = 6 \qquad (1+2k) = -3.$$

$$2k = -4$$

$$k = -2.$$

$\therefore$ Particular soln $u = 6e^{-3x} \cdot e^{-2t}.$

Ex:- Solve by separation of variables $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

→ Let $z = XY$ where $X = X(x)$ and $Y = Y(y)$

Substituting in above eqn we get

$$\frac{\partial^2 (XY)}{\partial x^2} - 2\frac{\partial (XY)}{\partial x} + \frac{\partial (XY)}{\partial y} = 0.$$

$$Y\frac{\partial^2 X}{\partial x^2} - 2Y\frac{\partial X}{\partial x} + X\frac{\partial Y}{\partial y} = 0$$

$$YX'' - 2YX' + XY' = 0$$

$$YX'' - 2YX' = -XY'$$

$$Y(X'' - 2X') = -XY'$$

$$\frac{X'' - 2X'}{X} = \frac{-Y'}{Y}.$$

$$\frac{X'' - 2X'}{X} = k \qquad \frac{-Y'}{Y} = k.$$

$$\frac{X''}{X} - \frac{2X'}{X} = k. \qquad \frac{Y'}{Y} = -k$$

$$\frac{\partial^2 X}{\partial x^2}\left(\frac{1}{X}\right) - \frac{2}{X}\left(\frac{\partial X}{\partial x}\right) = k \qquad \frac{1}{Y}\frac{\partial Y}{\partial y} = -k.$$

$$\frac{\partial^2 X}{\partial x^2} - 2\frac{\partial X}{\partial x} = kX \qquad \int \frac{1}{Y}\,\partial Y = \int -k\,dy$$

$$\log Y = -ky + c_2$$

$$Y = e^{-ky + c_2}$$

$$\frac{\partial^2 X}{\partial x^2} - 2\frac{\partial X}{\partial x} - kX = 0 \qquad Y = c_3 e^{-ky} \quad (\text{where } c_3 = e^{c_2})$$

(it is linear diff eqn)

Using operator D the above eqn becomes $(D = d/dx)$

$$(D^2 - 2D - k)X = 0.$$

A. E $D^2 - 2D - k = 0$

$$D = 1 \pm \sqrt{1+k}.$$

(the roots are real & distinct)

$$X = c_1 e^{-(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}$$

$$z = XY$$

$$z = \left[c_1 e^{-(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}\right](c_3 e^{-ky})$$

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9. $4\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 3u$, $t=0$ $u = 3e^{-x} - e^{-5x}$

$\rightarrow$ $u = XT$

$4\frac{\partial (XT)}{\partial t} + \frac{\partial}{\partial x}(XT) = 3(XT)$

$4X\frac{\partial T}{\partial t} + T\frac{\partial X}{\partial x} = 3(XT)$

$4X\frac{\partial T}{\partial t} = 3XT - T\frac{\partial X}{\partial x}$.

$4XT' = 3XT - TX'$

$4XT' = T(3X - X')$

$\frac{4T'}{T} = \frac{3X - X'}{X}$

$\frac{4T'}{T} = k$, $\frac{3X - X'}{X} = k$

$\frac{4}{T}\frac{\partial T}{\partial t} = k$ | $3 - \frac{1}{X}\frac{\partial X}{\partial x} = k$.

$\frac{4}{T}\partial T = k\,\partial t$. | $3 - k = \frac{1}{X}\frac{\partial X}{\partial x}$

$\int \frac{4}{T}\partial T = \int k\,\partial t$ | $\int (3-k)\,dx = \int \frac{1}{X}\partial X$

$4\log T = kt + c_1$ | $3x - kx + c = \log X$

$T = c_2 e^{kt/4}$ | $c_4 e^{3x - kx} = X$

$u = c_2 c_4 e^{3x - kx} \cdot e^{kt/4}$

$3e^{-x} - e^{-5x} = c_2 c_4 e^{3x - kx}$

<u>Vibration of a stretched string</u> (<u>One Dimensional wave eqn</u>) :-

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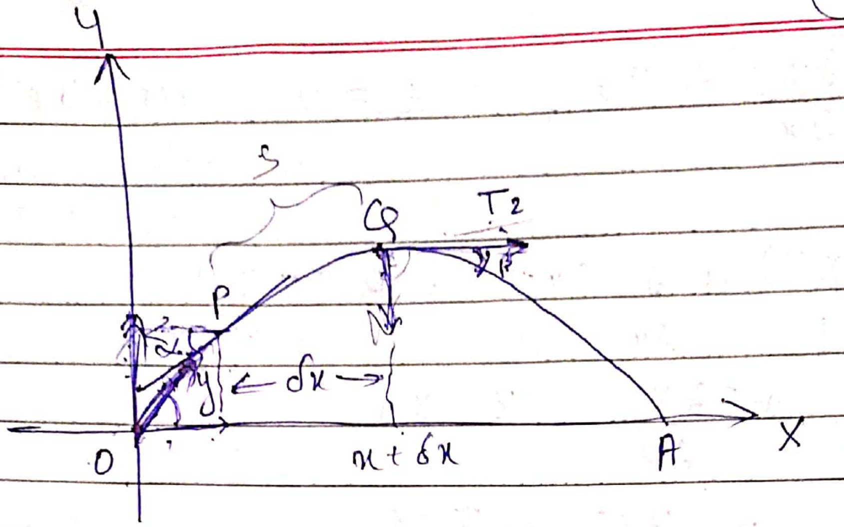
(0, 0)



Consider a uniform elastic string of length 'l' stretched tightly between 2 points O & A & displays slightly from its equilibrium position OA. Taking the end O as the Origin and OA as the x axis and a $\perp^r$ as a function of distance x and time t.

We shall obtain the eqn of the motion of the string by the following assumptions.

1) The motion takes place entirely in the x-y plane & each particle of the string moves $\perp^r$ to the equilibrium position OA of the string
2) The string is perfectly flexible & does not offer resistance to bending
3) The tension in the string is so large that forces due to weight of the string can be neglected.
4) The displacement y and the slope $\frac{dy}{dx}$ are small enough so that the higher powers can be neglected
5) Let m be the mass per unit length of the string.

Consider the motion of an element PQ of length ~~Δx~~ δs, since the string does not offer resistance to bending (by assumption), the tension T_1 and T_2 at P and Q respectively are tangential to the curve.

Since there is no ~~force~~ motion in the horizontal direct we have $T_1 \cos\alpha = T_2 \cos\beta = T$ (constant) $\longrightarrow$ (1)

Mass of element PQ is $m.\delta s$. By newton's second law of motion, the eqn of the motion in the vertical direction is

$$(m.\delta s)\frac{\partial^2 y}{\partial t^2} = T_2 \sin\beta - T_1 \sin\alpha$$

$$(m)(a) = F$$

$$\frac{m.\delta s}{T}\frac{\partial^2 y}{\partial t^2} = \frac{T_2 \sin\beta}{T_2\cos\beta} - \frac{T_1 \sin\alpha}{T_1\cos\alpha} \quad \text{(using T)}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m\delta s}(\tan\beta - \tan\alpha)$$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{T}{m\delta s}\left[\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

Since $\delta s = \delta x$, to a first approximation and $\tan\alpha$ and $\tan\beta$ are the slopes of the curve of the string at x and $x+\delta x$.

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left[\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\delta x}\right]$$

$$= \frac{T}{m} \times \frac{\partial^2 y}{\partial^2 x}, \quad \text{as } \delta x \to 0.$$

$$\frac{\partial^2 y}{\partial x^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad \text{where } c^2 = \frac{T}{m}$$

This is partial differential eqn giving the transfers vibration of the string. It is also called as one dimensional wave eqn.

12M The Boundary conditions with the eqn $\frac{\partial^2 y}{\partial t^2} = c^2.\frac{\partial^2 y}{\partial x^2}$ has to satisfy

i) $y = 0$, when $x = 0$

ii) $y = 0$, when $x = l$

$\Big\}$ Boundary conditions

If the string is made to vibrate by pulling it into the curve $y = f(x)$ and then releasing it, the initial conditions are

i) $y = f(x)$, when $t = 0$
ii) $\frac{\partial y}{\partial t} = 0$, when $t = 0$
} Initial Condition

Solution of wave eqn :

The wave eqn is $\frac{\partial^2 y}{\partial t^2} = c^2 \cdot \frac{\partial^2 y}{\partial x^2} \longrightarrow (1)$.

Let $y = XT$ where $X = X(x)$ and $T = T(t)$, be the soln of eqn (1) → (2)

Substituting in (1) we get

$$\frac{\partial^2 XT}{\partial t^2} = c^2 \cdot \frac{\partial^2 XT}{\partial x^2}$$

$$X \frac{\partial^2 T}{\partial t^2} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

$$X T'' = c^2 T X''$$

Separating the variables we have.

$$\frac{X''}{X} = \frac{T''}{T} \cdot \frac{1}{c^2} \longrightarrow (3)$$

Equating to Common constant k

$\frac{X''}{X} = k$ | $\frac{1}{c^2} \cdot \frac{T''}{T} = k$

$X'' - Xk = 0$ | $T'' - kc^2 T = 0 \longrightarrow (4)$

$D = k$ | $D^2 = kc^2$

Soln of eqn (4), we get

(i) when k is $+ve$ and equal to p^2 (say)

$(D^2 - k)X = 0$. | $(D^2 - k^2c^2)T = 0$

$(D^2 - p^2)X = 0$ | $(D^2 - p^2c^2)T = 0$

$D^2 - p^2 = 0$ | $D = \pm pc$.

$D^2 = p^2$

(i) $X = c_1 e^{px} + c_2 e^{-px}$ $\qquad T = c_3 e^{cpt} + c_4 e^{-cpt}$

(ii) when k is $-ve$, & $k = -p^2$ (say)

$(D^2 - k)X = 0$ $\qquad (D^2 + kc^2)T = 0$

$D^2 + p^2 = 0$ $\qquad (D^2 + p^2c^2)T = 0$

$D^2 = -p^2$ $\qquad D = \pm jpc$.

$X = c_1 \cos px + c_2 \sin px$ $\qquad T = c_3 \cos cpt + c_4 \sin cpt$.

(iii) when $k = 0$

$X = c_1 x + c_2$ $\qquad T = c_3 t + c_4$ $\quad (\because (c_1x + c_2)e^{x}$ where $x = 0)$

$\therefore$ Various possible solns of eqn ① are

$y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt})$

$y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$

$y = (c_1 x + c_2)(c_3 t + c_4)$.

Of these 3 solns, we have to choose that soln which is consistent with the physical nature of the problm. Since we are dealing with problm on vibrations, y must be periodic fn of x and t.

$\therefore$ Soln must involve trignometric terms

$\therefore y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \to ⑤$

is the only soln of the wave eqn.

Now applying boundary conditions that $y=0$ when $x=0$ and $y=0$ where $x=l$.

$\Rightarrow 0 = c_1 [c_3 \cos cpt + c_4 \sin pt] \longrightarrow ⑥$.

$0 = (c_1 \cos pl + c_2 \sin pl)(c_3 \cos cpt + c_4 \sin cpt) \to ⑦$.

from eqn ⑥, we have

$c_1 = 0$.

and eqn ⑦ reduces to

$c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt) = 0$.

which is satisfied when $\sin pl = 0$ or $pl = n\pi$

and $p = \frac{n\pi}{l}$, when $n = 1, 2, 3 \dots$

i.e. the soln of the wave eqn satisfying boundary condition is

$$y = c_2 \left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

$$= \left[a_n \cos \left(\frac{n\pi c t}{l} \right) + b_n \sin \left(\frac{n\pi ct}{l} \right) \right) \sin \frac{n\pi x}{l}$$

where $c_2 c_3 = a_n$, $c_2 c_4 = b_n$.

Adding up the soln for different values of n we get

$$y = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \rightarrow (8)$$

is also a solution.

Now applying the initial condition i.e
$y = f(x)$ & $\frac{\partial y}{\partial t} = 0$, when $t = 0$.

We have, $$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \rightarrow (9).$$

$$0 = \sum_{n=1}^{\infty} \frac{n\pi c}{l} b_n \sin \frac{n\pi x}{l} \rightarrow (10).$$

$\left(\because f(x) = y \; \& \; \frac{\partial y}{\partial t} = 0 \right)$

~~from~~ eqn (9) that represents fourier series of $f(x)$, we have

$$a_n = \frac{2}{l} \int_0^l f(x) \cdot \sin \frac{n\pi x}{l} \, dx \rightarrow (11)$$

From (10),

$$b_n = 0, \quad \forall \text{'n'}$$

Substitute in eqn (8) we have

$$y = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi ct}{l} \cdot \sin \frac{n\pi x}{l} \rightarrow (12)$$

where a_n is given by eqn (11) and when $f(x)$ i.e $y(x, 0)$ is known.

Ex:- A tightly stretched string of length l with fixed ends is initially in the equilibrium position. It is set vibrating by giving each point a velocity $V_0 \sin^3 \frac{\pi x}{l}$. Find the displacement $y(x,t)$.

→ Here $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

Here boundary conditions are $y(0,t) = 0$ & $y(l,t) = 0$.

And initial conditions $y(x,0) = 0$ & $\left(\frac{\partial y}{\partial t}\right)_{t=0} = V_0 \sin^3 \frac{\pi x}{l}$

we have $y(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \times \sin \frac{n\pi x}{l} \longrightarrow (1)$

Since the string was at rest initially i.e $y(x,0) = 0$.

$\Rightarrow y(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$

$\Rightarrow$

$a_n = 0 \quad \because \sin \frac{n\pi x}{l} \neq 0.$

$\Rightarrow y(x,t) = \sum b_n \sin \frac{n\pi ct}{l} \cdot \sin \frac{n\pi x}{l} \longrightarrow (2)$.

Differentiate partially with respect to t.

$\frac{\partial y}{\partial t} = \frac{\pi c}{l} \sum_{n=1}^{\infty} n b_n \cdot \cos \frac{n\pi ct}{l} \times \sin \frac{n\pi x}{l}$

But $\frac{\partial y}{\partial t} = V_0 \cdot \sin^3 \frac{\pi x}{l}$.

$\Rightarrow V_0 \sin^3\left(\frac{\pi x}{l}\right) = \frac{\pi c}{l} \sum_{n=1} n b_n \cdot \sin \frac{n\pi x}{l}$

we have

(using $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
$\sin^3\theta = \frac{1}{4}[3\sin\theta - \sin 3\theta$

$\frac{V_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$

$= \frac{\pi c}{l} \sum_{n=1}^{\infty} n b_n \sin \frac{n\pi x}{l}$

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$$\frac{V_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] = \frac{\pi c}{l}\sum_{n=1}^{\infty} n\, b_n \sin\frac{n\pi x}{l}$$

$$= \frac{\pi c}{l} b_1 \sin\frac{\pi x}{l} + \frac{2\pi c}{l} b_2 \sin\frac{2\pi x}{l} + \frac{3\pi c}{l} b_3 \sin\frac{3\pi x}{l} + \cdots$$

Equating the coefficients from both sides we have:

$$\frac{3V_0}{4}\cancel{\sin\frac{\pi x}{l}} = \frac{\pi c}{l} b_1 \cancel{\sin\frac{\pi x}{l}}$$

$$0 = \frac{2\pi c}{l} b_2 \sin\frac{2\pi x}{l}$$

$$-\frac{V_0}{4} = \frac{3\pi c}{l}\cdot b_3 \cdots$$

$$\therefore\ b_1 = \frac{3l}{4\pi c}\cdot V_0 \qquad b_2 = 0,$$

$$b_3 = \frac{-l}{12\pi c}\cdot V_0 \quad \cdots$$

Substituting in (2) we get

$$y(x,t) = \frac{3l}{4\pi c} V_0 \sin\frac{\pi ct}{l}\sin\frac{\pi x}{l} - \frac{lV_0}{12\pi c}\sin\frac{3\pi ct}{l}\sin\frac{3\pi x}{l}$$

Q:- A string is stretched and fixed (fastened) to 2 fixed points, distance l apart. Motion is started by displacing the string to form $y = \mu x(l-x)$ from which it is released at time $t=0$. Find the displacement y of any point of the string at a distance x from one end at time t.

→ Here the initial displacement is $f(x) = \mu x(l-x)$ & the initial velocity $g(x) = 0$. As such the required displacement is $y = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi c}{l}\right)t \cdot \sin\frac{n\pi x}{l}$ → (1)

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where $a_n = \frac{2}{l}\int_0^l \{\mu x(l-x)\} \sin\frac{n\pi x}{l}\,dx$

$$a_n = \frac{2\mu}{l}\left[x(l-x)\left\{-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right\}\right]_0^l - \int_0^l (l-2x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)dx\Big]$$

$$a_n = \frac{2\mu}{n\pi}\left[\int_0^l (l-2x)\cos\frac{n\pi x}{l}\,dx\right.$$

$$= \frac{2\mu}{n\pi}\left[\left[(l-2x)\frac{l}{n\pi}\sin\left(\frac{n\pi x}{l}\right)\right]_0^l - \int_0^l(-2)\frac{l}{n\pi}\sin\frac{n\pi x}{l}\,dx\right.$$

$$= \frac{4\mu l}{n^2\pi^2}\int_0^l \sin\frac{n\pi x}{l}\,dx.$$

$$= -\frac{4\mu l^2}{n^2\pi^3}\left\{\cos\frac{n\pi . x}{l}\right\}_0^l$$

$$= \frac{-4\mu l^2}{n^3\pi^3}\{\cos n\pi - 1\} = \frac{4\mu l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

Using in (1)

$$y(x,t) = \frac{4\mu l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{[1-(-1)^n\}}{n^3}\sin\frac{n\pi x}{l}.\cos\frac{n\pi c}{l}$$

Eu:- A string is stretched and fastened to 2 fixed end points l distance apart, if the motion is started by displacing the string in the form $y = a.\sin\frac{\pi x}{l}$ from which it is released at time $t=0$. Show that the displacement of any point at a string at a distance x from one end is given by

$$y(x,t) = a\sin\frac{\pi x}{l}.\cos\frac{\pi ct}{l}$$

$\Rightarrow$ $y(x,t) = \sum_{n=1}^{\infty} a_n\cos\frac{n\pi ct}{l}.\sin\frac{n\pi x}{l}$

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where $a_n = \frac{2}{l}\int_0^l f(x)\cdot \sin\frac{n\pi x}{l}\,dx$.

and for $f(x) = u(x,0)$, Here $u(x,0) = a\sin\frac{\pi x}{l}$

$$\therefore a_n = \frac{2a}{l}\int_0^l a\sin\frac{n\pi x}{l}\cdot \sin\frac{n\pi x}{l}\,dx$$

which vanishes for all values of 'n' except $n=1$

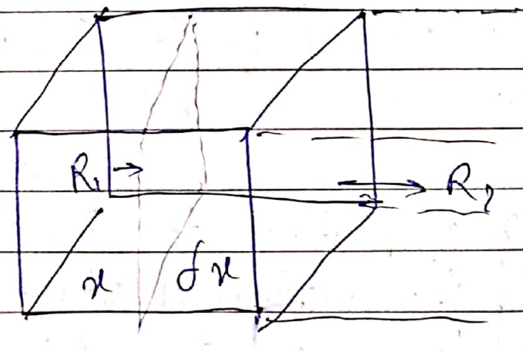
$$a_1 = \frac{2a}{l}\int_0^l \sin^2\frac{\pi x}{l}\,dx = \frac{a}{l}\int_0^l \left(1-\cos\frac{2\pi x}{l}\right)dx$$

$$= \frac{a}{l}\left[x - \frac{l}{2\pi}\sin\frac{2\pi x}{l}\right]_0^l = \frac{a}{l}(l) = a.$$

Hence $y(x,t) = a_1\cos\frac{n\pi ct}{l}\sin\frac{\pi x}{l}$

$$= a\cos\frac{n\pi t}{l}\cdot\sin\frac{\pi x}{l}$$

One-dimensional heat flow.



Consider the homogeneous path of the uniform cross section α (cm^2). Suppose that sides are covered with the material improvised to read so that stream lines of the head flow are all parallel & $\perp^r$ to the area α.
Take one and end of path at the origin and direction of flow has +ve x axis.

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Let ρ be the density, s specific heat and k thermal conductivity,
Let $u(x,t)$ be the temp at the distance x from 0. If δu be the temp change in a slab of thickness δx of the path.

The fundamental principles involved in the problem of the head conduction are
i) heat flows from higher temperature to the lower temperature
ii) The quantity of the heat in a body is proportional to its mass and temp
iii) The rate of heat flow across the area is proportional to the area and rate of change of temp with respect to its distance normal to area i.e

$$R_1 = -k\alpha\left(\frac{\partial u}{\partial x}\right)_x, \quad R_2 = -k\alpha\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

(-ve sign is bcz as x increases u decreases)
Hence the rate of increase of heat in this slab is
$R_1 - R_2 = \rho \cdot \alpha \cdot s \cdot \delta x \dfrac{\partial u}{\partial t}$. where s is specific heat.

R_1 and R_2 are respectively inflow and outflow of the heat.

$$\Rightarrow s\rho\alpha\,\delta x\frac{\partial u}{\partial t} = -k\alpha\left(\frac{\partial u}{\partial x}\right)_x + k\alpha\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$\Rightarrow s\rho\alpha\frac{\partial u}{\partial t} = \frac{k\alpha\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - k\alpha\left(\frac{\partial u}{\partial x}\right)_x}{\delta x}$$

as $\delta x \to 0$

$$s\rho\alpha\frac{\partial u}{\partial t} = k\alpha\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = c^2 \cdot \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \frac{k}{s\rho},$$

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This is called one dimensional heat flow eqn, where c^2 is called the diffusivity of the material of path.

Solution of heat equation:

We have $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \rightarrow (1)$.

Let us solve this p.d.e by method of separation of variables.

Let $u = XT$, $X = X(x)$ and $T = T(t)$

$\therefore \frac{\partial u}{\partial t} = XT'$, $\frac{\partial u}{\partial x} = X'T$, $\frac{\partial^2 u}{\partial x^2} = X''T$

$\therefore$ eqn (1) becomes

$XT' = c^2 X'' T$.

~~$XT' - c^2 X'' T = 0$~~

$$\frac{X''}{X} = \frac{T'}{T} \times \frac{1}{c^2}$$

equating to Common Constant k.

We have $X'' = kX$

$X'' - kX = 0$

this is 2nd order diff eqn

$\frac{\partial}{\partial x} = D$

$\therefore (D^2 - k)X = 0$

A.E: $D^2 - k = 0$

$D^2 = k$

$\therefore D = \pm\sqrt{k}$

$\frac{1}{c^2}\frac{T'}{T} = k$

$\frac{T'}{T} = kc^2$

$\frac{dT}{dt}\left(\frac{1}{T}\right) = kc^2$

$\int \frac{dT}{\cancel{dt}}\left(\frac{1}{T}\right) = \int kc^2\, dt$

$\log T = kc^2 t + \log c$

$T = c_3 e^{c^2 kt}$

case 1: when k is +ve and equal to p^2

$X = (c_1 e^{px} + c_2 e^{-px})$, $T = c_3 e^{c^2 p^2 t}$

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case 2 : when k is $-ve$ and equal to $-p^2$
$X = C_1 \cos px + C_2 \sin px$, $T = C_3 e^{-c^2p^2t}$.

case 3 : when k is zero.
$X = (C_1 + C_2 x)$, $T = C_3$

$\therefore$ The possible soln of eqn (1) are
$u = (C_1 e^{px} + C_2 e^{-px})(C_3 e^{c^2p^2t}) \rightarrow$ (2)
$u = (C_1 \cos px + C_2 \sin px)(C_3 e^{-c^2p^2t}) \rightarrow$ (3)
$u = (C_1 + C_2 x)(C_3) \longrightarrow$ (4).

In (2) as temp $\uparrow$, u is also increasing bt in reali-ty it is opp. and (4) is linear.

Out of this 3 soln eqn (3) will be suitable
$\therefore$ $u = (C_1 \cos px + C_2 \sin px)(C_3 e^{-c^2p^2t})$
Since u decreases as the time t increases

Ex:- Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with initial condition
$u(x,0) = 3 \sin \frac{n\pi x}{l}$ and $u(0,t) = 0$, $u(l,t) = 0$
$0 < x < l$, $t > 0$

$\rightarrow$ The suitable soln is
$u(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 e^{-c^2p^2t}) \rightarrow$ (1)

By initial condition $u(0,t) = 0$.
$\Rightarrow C_1 e^{-c^2p^2t^2} = 0$.
$\therefore C_1 = 0$

And $u(l,t) = 0$
$\Rightarrow (C_1 \cos pl + C_2 \sin pl) e^{-c^2p^2t} = 0$ (Substitute $l = x$ and $C_1 = 0$)
$e^{-c^2p^2t} \cdot C_2 \sin pl = 0$.
$\sin pl = 0$ $\Rightarrow pl = n\pi$
$p = n\pi/l$

Substituting In eqn (1) we have

10 or 8 m

Ex:- An insulated rod of length 'l' or its ends A & B are ~~melted~~ maintaind at 0°C and 100°C respectively until study state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.

→ The heat eqn is given by

$$\frac{\partial u}{\partial t} = c^2 \cdot \frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$$

Before there is a change of temp at B, there is a steady state condition i.e temp depends only on x, not on time t. i.e eqn (1) becomes

$$\frac{\partial^2 u}{\partial x^2} = 0.$$

$\Rightarrow m^2 = 0$

i.e $m = 0, 0$.

$\therefore u = ax + b$ where a & b are arbitory constants.

when $x = 0$, $u = 0$.

$b = 0$

when $x = l$, $u = 100$, $100 = al$

$a = \frac{100}{l}$.

$\therefore$ The initial condition is expressed by

$$u(x,0) = \frac{100}{l} x.$$

$\therefore$ The boundary conditions are $u(0,t) = 0$ & $u(l,t) = 0$.

$\therefore$ The suitable soln is $u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cdot e^{-\frac{c^2 n^2 \pi^2}{l^2} t}$.

By initial condition

$$u(x,0) = \sum_{n=1}^{\infty} b_n \cdot \sin\frac{n\pi x}{l}$$

which is half range sine series.

where $b_n = \frac{2}{l}\int_0^l u(x,0)\cdot \sin\frac{n\pi x}{l}\,dx$.

$$b_n = \frac{2}{l}\int_0^l \frac{100x}{l}\cdot \sin\frac{n\pi x}{l}\,dx.$$

$$= \frac{2}{l^2}\int_0^l 100x\left[\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right] - (100)\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)$$

put $x = l$ and $x = 0$

$$= \frac{2}{l^2}\left[-100l^2\left(\frac{\cos n\pi}{n\pi}\right)\right]$$

$$= \frac{200}{n\pi}\left[-(-1)^n\right] = \frac{200}{n\pi}\left[(-1)^{n+1}\right]$$

$\therefore$ The temp is

$$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}\cdot e^{-\frac{c^2n^2\pi^2}{l^2}t}$$

Ex:- Solve $\frac{\partial u}{\partial t} = c^2\cdot\frac{\partial^2 u}{\partial x^2}$, $0 \le x \le 1$.

$u(0,t) = u(1,t) = 0$ and $u(x,0) = (x - x^2)$.

$\rightarrow$ The suitable soln is $u(x,t) = (c_1\cos px + c_2\sin px)c_3e^{-c^2p^2t}$

using Boundary Conditions

$u(0,t) = 0 \Rightarrow u(0,t) = c_1c_3e^{-c^2p^2t}$.

$\Rightarrow c_1 = 0$. ($\because c_3$ can't be zero)

$u(1,t) = 0 \Rightarrow u(1,t) = (c_2\sin p)(c_3e^{-c^2p^2t}$

$\Rightarrow \sin p = 0 \Rightarrow p = n\pi$

$\therefore u(x,t) = (C_2 \sin n\pi x)\, C_3 e^{-c^2p^2t}$.

$u(x,t) = a_n \sin n\pi x\, e^{-c^2p^2t}$.

Adding for different values of n we have

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin \pi n x\, e^{-c^2p^2t}.$$

using initial condition we have $u(x,0) = \sum_{n=1}^{\infty} a_n \sin n\pi x$

which is half range fourier sine series

where $a_n = \frac{2}{l}\int_0^l u(x,0) \cdot \sin \frac{n\pi x}{l}\, dx$

$\therefore a_n = 2\int_0^1 (x - x^2) \sin n\pi x\, dx \quad (\because l = 1)$

$$a_n = 2\left[(x - x^2)\left(\frac{-\cos n\pi x}{n\pi}\right) - \frac{\sin n\pi x}{n^2\pi^2}(1-2x)\right]_0^1$$

$$\cancel{a_n} = \cancel{2}\left[\quad + (-2)\frac{\cos n\pi x}{n^3\pi^3}\right]_0^1$$

$$a_n = 2\left[+\frac{\sin n\pi}{n^2\pi^2} - \frac{2\cos n\pi}{n^3\pi^3} + \frac{2\cos 0}{n^3\pi^3}\right]$$

$$a_n = 2\left[\frac{-2\cos n\pi}{n^3\pi^3} + \frac{2}{n^3\pi^3}\right]$$

Take out -2 outside

$$a_n = -4\left[\frac{\cos n\pi}{n^3\pi^3} - \frac{1}{n^3\pi^3}\right]$$

$\cancel{a_n}$ $\therefore$ Required temp is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{n^3\pi^3}\left[(-1)^n \mp 1\right] \sin n\pi x \cdot e^{-c^2p^2t}.$$

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Ex:- A rod of length 1 with insulated sides is initial at uniform temp u_0. Find the temp $u(x,t)$

→ Given : $\frac{\partial u}{\partial t} = c^2 \cdot \frac{\partial^2 u}{\partial x^2}$ and Boundary Condn are $u(0,t) = u(1,t) = 0$.

Initial cond $u(x,0) = u_0$

∴ Suitable soln is $u(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2p^2t}$

using Boundary Condn

$u(x,t) = u(0,t) = c_1 c_3 e^{-c^2p^2t} \Rightarrow c_1 = 0$

$= u(1,t) = (c_2 \sin p) c_3 e^{-c^2p^2t}$

$\Rightarrow \sin p = 0 \Rightarrow p = n\pi$.

∴ $u(x,t) = (c_2 \sin n\pi x) c_3 e^{-c^2p^2t}$.

$u(x,t) = a_n \sin n\pi x \, e^{-c^2p^2t}$.

Adding for diff values of n, we have

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin n\pi x \, e^{-c^2p^2t}.$$

Now using initial condn $u(x,0)$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin n\pi x$$

which is half range fourier sine series where

$$a_n = \frac{2}{l} \int_0^l u(x,0) \sin \frac{n\pi x}{l} \, dx.$$

$$a_n = \frac{2}{1} \int_0^1 u_0 \sin n\pi x \, dx$$

$$a_n = 2u_0 \int_0^1 \sin n\pi x \, dx$$

$$a_n = 2u_0 \left[\frac{-\cos n\pi x}{n\pi} \right]_0^1$$

$$a_n = 2u_0\left[\frac{-\cos n\pi}{n\pi} + \frac{1}{n\pi}\right]$$

$$a_n = \frac{2u_0}{n\pi}\left[-(-1)^n + 1\right]$$

$\therefore$ The required temp is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2u_0}{n\pi}\left[1-(-1)^n\right] \sin n\pi x \, e^{-c^2p^2t}$$

∂

$\frac{\partial u}{\partial t} = 0$, and the above eqn reduces to

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, which is well known as Laplace eqn in 2 dimension.

Soln of Laplace Eqn :

We have $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \longrightarrow (1)$

Let us solve by separation of variables

$u = XY$ where $X = X(x)$ and $Y = Y(y)$.

Substituting in eqn ① we get

$$X''Y + XY'' = 0$$

$$X''Y = -XY''$$

$$\frac{X''}{X} = \frac{-Y''}{Y}$$

Equating to Common Constant k, we have

$X'' - kX = 0$ | $Y'' + kY = 0$

$(D^2 - k) = 0$ | $(D^2 + k) = 0$

$D^2 = k$ | $D^2 = -k$

$D = \pm\sqrt{k}$ | $D = \pm i\sqrt{k}$

Case 1 : when k is +ve, say $k = p^2$

$\therefore X = C_1 e^{px} + C_2 e^{-px}$ $\qquad Y = C_3 \cos py + C_4 \sin py$

case 2 : when k is -ve, say $k = -p^2$

$\therefore X = C_1 \cos px + C_2 \sin px$ $\qquad Y = C_3 e^{py} + C_4 e^{-py}$

case 3 : when k is 0.

$X = C_1 x + C_2$ $\qquad Y = C_3 y + C_4$

$\therefore$ The various possible solns are

$$u = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \rightarrow ①$$

$$u = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \rightarrow ②$$

$$u = (C_1 x + C_2)(C_3 y + C_4). \rightarrow ③$$

Out of these solns, we take that soln which is consis-tent with the boundary condns.

Note: ① For u to be non trivial soln, atleast one of C_1 and C_2, and atleast one of C_3 and C_4

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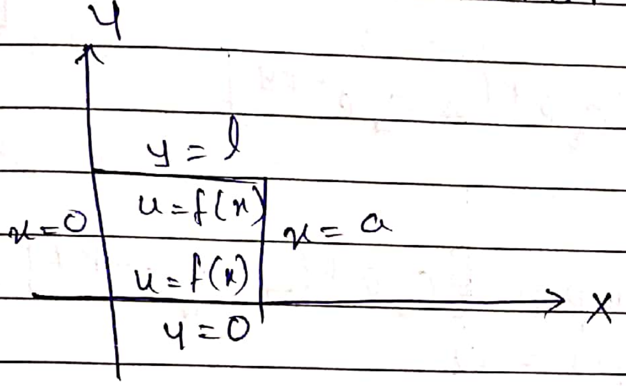
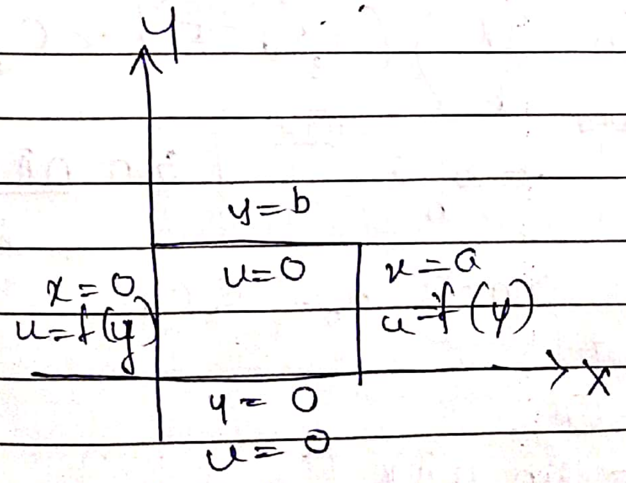
are to be non zero.

2) The simplest regions in which the eqn is solved are the regions bounded by straight lines parallel to the Coordinate axis.

3) The simplest boundary Condn is the one in which u is specified to be non zero on one of the boundary line and zero to the other boundary line.



If the boundary line on which u is specified to be non zero parallel to x axis (y constant), the soln of the form ~~which invo~~ eqn 2 which involves the cosine & sine fn of x is used.



If the boundary line on which u is specified to be non zero parallel to y axis (x constant), the soln of the form eqn 1 which involves the cosine & sine fn of y is used.

as x is varying ... sine of x^2 term is there

Ex:- Solve the laplace eqn $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the condn $u(0, y) = 0 = u(l, y)$ & $u(x, 0) = 0$ and $u(x, a) = \sin\frac{n\pi x}{l}$

$\rightarrow$ The suitable soln is $u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$ $\rightarrow$ (A)

when $u(0, y) = 0$

$\Rightarrow 0 = c_1 [c_3 e^{py} + c_4 e^{-py}]$

$c_1 = 0$

$u(l, y) = c_2 \sin pl \,[c_3 e^{py} + c_4 e^{-py}] = 0$.

$\Rightarrow \sin pl = 0$

$\therefore pl = n\pi$

$p = \frac{n\pi}{l}$

Substituting in eqn (A) we get

$u(x, y) = \left(c_2 \sin \frac{n\pi x}{l}\right)\left(c_3 e^{\frac{n\pi}{l}y} + c_4 e^{-\frac{n\pi}{l}y}\right)$

$u(x, y) = \left[a_n e^{\frac{n\pi y}{l}} + b_n e^{-\frac{n\pi y}{l}}\right] \sin\frac{n\pi x}{l}$ $\rightarrow$ (B)

where $a_n = c_2 c_3$ and $b_n = c_2 c_4$.

when $u(x, 0) = 0$.

~~$c_1 \cos px$~~ $0 = \sin\frac{n\pi x}{l}[a_n + b_n]$

$\Rightarrow a_n + b_n = 0$

$\therefore b_n = -a_n$

eqn (B) becomes.

$u(x, y) = \sin\frac{\pi n x}{l}\left[a_n e^{\frac{n\pi y}{l}} - a_n e^{-\frac{n\pi y}{l}}\right]$

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$$u(x,y) = a_n \sin\frac{n\pi x}{l}\left[e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right]$$

$$= a_n \sin\frac{n\pi x}{l}\left[2\sinh\frac{n\pi y}{l}\right]$$

Adding all these solns for different values of n we get

$$u(x,y) = \sum_{n=1}^{\infty} 2a_n \sinh\frac{n\pi y}{l} \cdot \sin\frac{n\pi x}{l}$$

when $u(x,a) = \sin\frac{n\pi x}{l}$

$$u(x,a) = \sum_{n=1}^{\infty} 2a_n \sinh\frac{n\pi a}{l}\sin\frac{n\pi x}{l}$$

$$\sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} a_n' \sin\frac{n\pi x}{l} \quad \text{where } a_n' = 2a_n \sinh\frac{n\pi a}{l}$$

$\Rightarrow a_n' = 1$

$$2a_n \sinh\frac{n\pi a}{l} = 1$$

$$a_n = \frac{1}{2\sinh\frac{n\pi a}{l}}$$

$\therefore$ The required soln is

$$u(x,y) = \sum_{n=1}^{\infty} \cancel{2} \times \frac{1}{\cancel{2}\sinh\frac{n\pi a}{l}} \cdot \sinh\frac{n\pi y}{l} \cdot \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} \frac{\sinh\frac{n\pi y}{l} \cdot \sin\frac{n\pi x}{l}}{\sinh\frac{n\pi a}{l}}$$

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Ex:- Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the condns

$u(0, y) = 0$ $u(x, 0) = 0$, $u(x, l) = 0$ and

$u(a, y) = \begin{cases} y, & 0 \le y \le l/2 \\ l-y, & l/2 \le y \le l \end{cases}$

→ The suitable soln is

$u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \to (A)$

when $u(x, 0) = 0$.

$0 = (c_1 e^{px} + c_2 e^{-px}) c_3$

$\Rightarrow c_3 = 0$,

when $u(x, l) = 0$.

$0 = (c_1 e^{px} + c_2 e^{-px})(c_4 \sin pl)$

$\sin pl = 0$

$pl = n\pi$

$p = \frac{n\pi}{l}$.

$\therefore$ eqn (A) becomes

$u(x, y) = (c_1 e^{\frac{n\pi x}{l}} + c_2 e^{\frac{-n\pi x}{l}})(c_4 \sin \frac{n\pi y}{l})$

$u(x, y) = \sin \frac{n\pi y}{l} (a_n e^{\frac{n\pi x}{l}} + b_n e^{\frac{-n\pi x}{l}})$

where $a_n = c_1 c_4$ and $b_n = c_2 c_4$

adding for diff values of n, we get

$u(x, y) = \sum_{n=1}^{\infty} \sin \frac{n\pi y}{l} [a_n e^{\frac{n\pi x}{l}} + b_n e^{\frac{-n\pi x}{l}}] \to (B)$.

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when $u(0,y) = 0$

$$u(0,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi y}{l}\,[a_n + b_n]$$

$\Rightarrow a_n + b_n = 0 \quad \therefore b_n = -a_n.$

Now eqn (3) becomes.

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi y}{l}\left[e^{\frac{n\pi x}{l}} - e^{-\frac{n\pi x}{l}}\right]$$

$$= \sum_{n=1}^{\infty} 2a_n \sin\frac{n\pi y}{l} \cdot \sinh \frac{n\pi x}{l}.$$

which is half range sine series with range $(0, l)$

$$\therefore 2a_n \cdot \sin\frac{n\pi a}{l} = \frac{2}{l}\int_0^l u(a,y) \cdot \sin\frac{n\pi y}{l}\,dy$$

$$\Rightarrow 2a_n \sin\frac{n\pi a}{l} = \frac{2}{l}\left[\int_0^{l/2} y \sin\frac{n\pi y}{l} + \int_{l/2}^{l} (l-y)\sin\frac{n\pi y}{l}\,dy\right]$$

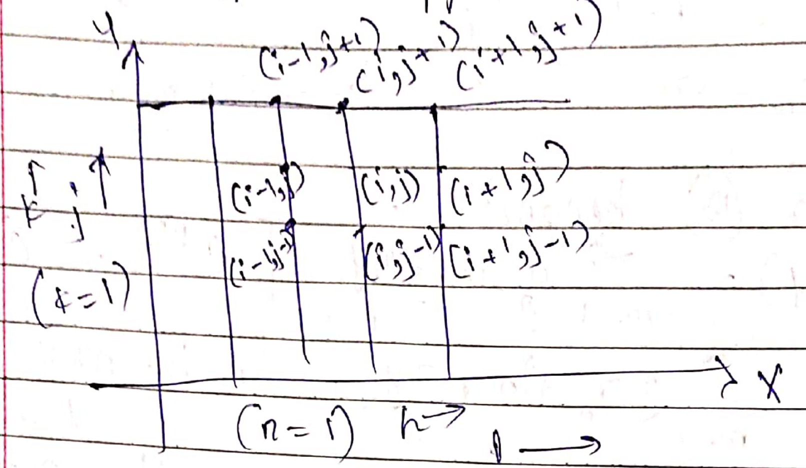
$$= \frac{2}{l}\left[\sin\frac{n\pi y}{l} - \frac{y\cos\frac{n\pi y}{l}}{\frac{n\pi}{l}}\right]_0^{l/2} + \left[-\sin\frac{n\pi y}{l} - (l-y)\frac{\cos\frac{n\pi y}{l}}{\frac{n\pi}{l}}\right]_{l/2}^{l}$$

$\frac{n\pi}{2l}$

$$= \frac{2}{l}\left[\sin\frac{n\pi\, l/2}{l} - \frac{y\cos\frac{n\pi\, l/2}{l}}{n\pi/l}\right] + \left\{\frac{y}{\frac{n\pi}{l}}\right.$$

solve it

Numerical soln to P.D.E:
Finite Difference approximation.



Here u is a fn of x, y i.e $u = f(x, y)$
$i \to x$ - scale, where $x_i = x_0 + ih$, $i = 1, 2 \ldots n$
$j \to y$ - scale, where $y_j = y_0 + jk$, $j = 1, 2 \ldots n$

The x, y plane is divided into series of rectangles of sides $\Delta x = h$, $\Delta y = k$
By equidistant line drawn ||rl to the axes of the co ordinates.
Since $u = f(x, y)$, the value of $u(x, y)$ at the point (i, j) is denoted by u_{ij}
$\therefore u(x, y) = u_{i,j}$
Hence the first finite difference approximation for the partial derivative is as follows.

$$u_x = \frac{u_{i+1,j} - u_{i,j}}{h} \quad \text{[forward difference]}$$

$$= \frac{u_{i,j} - u_{i-1,j}}{h} \quad \text{[backward difference]}$$

And $u_y = \frac{\partial u}{\partial y} = \frac{u_{i,j+1} - u_{i,j}}{k}$ [forward diff]

$$= \frac{u_{i,j} - u_{i,j-1}}{k} \quad \text{[backward diff]}$$

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Similarly 2^{nd} order diff is as follows.

$$\frac{\partial^2 u}{\partial x^2} = u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

$$\frac{\partial^2 u}{\partial y^2} = u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

And also we have

$$u_x = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

$$u_y = \frac{u_{i,j+1} - u_{i,j-1}}{2k}$$

The point (i,j) is called gritted point / mesh / lattice point. i.e points of intersection of these families of lines

Numerical soln for Laplace eqn [Elliptic eqn]:

Given: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \longrightarrow (1)$

Replacing the derivatives in (1) by finite differences we have

$$\left[\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\right] + \left[\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}\right] = 0$$

By taking square mesh i.e $h = k$ in the above eqn we get

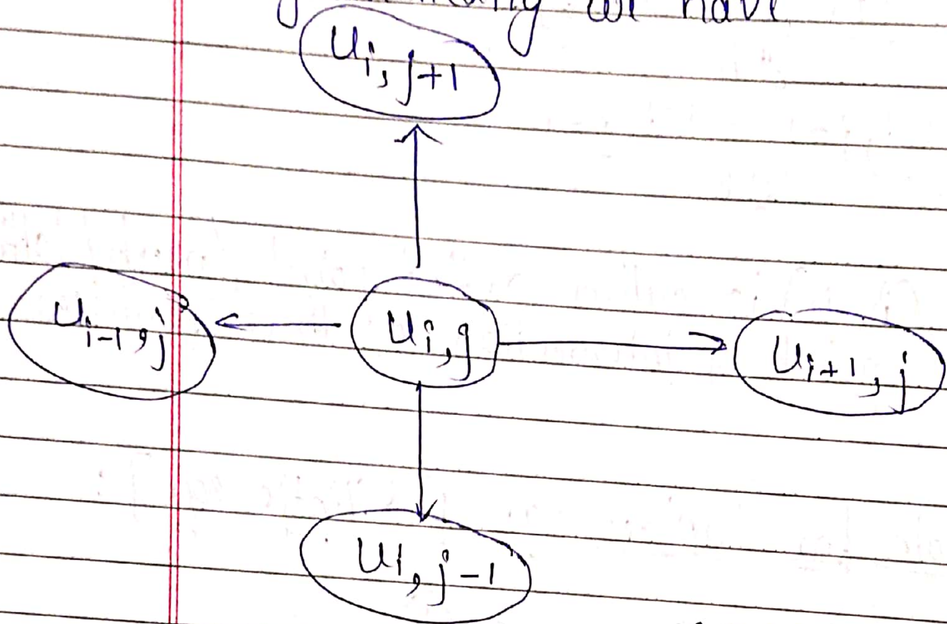
$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right] \rightarrow (2)$$

i.e the value of u at any interior point is the arithmetic mean of the values of u at 4 lattice points (neighbouring points). This is called standard

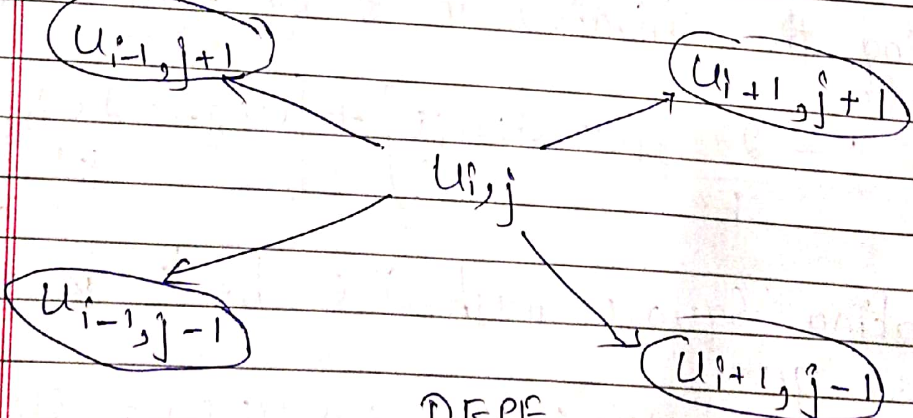
Five point formula (SFPF)
Instead of formula (2) we can use

$$u_{i,j} = \frac{1}{4}\left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1}\right] \quad (3)$$

These is called diagonal five point formula (DFPF)
Diagramatically we have

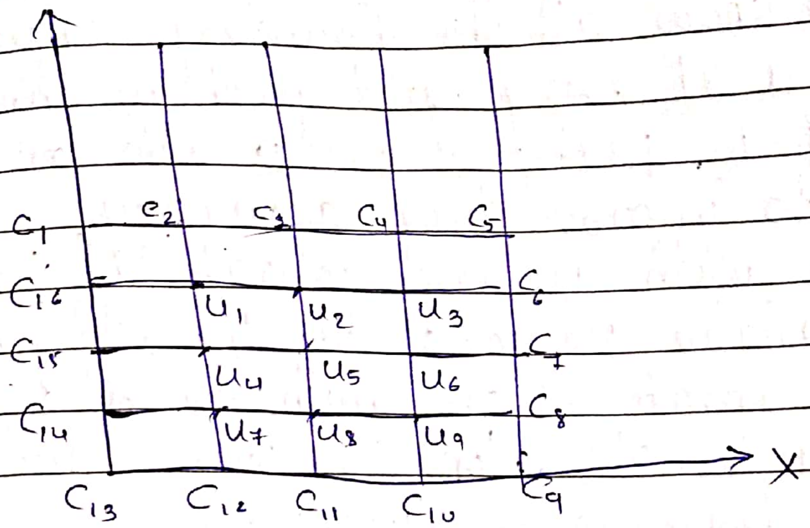


SFPF



DFPF

Liebmann's Iteration process:-



To solve laplace eqn $u_{xx} + u_{yy} = 0$ in a bounded square region R with a boundary C. let R be divided into network of small squares of side h. The boundary values of u at the grid points are given by $c_1, c_2 \ldots\ldots c_{16}$. The values of u at the interior points are given by $u_1, u_2 \ldots\ldots u_9$ we start the iteration process, initially we find the rough values at the interior points & then we imp-rove it by iteration process using SFPF

First we find u_5 at the centre of the square by taking SFPF. i.e $u_5 = \frac{1}{4}[c_{15} + c_3 + c_7 + c_{11}]$

Next, we find the initial values at the centres of the 4 large inner square using DFPF and the values of remaining interior points by SFPF.

DFPF

$u_1 = \frac{1}{4}[c_{15} + c_1 + c_3 + c_5]$
$u_3 = \frac{1}{4}[u_5 + c_3 + c_5 + c_7]$
$u_7 = \frac{1}{4}[c_{13} + c_{15} + u_5 + c_{11}]$
$u_9 = \frac{1}{4}[c_{11} + u_5 + c_7 + c_9]$

SFPF

$u_2 = \frac{1}{4}[u_1 + c_3 + u_3 + u_5]$
$u_4 = \frac{1}{4}[u_1 + u_5 + u_7 + c_{15}]$
$u_6 = \frac{1}{4}[u_5 + u_3 + u_7 + u_9]$
$u_8 = \frac{1}{4}[u_7 + u_5 + u_9 + c_{11}$

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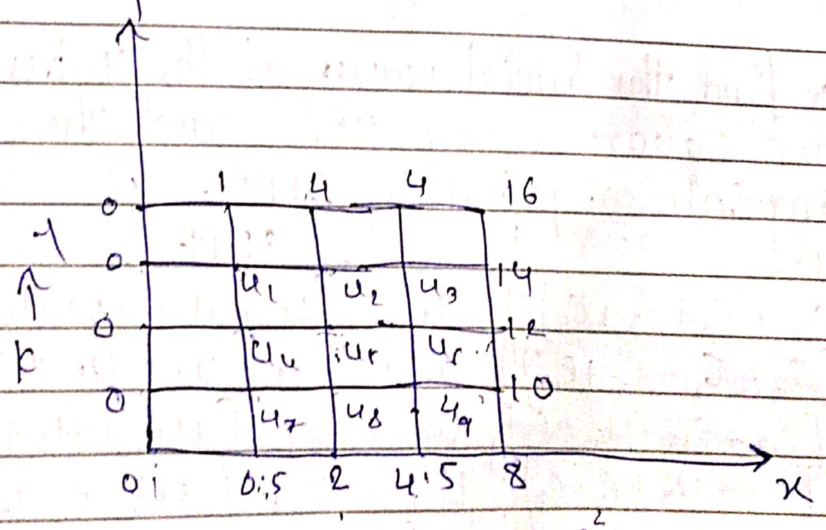
Now we know all the Boundary values and rough values of u at every grid point in the Interior region of R. Next, we proceed with an iteration process to improve their accuracy.
We start with u_1 and iterate it by using the lattest variable values of its 4 adjacent points. Thus we iterate all the mesh points systematically from left to right along successive rows. The iteration formula is given by

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n)} + u_{i,j-1}^{(n+1)} \right]$$

This is known as Liebmann's iteration process. This process is stopped once we get the values with desired accuracy.

1) ~~Uxx~~ Given that $u_{xx} + u_{yy} = 0$ in $0 \leq x \leq 4$, $0 \leq y \leq 4$, $u(0, y) = 0$, $u(4, y) = 8 + 2y$, $u(x, 0) = \frac{x^2}{2}$, $u(x, 4) = x^2$, $h = k = 1$.
Obtain the result correct to 1 decimal places, carry out 2 iterations.



$h \to x \quad \frac{2^2}{2} = 2$

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Let us divide the region R i.e $0 \le x \le 4$, $0 \le y \le 4$ into 16 square bits the numerical values of the boundary using the given analytical expressions are calculated & shown in the above diagram.

Let $u_1, u_2 \ldots\ldots u_n$ be the values of u at the interior lattice points, Now the initial values of u_i are calculated either by SFPF or DFPF. We have

$$u_5^{(0)} = \frac{1}{4}[4+12+0+2] = \frac{18}{4} = 4.5 \quad [SFPF]$$

$$u_1^{(0)} = \frac{1}{4}[0+4+4.5+0] = 2.125 \quad [DFPF]$$

$$u_3^{(0)} = \frac{1}{4}[4.5+4+16+12] = 9.125 \quad [DFPF]$$

$$u_9^{(0)} = \frac{1}{4}[8+2+4.5+12] = 6.625 \quad [DFPF]$$

$$u_7^{(0)} = \frac{1}{4}[0+0+4.5+2] = 1.625 \quad [DFPF]$$

$$u_2^{(0)} = \frac{1}{4}[4+4.5+9.125+2.125] = 4.937 \quad [SFPF]$$

$$u_4^{(0)} = \frac{1}{4}[0+2.125+4.5+1.625] = 2.063 \quad [SFPF]$$

$$u_6^{(0)} = \frac{1}{4}[12+4.5+9.125+6.625] = 8.062 \quad [SFPF]$$

$$u_8^{(0)} = \frac{1}{4}[6.625+4.5+1.625+2] = 3.6\cancel{78}5 \quad [SFPF]$$

Now we use Liebmann's iterative formula

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left\{u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}\right\}$$

when we use these formula at the points $u_1, u_2, \ldots u_9$, we get the following eqns for iterations.

$$u_1^{(n+1)} = \frac{1}{4}\left[0 + u_2^{(n)} + u_4^{(n)} + 1\right]$$

$$u_2^{(n+1)} = \frac{1}{4}\left[u_1^{(n+1)} + u_3^{(n)} + u_5^{(n)} + 4\right]$$

$$u_3^{(n+1)} = \frac{1}{4}\left[u_2^{(n+1)} + 14 + u_6^{(n)} + 9\right]$$

$$u_4^{(n+1)} = \frac{1}{4}\left[0 + u_5^{(n)} + u_7^{(n)} + u_1^{(n+1)}\right]$$

$$u_5^{(n+1)} = \frac{1}{4}\left[u_4^{(n+1)} + u_6^{(n)} + u_8^{(n)} + u_2^{(n+1)}\right]$$

$$u_6^{(n+1)} = \frac{1}{4}\left[u_5^{(n+1)} + 12 + u_9^{(n)} + u_3^{(n+1)}\right]$$

$$u_7^{(n+1)} = \frac{1}{4}\left[0 + u_8^{(n)} + 0.5 + u_4^{(n+1)}\right]$$

$$u_8^{(n+1)} = \frac{1}{4}\left[u_7^{(n+1)} + u_9^{(n)} + 2 + u_5^{(n+1)}\right]$$

$$u_9^{(n+1)} = \frac{1}{4}\left[u_8^{(n+1)} + 10 + 4.5 + u_6^{(n+1)}\right]$$

step1 : 1st iteration, put $n=0$ in the above eqn.

$$u_1^{(1)} = \frac{1}{4}\left[0 + u_2^{(0)} + u_4^{(0)} + 1\right] = \frac{1}{4}\left[0 + 4.93\right.$$
$$\left. + 2.063 + 1\right] = 2$$

$$u_2^{(1)} = \frac{1}{4}\left[u_1^{(1)} + u_3^{(0)} + u_5^{(0)} + 4\right] = 4.906$$

$$u_2^{(1)} = \frac{1}{4}\left[2 + 9.125 + 4.5 + 4\right] = 4.906$$

$u_3(1) = \frac{1}{4}[4.906 + 14 + 4.5 + 9] = 8.992$

$u_4(1) = \frac{1}{4}[0 + 2 + 4.5 + 1.625] = 2.031$

$u_5(1) = \frac{1}{4}[u_4^{(1)} + u_6^{(0)} + u_8^{(0)} + u_2^{(1)}] = 4.671$

$u_6^{(1)} = \frac{1}{4}[4.671 + 12 + 6.625 + 8.992] = 8.072$

$u_7(1) = \frac{1}{4}[0 + 2.031 + 3.687 + 0.5] = 1.555$

$u_8(1) = \frac{1}{4}(1.555 + 2 + 6.625 + 4.671] = 3.712$

$u_9^{(1)} = \frac{1}{4}[10 + 4.5 + 3.712 + 8.072] = 6.571$

2nd iteration (put n = 1)

$u_1^{(2)} = \frac{1}{4}[0 + 1 + 4.906 + 2.031] = 1.984$

$u_2^{(2)} = \frac{1}{4}[1.984 + 4 + 8.992 + 4.671] = 4.911$

$u_3^{(2)} = \frac{1}{4}[4.911 + 9 + 14 + 8.072] = 8.995$

$u_4^{(2)} = \frac{1}{4}[0 + 1.984 + 4.671 + 1.555] = 2.052$

$u_5^{(2)} = \frac{1}{4}[4.911 + 8.072 + 2.052 + 3.712] = 4.687$

$u_6^{(2)} = \frac{1}{4}[4.687 + 12 + 6.571 + 8.995] = 8.063$

$u_7^{(2)} = \frac{1}{4}[0 + 0.5 + 3.712 + 2.052] = 1.566$

$u_8^{(2)} = 1/4[1.566 + 2 + 6.571 + 4.687] = 3.706$

$u_9^{(2)} = 1/4[10 + 4.5 + 3.706 + 8.063] = 6.567$

Ex:- Solve the laplace eqn by applying 5 point formula which satisfies the flling Boundary condns

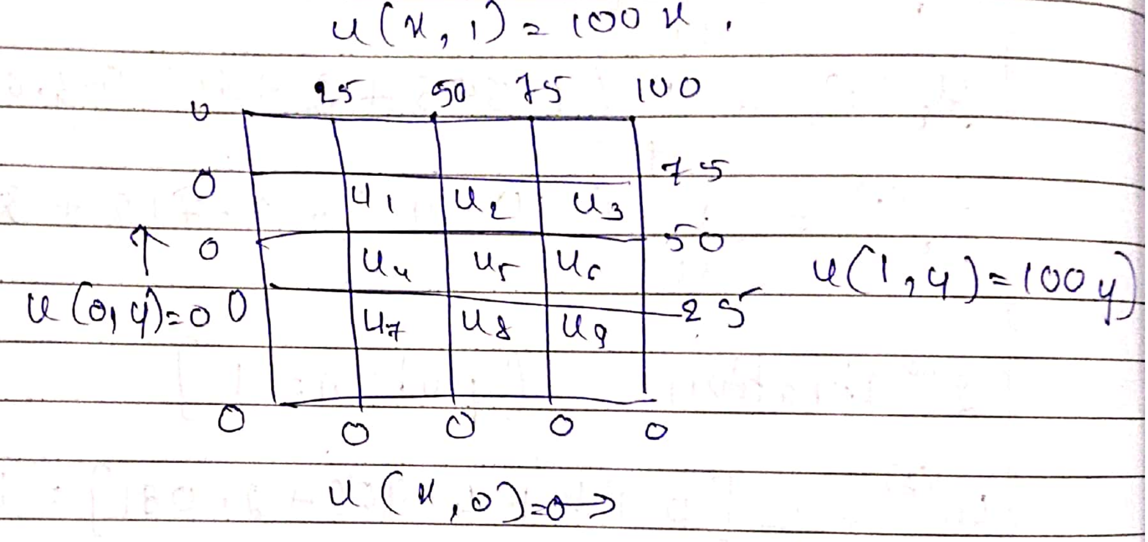
$u_{xx} + u_{yy} = 0$

$u(0, y) = 0$, $u(x, 0) = 0$, $u(x, 1) = 100x$, $u(1, y) = 100y$

from the soln of boundary condn the region of computation is a square bounded by $x \geq 0$, $y \geq 1$

Taking uniform bit spacing $\Delta x = \Delta y = 1/4 = h$

We have.



Now, initial values of u_i are calculated either by SFPF or DFPF.

$$u_5^{(0)} = \frac{1}{4}[50 + 50 + 0 + 0] = 25$$

$$u_1^{(0)} = \frac{1}{4}[\cancel{25 + 7} \; 0 + 50 + 25 + 0] = 18.75$$

$$u_3^{(0)} = \frac{1}{4}[50 + 100 + 50 + 25] = 56.25$$

$$u_7^{(0)} = \frac{1}{4}[0 + 25 + 0 + 0] = 6.25$$

$$u_9^{(0)} = \frac{1}{4}[50 + 25 + 0 + 0] = 18.75$$

$u_1^{(0)} = \frac{1}{4}[50 + 56.25 + 18.75 + 25]$

$= 37.5$

$u_4^{(0)} = \frac{1}{4}[25 + 18.75 + 0 + 6.25] = 12.5$

$u_6^{(0)} = \frac{1}{4}[50 + 56.25 + 25 + 18.75] = 37.5$

$u_8^{(0)} = \frac{1}{4}[0 + 18.75 + 25 + 6.25] = 12.5$

Now we use Liebmann's iterative formula

$$u_{ij}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n} + u_{i,j+1}^{(n+1)}\right]$$

$u_1^{(n+1)} = \frac{1}{4}[0 + 25 + u_2^{(n)} + u_4^{(n)}]$

$= \frac{1}{4}[0 + 25 + 37.5 + 12.5] = 18.750$

$u_2^{(1)} = \frac{1}{4}[50 + 18.750 + 56.25 + 25] = 37.5$

$u_3^{(1)} = \frac{1}{4}[75 + 75 + 37.5 + 37.5] = 56.250$

$u_4^{(1)} = \frac{1}{4}[0 + 18.75 + 25 + 6.25] = 12.5$

$u_5^{(1)} = \frac{1}{4}[37.5 + 12.5 + 37.5 + 12.5] = 25.0$

$u_6^{(1)} = \frac{1}{4}[50 + 56.25 + 25 + 18.75] = 37.5$

$u_7^{(1)} = \frac{1}{4}[0 + 0 + 12.5 + 12.5] = 6.25$

$u_8^{(1)} = 1/4[25 + 18.75 + 0 + 6.25] = 12.5$

$u_9^{(1)} = 1/4[0 + 25 + 37.5 + 12.5] = 18.75$

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$u_1^{(2)} = \frac{1}{4}[25 + 0 + 37.5 + 12.5] = 18.75$

$u_2^{(2)} = 1/4\,[50 + 18.75 + 56.25 + 25] = 37.5$

$u_3^{(2)} = 1/4\,[75 + 75 + 37.5 + 37.5] = 56.25$

$u_4^{(2)} = 1/4\,[0 + 18.75 + 6.25 + 25] =$

$u_5^{(2)} = 1/4\,[$

$u_6^{(2)}$

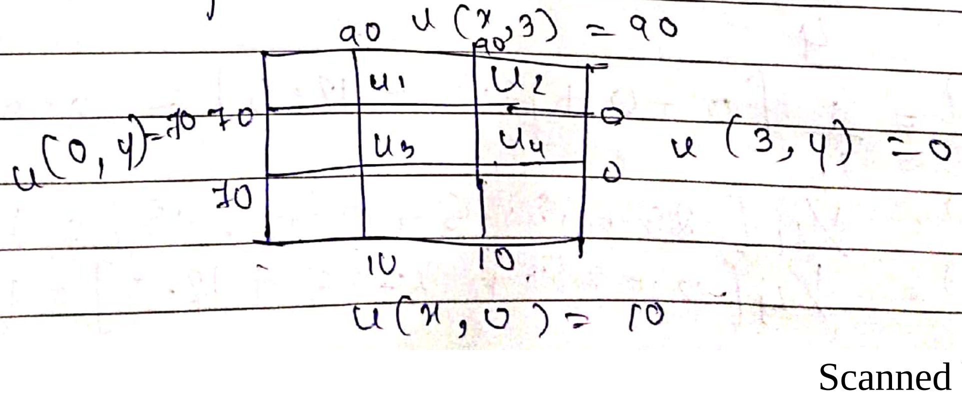
$u_7^{(2)}$

$u_8^{(2)}$

$u_9^{(2)}$

Ex:- Determine the system of 4 eqns in 4 unknowns u_1, u_2, u_3, u_4 for computing approximation for harmonic fns, ~~tapt~~ $u(x, y)$ in a rectangle $R = \{(x, y) ; 0 \leq x \leq 3,\ 0 \leq y \leq 3\}$ and the Boundary condns are $u(x, 0) = 10$; $u(x, 3) = 90$, $0 \leq x \leq 3$, $u(0, y) = 70$, $u(3, y) = 0$, $0 \leq y \leq 3$.

→ We have from above data



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We have $u_1 = \frac{1}{4}[70 + 90 + u_2 + u_3]$

$u_2 = \frac{1}{4}[u_1 + 90 + 0 + u_4]$

$u_3 = \frac{1}{4}[70 + u_1 + u_4 + 10]$

$u_4 = \frac{1}{4}[u_3 + u_2 + 0 + 10]$

System of 4 eqns in '4' unknowns.

Assume $u_4^{(0)} = 0$

$u_2 = (u_1 + 90)/4$

$4u_2 = u_1 + 90$.

$u_3 = (u_1 + 80)/4 \Rightarrow 4u_3 = u_1 + 80$,

$u_1 = \frac{1}{4}\left(70 + 40 + \frac{1}{4}[u_1 + 90] + \frac{1}{4}[u_1 + 80]\right)$

$16u_1 = 280 + 360 + u_1 + 90 + u_1 + 80$

$16u_1 = 810 + 2u_1$

$14u_1 = 810$

$u_1^{(0)} = 57.8571$

$u_2^{(0)} = \frac{1}{4}[57.8571 + 90] = 36.964$

$u_3^{(0)} = \frac{1}{4}[57.8571 + 70 + 10] = 34.464$

By applying Liebmann's iteration formula

$u_{i,j}^{(n+1)} = \cancel{u_{i-1,j}^{(n)} + u}$

$u_1^{(n+1)} = \frac{1}{4}[90 + 70 + u_2^{(n)} + u_3^{(n)}]$

$= \frac{1}{4}[90 + 70 + 36.964 + 34.464] = 57.857$

$u_2^{(n+1)} = \frac{1}{4}[90 + 0 + u_1^{(n+1)} + u_4^{(n)}]$

$= \frac{1}{4}[90 + 0 + 57.857 + 0] = 36.964$

$u_3^{(n+1)} = \frac{1}{4}[70 + 10 + u_1^{(n+1)} + u_4^{(n)}]$

$= \frac{1}{4}[70 + 10 + 57.857 + 0] = 34.464$

$u_4^{(n+1)} = \frac{1}{4}[0 + 10 + u_2^{(n+1)} + u_3^{(n+1)}]$

$= \frac{1}{4}[0 + 10 + 36.964 + 34.464] = 20.357$

$u_1^{(2)} = \frac{1}{4}[90 + 70 + 36.964 + 34.464] = 57.856$
$u_2^{(2)} = \frac{1}{4}[0 + 90 + 57.856 + 20.357] = \cancel{39.5}\ 42.05$
$u_3^{(2)} = \frac{1}{4}[70 + 10 + 20.357 + 57.856] = 39.553$
$u_4^{(2)} = \frac{1}{4}[0 + 10 + 39.553 + 42.053] = 22.902$

Practice Ex 17 of 28 1, 3, Solving (direct integration, variable separable, lagranges) → 4 (direct), 5 (assume z is fn of x alone), 6 (lagranges), 7 (variable separable)
one dimensional wave eqn (soln & derivation) → 8, 9

15 of 28 → 12, 13, 14 one dimensional wave eqn, one dimensional heat eqn (soln & derivation) 17, 16, 19
20, 21, 22 (laplace), numerical method

p.g no 20 of 28 → 10, 11
21 of 28 → 6, 7

Wave eqn (Hyperbolic)
We have $\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2} \longrightarrow (A)$.
Substituting u_{xx} and u_{tt} in terms of finite difference, we get

$$u_{i,j+1} = 2(1-\lambda^2 c^2) \cdot u_{i,j} + \lambda^2 c^2 [u_{i+1,j} + u_{i-1,j}] - u_{i,j-1} \longrightarrow (B)$$

where $\lambda = k/h$.
This is called explicit formula
If $\lambda^2 = 1/c^2$, then we get,

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \longrightarrow (C)$$

This is called recurence relation for wave eqn

Note

For $\lambda = 1/c$, the soln for the wave eqn is stable
For $\lambda > 1/c$, the soln is unstable.

Formula B converges for $\lambda \le 1$ (i.e $k \le h$) & we have $\left(\frac{\partial u}{\partial t}\right)_{(x,0)} = 0$ or $u_t(x,0) = 0$

$u_{i,j+1} = u_{i,j-1}$

put $j = 0$, subn in (B)

$u_{i,1} = 2(1-\lambda^2c^2)u_{i,0} + \lambda^2c^2[u_{i+1,j} + u_{i-1,j}] - u_{i,1}$

$\therefore u_{i,1} = (1-\lambda^2c^2)u_{i,0} + \frac{1}{2}\lambda^2c^2(u_{i+1,j} + u_{i-1,j}) \rightarrow$ (D)

Eqn (B) will generate 2nd row of the table

Ex:- The transfers displacement u of a particle of distance x from one end and at any time 't' of a vibrating string satisfying the eqn

(1) $\frac{\partial^2 u}{\partial t^2} = 4 \cdot \frac{\partial^2 u}{\partial x^2}$ with boundary conditions

$u = 0$ at $x = 0$, $t > 0$ & $u = 0$ at $x = 4$, $t > 0$ & initial conditions $u = x(4-x)$ & $\partial u/\partial t = 0$ at $t = 0$, $0 \le x \le 4$. Solve this eqn numerically for half period of vibration by taking $h = 1$ & $k = 1/2$.

→ We have $a^2 = 4$, $a = 2$

Period of vibration $= \frac{2l}{a} = \frac{2 \times 4}{2} = 4$

We find till $t = 2$

Given $h = 1$ and $k = 1/2$

$\therefore \lambda = k/h = 1/2$, $\lambda^2 = 1/c^2 = 1/4$

$\therefore$ We use recurrance relation which is given by

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \longrightarrow (A)$$

By using $u_t(x,0) = 0$.

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0.$$

$$\boxed{u_{i,j+1} = u_{i,j-1}}.$$

Put $j=0$ we have $u_{i,1} = u_{i,-1} \longrightarrow (1)$.

At $j=0$ in eqn (A) we have

$$u_{i,1} = u_{i+1,0} + u_{i-1,0} - u_{i,-1} \longrightarrow \cancel{(2)}$$

$$\Rightarrow u_{i,1} = u_{i+1,0} + u_{i-1,0} - u_{i,1} \quad (\text{from } (1))$$

$$\Rightarrow 2u_{i,1} = u_{i+1,0} + u_{i-1,0}$$

$$\Rightarrow u_{i,1} = \frac{1}{2}\left[u_{i+1,0} + u_{i-1,0}\right] \longrightarrow (2)$$

$h=1$

$t = 1/2$ $j \downarrow$ $i \longrightarrow$

$t \backslash x$	0	1	2	3	4
0	0 $u_{0,0}$	3 $u_{1,0}$	4 $u_{2,0}$	3 $u_{3,0}$	0
0.5	0 $u_{0,1}$	2 $u_{1,1}$	3 $u_{2,1}$	2 $u_{3,1}$	0 $u_{4,1}$
1	0 $u_{0,2}$	0 $u_{1,2}$	0 $u_{2,2}$	0 $u_{3,2}$	0 $u_{4,2}$
1.5	0 $u_{0,3}$	-2 $u_{1,3}$	-3 $u_{2,3}$	-2 $u_{3,3}$	0
2	0	-3	-4	-3	0

To generate 2nd row we use

$$u_{i,1} = \frac{1}{2}\left[u_{i+1,0} + u_{i-1,0}\right]$$

put $i = 1, 2, 3$ in (2)

$$u_{1,1} = \frac{1}{2}\left[u_{2,0} + u_{0,0}\right] = \frac{1}{2}[4+0] = 2$$

$$u_{2,1} = \frac{1}{2}\left[u_{3,0} + u_{1,0}\right] = \frac{1}{2}[3+3] = 3$$

$$u_{3,1} = \frac{1}{2}\left[u_{4,0} + u_{2,0}\right] = \frac{1}{2}[4+0] = 2$$

- if k & h are not give u can assume,
- if they ask to calculate for 2 iterations then leave j = 0 & calculate for j = 1, 2

To generate 3rd row put j = 1 in (A)

$u_{i,2} = u_{i+1,1} + u_{i-1,1} - u_{i,0} \longrightarrow$ (B)

put i = 1, 2, 3.

$u_{1,2} = u_{2,1} + u_{0,1} - u_{1,0}$

$= 3 + 0 - 3 = 0$

$u_{2,2} = 2 + 2 - 4 = 0$

$u_{3,2} = 3 + 0 - 3 = 0$

To generate 4th row put j = 2 in (A)

$u_{i,3} = u_{i+1,2} + u_{i-1,2} - u_{i,1} \longrightarrow$ (C)

put i = 1, 2, 3

$u_{1,3} = 0 + 0 - 2 = -2$

$u_{2,3} = 0 + 0 - 3 = -3$

$u_{3,3} = 0 + 0 - 2 = -2$

To generate 5th row put j = 3 in (A)

$u_{i,4} = u_{i+1,3} + u_{i-1,3} - u_{i,2} \longrightarrow$ (D)

put i = 1, 2, 3.

$u_{1,4} = 0 - 3 - 0 = -3$

$u_{2,4} = -2 - 2 - 0 = -4$

$u_{3,4} = -3 + 0 - 0 = -3$

Ex: Solve the wave eqn $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ subject to the following conditions using finite difference method

$u(x,0) = \sin \pi x$, $\frac{\partial u}{\partial t}(x,0) = 0$ (initial condn)

$u(0,t) = 0$, $u(1,t) = 0$, $t \geq 0$ (BC)

$h = 1/4$, $k = 1/5$. Find the soln upto 3rd stage.

→ Here $c^2 = 1$

$\lambda = \frac{k}{h} = \frac{1}{5} \times 4 = \frac{4}{5}$, $\lambda^2 = \frac{16}{25}$ & $\frac{1}{c^2} = 1$

$\therefore \lambda^2 \neq 1/c^2$

So we use explicit formula as

$$u_{i,j+1} = 2(1-\lambda^2 a^2) u_{i,j} + \lambda^2 a^2 (u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}$$

$$u_{i,j+1} = \frac{18}{25} u_{i,j} + \frac{16}{25} (u_{i+1,j} + u_{i-1,j}) - u_{i,j-1} \rightarrow (A)$$

We have $u_t(x,0) = 0$, $u_{i,j+1} = u_{i,j-1}$

put $j = 0$

$\Rightarrow u_{i,1} = u_{i,-1} \longrightarrow (1)$.

put $j=0$ in eqn (A).

$$u_{i,1} = \frac{18}{25} u_{i,0} + \frac{16}{25} (u_{i+1,0} + u_{i-1,0}) - u_{i,-1}$$

$$\Rightarrow u_{i,1} = \frac{1}{2}\left[\frac{18}{25} u_{i,0} + \frac{16}{25}(u_{i+1,0} + u_{i-1,0})\right] \rightarrow (2)$$

To generate 2nd row we put $i = 1, 2, 3$.

$$u_{1,1} = 1/2 \left[u_{1,0} + 16/25 (u_{2,0} + u_{0,0})\right]$$
$$= 1/2 \left[18/25\, u_{1,0} + 16/25 [u_{2,0} + u_{00}]\right]$$
$$= 1/2 \left[18/25 \times 0.7071 + 16/25 [1 + 0.7071]\right]$$
$$= 0.57$$

$h = 1/4 \rightarrow$

$k = 1/5$

t \ x	0	1/4	2/4	3/4	1
0	0 $u_{0,0}$	0.7071 $u_{1,0}$	1 $u_{2,0}$	0.7071 $u_{3,0}$	0
1/5	0 $u_{0,1}$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	0
2/5	0 $u_{0,2}$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	0
3/5	0 $u_{0,3}$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	0

$$u_{2,1} = 1/2 \left[18/25\, u_{2,0} + 16/25 [u_{3,0} + u_{1,0}]\right]$$
$$= 0.81$$

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$u_{3,1} = \frac{1}{2}\left[\frac{18}{25} \times 1 + \frac{16}{25}[0.7071 + 0.7071] \right]$

$u_{3,1} = 0.57$.

To generate 3rd row put $j = 1$ in (A).

$$u_{i,2} = \frac{18}{25} u_{i,1} + \frac{16}{25}\left[u_{i+1,1} + u_{i-1,1} \right] - u_{i,0}$$

$u_{1,2} = 0.26$

$u_{2,2} = 0.45$

$u_{3,2} = 0.26$.

$u_{1,3} = -0.09$

$u_{2,3} = -0.211$

$u_{3,3} = -0.09$,

Scanned by CamScanner