MATRICES AND LINEAR EQUATIONS

Linear equations-

Many problems in the natural & social sciences as well as in engineering & physical sciences deals with eq^{ns} relating to sets of variables. It's of the type $AX=B$ expressing the variable B in terms of X and the constant A is called a linear equation. The word linear is used because graph of equation is straight line. Similarly equation $a_1x_1 + a_2x_2 + \cdots a_nx_n = b_n$ is also an equation. where $x_1, x_2, x_3, \cdots$ are the solution. {If $x_1 = x_2, \cdots = x_n = 0$ — trivial/obvious solution}.

System of linear equations: {Group of eq^{ns}}

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$

$\cdots\cdots\cdots$

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n$

~~Ca~~ These eq^n can be written as $AX = B$

If $b_1 = b_2 = b_3 = \cdots = b_n = 0$, the system is said to be homogenous else its non homogenous.

set of values $x_1, x_2, \cdots x_n$ which satisfy all the equations simultaneously is called solution of system of eq^{ns}.

System of linear equations said to be consistent if it possesses a solution otherwise system is said to be inconsistent.

Geometrical interpretation of linear equations-

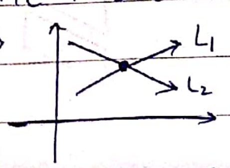
(*) [For 2 equations]:

$a_1x + a_2y = \cancel{b}c_1 \to l_1$ {$l_1, l_2 \to$ straight line}

$b_1x + b_2y = c_2 \to l_2$

If $x = s_1, y = s_2$ is a solution to the linear system then the point (s_1, s_2) lies on both the lines l_1, l_2

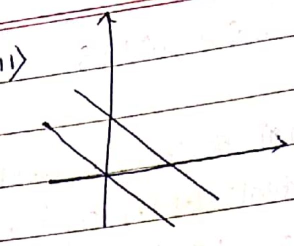
the there are 3 possibility:

i) 

{if lines intersect}

$$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$$

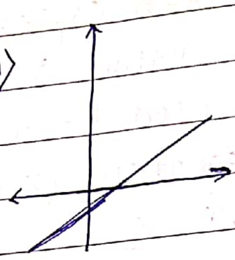
(a) unique sol^n

ii) 

{lines are parallel}

$\frac{a_1}{b_1} = \frac{a_2}{b_2} \neq \frac{c_1}{c_2}$

No solution

iii) 

{lines overlap}

$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{c_1}{c_2}$

∞ solutions

i) System has unique solutions ie lines l_1 & l_2 intersect at one point.

ii) System " no ——" —— don't " (ie $\parallel^{lel}$)

iii) " " ∞ many ——" —— coincides.

(*) [For three equations]

$a_1x + b_1x + c_1x = d_1 \rightarrow P_1$ {$P_1, P_2, P_3 \Rightarrow$ planes}

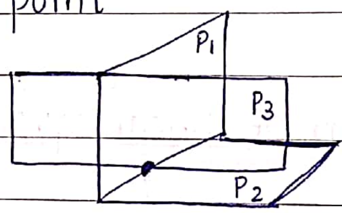
$a_2x + b_2x + c_2x = d_2 \rightarrow P_2$

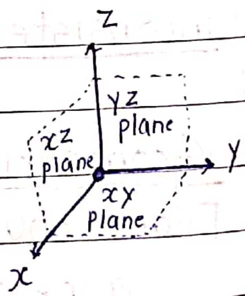
$a_3x + b_3x + c_3x = d_3 \rightarrow P_3$

a) Unique solutions :-

All planes intersect at a common point (ie eg origin)

Eg: Walls of a room intersect at one corner ie a unique point

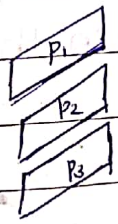




b) No solutions :-

All three planes are parallel

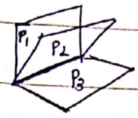
When the book is closed 3 pages of book appear $\parallel^{lel}$.



c) ∞ solutions

If all three planes overlap (at common straight line) then there are ∞'ly many solⁿ.

Eg: 3 pages of a book intersect along a straight line.



Solution of system of linear Equations-

Consider

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

The system of equations written in matrix form $AX = B$

where A is the coefficient matrix (having coefficients) of order 3x3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$[A:B]$ = Augmented matrix

$$= \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

{to be transformed to simplest form using only row operation}

Elementary Row operations :-

i) Interchange of any two rows $[R_i \leftrightarrow R_j]$

ii) Multiplication of a row by non zero constant k $[R_i \to kR_i]$

iii) Additⁿ of 'k' times i^{th} row to j^{th} row. $[R_j \to kR_i + R_j]$

By using elementary row operations reduce the augmented matrix to Row echelon form [REF] or Echelon form and reduced row echelon form [RREF].

- Echelon form [staircase form]

A rectangular matrix is in echelon form [staircase pattern] or Row echelon form (REF), if it satisfies following three properties:-

i) The pivots (leading entry) are the first non zero entries in their rows.

ii) Below each pivot is a coloumn of zeroes.

iii) Each pivot lies to right of pivot in the row above

this produces the staircase pattern and zero rows come last.

{non zero row - if any one element is non zero}

Eg: $\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ← zero row at last

Pivot

Above matrix is in REF or echelon form.

If a matrix is in row echelon form satisfies following additional conditions then it's in reduced row echelon form [RREF]

iv) The leading entry/pivot in each non zero row is '1'

v) Each leading entry/pivot 1 is the only non zero entry in it's column

Eg: $\begin{bmatrix} 1 & 0 & 5 & 0 & -7 \\ 0 & 1 & 4 & 0 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ - reducing previous matrix.

doesn't contain pivot

Matrix in RREF.

Pivot positions in REF, RREF:-

Pivot is a non zero number that is used in a pivot position to create zeroes or is changed into leading '1', which inturn is used to create zeroes.

A pivot position in a matrix A is a location in A that corresponds to leading entry in the row echelon form of A or '1' in the RREF of A.

A pivot column of A that contains a pivot position

Remark:- There is only one pivot in any row/column.

Basic variable-

Any variable that corresponds to a pivot column in the augmented matrix of the system.

Free variable-

Non basic variables, free means that we are free to chose any value for it.

Eg: $\left[\begin{array}{ccccc:c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$

Pivot columns- $1^{st}, 3^{rd}$ & 5^{th} column.

Basic variables- x_1, x_3, x_5

Free variables- x_2, x_4 ← if there are free variables then ∞ solutions

Rewriting as system of equations-

$$x_1 + 6x_2 + 3x_4 = 0$$
$$x_3 - 8x_4 = 5$$
$$x_5 = 7$$

Solution of system of equations by Gauss-Elimination method:

To solve system of linear equations of form $AX = B$,

Procedure-

(i) Write system of equations in matrix form $AX = B$

(ii) Form augmented matrix $[A:B]$

(iii) Apply elementary row transformations to reduce the coefficient matrix present in the augmented matrix to upper triangular form or echelon form.

(iv) Write the reduced system of equations and using back substitution method find the solution of the system.

Rank of matrix:
The rank of matrix A is denoted by $\rho[A]$ or $R(A)$.
The rank of matrix A is equal to the number of non zero rows present in the row echelon form or RREF. {Rank - to check consistency of system of eq^{ns}}

Consistency of system of linear equations using Rank -
Consider system of linear equations $AX=B$ in n unknowns and $[A:B]$ be the augmented matrix then -
<a> system is consistent (posses a soln.) iff $\rho[A] = \rho[A:B]$
(i) unique solution - $\rho[A] = \rho[A:B] = r =$ no. of unknowns.
(ii) Infinitely many solutions - if
$\rho(A) = \rho[A:B] = r < n$ {no. of unknowns}
<b> System is inconsistent (no solution) if $\rho[A] \neq \rho[A:B]$
To find those solutions:-

Examples - $r < n \Rightarrow n > r$

$n - r = k$

k ~~sho~~ variables should be assigned some arbitrary constant.

for eg:→ $r = 2$ & $n = 4$

$n - r = 2$

out of 4, '2' variables should be assigned some arbitrary constant.

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Q) Reduce the matrix to echelon form and locate pivot points of A.

first pivot entry, since it's zero make it nonzero.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -4 & -7 \end{bmatrix}$$

operate $R_1 \leftrightarrow R_4$

pivot 1

$$A = \begin{bmatrix} 1 & 4 & 5 & -4 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

operate $R_2' \rightarrow R_2 + R_1$ and $R_3' \rightarrow R_3 + 2R_1$

pivot(2)

$$A = \begin{bmatrix} \boxed{1} & 4 & 5 & -4 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

operate $R_3' \rightarrow 2R_3 - 5R_2$ & $R_4' = 3R_2 + 2R_4$

$$A = \begin{bmatrix} \boxed{1} & 4 & 5 & -4 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -10 & 0 \end{bmatrix}$$

operate $R_3 \leftrightarrow R_4$

$$A = \begin{bmatrix} \boxed{1} & 4 & 5 & -4 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & \boxed{-10} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

'A' is in row echelon form/echelon form with 1^{st}, 2^{nd} & 4^{th} as pivot columns.

Rank of matrix $A = \rho(A) = 3$

(LP3) Q) Reduce the following matrix to echelon form and it's RREF.

$$B = \begin{bmatrix} \boxed{3} & -1 & 2 \\ 2 & +1 & 1 \\ 1 & -3 & 0 \end{bmatrix}$$

operate $R_1 \leftrightarrow R_3$ {for ease}

$$B = \begin{bmatrix} 1 & -3 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & 2 \end{bmatrix}$$

operate $R_2' \to R_2 - 2R_1$ and $R_3' \to R_3 - 3R_1$

$$B = \begin{bmatrix} \boxed{1} & -3 & 0 \\ 0 & \boxed{+7} & 1 \\ 0 & 8 & 2 \end{bmatrix}$$

~~operate $R_2' \to \frac{1}{7} R_2$ & R_3'~~ operate $R_3' \to 7R_3 - 8R_2$

$$B = \begin{bmatrix} \boxed{+1} & -3 & 0 \\ 0 & \boxed{7} & 1 \\ 0 & 0 & \boxed{6} \end{bmatrix}$$ ← row echelon form of B.

operate $R_3' \to \frac{1}{6} R_3$

$$B = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

operate $R_2' \to R_2 - R_3$

$$B = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

operate $R_2' \to R_2 / 7$

$$B = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

operate $R_1' \to R_1 + 3R_2$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is in reduced row echelon form

LP4) Q) Find the rank of $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & -4 & -5 \\ 7 & 8 & -5 & -1 \\ 10 & 14 & -2 & 8 \end{bmatrix}$ by reducing it to row reduced echelon form (RREF)

$$A = \begin{bmatrix} \boxed{1} & 2 & 1 & 3 \\ 2 & 1 & -4 & -5 \\ 7 & 8 & -5 & -1 \\ 10 & 14 & -2 & 8 \end{bmatrix}$$

Operate $R_2' \to R_2 - 2R_1$, $R_3' \to R_3 - 7R_1$, $R_4' \to R_4 - 10R_1$

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -3 & -6 & -11 \\ 0 & -6 & -12 & -22 \\ 0 & -6 & -12 & -22 \end{bmatrix}$$

Operate $R_3' \to R_3 - 2R_2$ $R_4' \to R_4 - 2R_2$

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -3 & -6 & -11 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2' \to \frac{-1}{3} R_2$

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 11/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operate $R_1 \to R_1 - 2R_1$

$$A = \begin{bmatrix} 1 & 0 & -3 & -13/3 \\ 0 & 1 & 2 & 11/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R(A) = 2$

Since there are two non zero rows in the RREF $\therefore$ rank is 2.

Q) Find the rank of matrix by reducing it to RREF.

$$A = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

operate $R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 0 & 2 & 1 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

operate $R_2' \rightarrow R_2 - 2R_1$, $R_3' \rightarrow R_3 - R_1$, $R_4' \rightarrow R_4 - R_1$

$$A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & -2 & 0 \end{bmatrix}$$

operate $R_3' \rightarrow R_3 + R_2$ and $R_4' \rightarrow R_4 + R_2$

$$A = \begin{bmatrix} 2 & 1 & 3 & +4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & -6 & -7 \end{bmatrix}$$

operate $R_4' \rightarrow R_4 - 2R_3$

$$A = \begin{bmatrix} 2 & 1 & 3 & +4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

operate $R_1' \to R_1 - 4R_4$, $R_2' \to R_2 + 7R_4$, $R_3' \to R_3 + 4R_4$

$$A \simeq \begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & -2 & -4 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

operate $R_3' \to \frac{-1}{3} R_3$, $R_2' \to \frac{-1}{2} R_2$

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

operate $R_1' \to R_1 - 3R_3$, $R_2' \to R_2 - 2R_3$

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operate $R_1' \to R_1 - R_2$

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

operate $R_1' \to \frac{1}{2} R_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since there are 4 non zero rows, rank of matrix is 4.

(LP5) Q) Solve the system of equation using Gauss elimination method-

(1)
$$10x_1 - x_2 + 2x_3 = 4$$
$$x_1 + 10x_2 - x_3 = 3$$
$$2x_1 + 3x_2 + 20x_3 = 7$$

(ii) $x_1+x_2+x_3=6$

$3x_1+3x_2+4x_3=20$

$2x_1+x_2+3x_3=13$

(iii) $x+y+z=6$

$x-y+2z=5$ } Answer in pg 15

$3x+y+z=8$

Ans: (i)

$$10x_1-x_2+2x_3=4$$
$$x_1+10x_2-x_3=3$$
$$2x_1+3x_2+20x_3=7$$

The given system of matrix in $AX=B$

$$\begin{bmatrix}10 & -1 & 2\\ 1 & 10 & -1\\ 2 & 3 & 20\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}4\\ 3\\ 7\end{bmatrix}$$

Now form augmented matrix $[A:B]$

$$=\left[\begin{array}{ccc|c}10 & -1 & 2 & 4\\ 1 & 10 & -1 & 3\\ 2 & 3 & 20 & 7\end{array}\right]$$

operate $R_1 \leftrightarrow R_1$

$$=\left[\begin{array}{ccc|c}\boxed{1} & 10 & -1 & 3\\ 10 & -1 & 2 & 4\\ 2 & 3 & 20 & 7\end{array}\right]$$

operate $R_2' \to R_2 - R_1(10)$ $R_3' \to R_3 - 2R_1$

$$=\left[\begin{array}{ccc|c}1 & 10 & -1 & 3\\ 0 & \boxed{-101} & 12 & -26\\ 0 & -17 & 22 & 1\end{array}\right]$$

operate $R_3' \to 101R_3 - 17R_2$

$$=\left[\begin{array}{ccc|c}1 & 10 & -1 & 3\\ 0 & -101 & 12 & -26\\ 0 & 0 & 2018 & 543\end{array}\right] \Leftarrow \text{REF}$$

$\therefore$ reduced system of eq^{ns}:-

$$x_1+10x_2-x_3=3$$
$$-101x_2+12x_3=-26$$
$$2018x_3=543$$

From equations - {using back substitution}

$x_3 = 0.2691$

$x_2 = 0.2894$

$x_1 = 0.3751$

(ii) Given system of equations in matrix $AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 13 \end{bmatrix}$$

augmented matrix: $[A:B]$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{array}\right]$$

operate $R_2' \to R_2 - 3R_1$ & $R_3' \to R_3 - 2R_1$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

converting the above REF form to system of eq^ns -

$x_1 + x_2 + x_3 = 6$

$x_2 = 2$

$-x_2 + x_3 = 1$

From above equations -

$\boxed{x_2 = 2}$

$x_3 = 1 + x_2$

$\boxed{x_3 = 3}$

$x_1 = 6 - x_2 - x_3$

$\boxed{x_1 = 3}$

(LPT)

Q) Find the pivots & solution for these 4 equations:-

$2x+y=0$

$x+2y+z=0$

$y+2z+t=0$

$z+2t=5$

- Following system of equations in matrix form AX:

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$$

Augmented matrix $= [A:B]$

$$= \left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right]$$

$R_2' \to 2R_2 - R_1$

$$= \left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right]$$

operate $R_3' \to 3R_3 - R_2$

$$= \left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right]$$

operate $R_4' \to 4R_4 - R_3$

$$= \left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 \\ 0 & 0 & 0 & 5 & 20 \end{array}\right]$$

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Writing system of equations -

$2x+y=0$

$3y+2z=0$

$4z+3t=0$

$5t=20$

From above equations -

$\boxed{t=4}$

$\boxed{z=-3}$

$\boxed{y=2}$

$\boxed{x=-1}$

LPQ5) ~~LP2~~ iii)

$x+y+z=6$

$x-y+2z=5$

$3x+y+z=8$

Writing the system of equations in matrix form. $AX=B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}$$

Augmented matrix:

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \end{array}\right]$$

operate $R_2' \to R_2 - R_1$ & $R_3' \to R_3 - 3R_1$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \end{array}\right]$$

operate $R_3' \to R_3 - R_2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

rewriting as system of equations -

~~x_1+x_2~~ $x+y+z=6$

$-2y+z=-1$

$-3z=-9$

From above equations-

$\boxed{z=3}$
$\boxed{y=2}$
$\boxed{x=1}$

(LP Q2) For which three numbers of 'a' will elimination fail to give three pivots?

$ax+2y+3z=b$, $ax+ay+4z=c$, $ax+ay+az=d$

- Writing the system of equations in matrix form $AX=B$

$$\begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ c \\ d \end{bmatrix}$$

Augmented matrix:

$$= \left[\begin{array}{ccc:c} a & 2 & 3 & b \\ a & a & 4 & c \\ a & a & a & d \end{array}\right]$$

operate $R_2' \to R_2 - R_1$ & $R_3' \to R_3 - R_1$

$$= \left[\begin{array}{ccc:c} a & 2 & 3 & b \\ 0 & -2+a & 1 & c-b \\ 0 & a-2 & a-3 & d-b \end{array}\right]$$

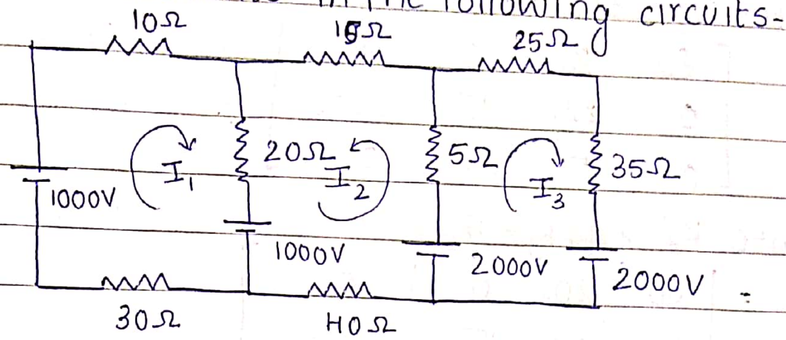
operate $R_3' \to R_3 - R_2$

$$= \left[\begin{array}{ccc:c} \boxed{a} & 2 & 3 & b \\ 0 & \boxed{a-2} & 1 & c-b \\ 0 & 0 & \boxed{a-4} & d-c \end{array}\right]$$

In REF, the pivots should be non zero

$\therefore$ if $a=0$, $a=2$ and $a=4$; the elimination method fails to give 3 pivots.

LPQ6) (a) Find the currents in the following circuits-



For loop 1:

$$1000-10I_1-20(I_1+I_2)-1000-30I_1=0$$

$$10I_1+20(I_1+I_2)+30I_1=0 \text{ --- } (x)$$

$$\cancel{30}\ 60I_1+20I_2=0 \text{ --- } (1)$$

For loop 2:

$$2000-5(I_2+I_3)\cancel{-20I_2}-20(I_2+I_1)-1000-40I_2-15I_2=0$$

$$1000-5I_2-5I_3-20I_2-20I_1-40I_2-15I_2=0$$

$$-20I_1-80I_2-5I_3+1000=0$$

$$20I_1+80I_2+5I_3=1000 \text{ --- } (2)$$

For loop 3:

$$2000-5I_2-5I_3-25I_3-35I_3-2000=0$$

$$-5I_2-65I_3=0$$

$$5I_2+65I_3=0 \text{ --- } (3)$$

From (1), (2) & (3)

system of equations are :-

$$60I_1+20I_2=0$$

$$20I_1+80I_2+5I_3=1000$$

$$5I_2+65I_3=0$$

Writing system of equations in matrix form $AX=B$

$$\begin{bmatrix} 60 & 20 & 0 \\ 20 & 80 & 5 \\ 0 & 5 & 65 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1000 \\ 0 \end{bmatrix}$$

Form augmented matrix :-

$$=[A:B]=\left[\begin{array}{ccc|c} 60 & 20 & 0 & 0 \\ 20 & 80 & 5 & 1000 \\ 0 & 5 & 65 & 0 \end{array}\right]$$

operate $R_2' \rightarrow 3R_2 - R_1$

$$= \left[\begin{array}{ccc|c} 60 & 20 & 0 & 0 \\ 0 & 220 & 15 & 3000 \\ 0 & 5 & 65 & 0 \end{array}\right]$$

operate $R_3' \rightarrow 44R_3 - R_2$

$$= \left[\begin{array}{ccc|c} 60 & 20 & 0 & 0 \\ 0 & 220 & 15 & 3000 \\ 0 & 0 & 2845 & -3000 \end{array}\right]$$

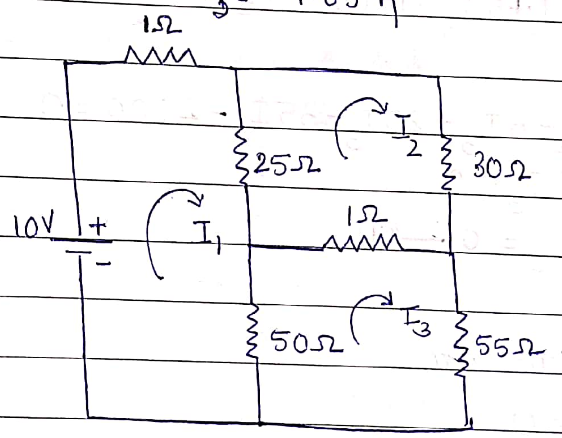
Write the reduced system of equations-

$$60I_1 + 20I_2 = 0$$
$$220I_2 + 15I_3 = 3000$$
$$2845I_3 = -3000$$

solving the equations-

$$I_1 = -4.56$$
$$I_2 = 13.708$$
$$I_3 = -1.054$$

• (ii)



$$10 - I_1 - 25(I_1 - I_2) - 50(I_1 - I_3) = 0$$
$$10 - 76I_1 + 25I_2 + 50I_3 = 0$$
$$76I_1 - 25I_2 - 50I_3 = 10 \quad \text{—(1)}$$

$$-25(I_2 - I_1) - 30I_2 - (I_2 - I_3) = 0$$
$$25I_1 - 56I_2 + I_3 = 0 \quad \text{—(2)}$$

$$-50(I_3 - I_1) - (I_3 - I_2) - 55I_3 = 0$$
$$50I_1 + I_2 - 106I_3 = 0 \quad \text{—(3)}$$

System of equations are-

$$76I_1 - 25I_2 - 50I_3 = 10$$
$$25I_1 - 56I_2 + I_3 = 0$$
$$50I_1 + I_2 - 106I_3 = 0$$

Writing system of ~~matr~~ eq^{ns} in matrix form $AX = B$.

$$\begin{bmatrix} 76 & -25 & -50 \\ 25 & -56 & 1 \\ 50 & 1 & -106 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix $= [A : B]$

$$= \left[\begin{array}{ccc:c} 76 & -25 & -50 & 10 \\ 25 & -56 & 1 & 0 \\ 50 & 1 & -106 & 0 \end{array}\right]$$

operate $R_2' \to 76R_2 - 25R_1$ & $R_3' \to 76R_3 - 50R_1$

$$= \left[\begin{array}{ccc:c} 76 & -25 & -50 & 10 \\ 0 & -3631 & 1326 & -250 \\ 0 & 1326 & -5556 & -500 \end{array}\right]$$

Operate $R_3' \to 3631R_3 + 1326R_2$

$$= \left[\begin{array}{ccc:c} 76 & -25 & -50 & 10 \\ 0 & -3631 & 1326 & -250 \\ 0 & 0 & 18415560 & 21470000 \end{array}\right]$$

$I_3 = 0.116$
$I_2 = 0.111$
$I_1 = 0.2449$

$$I_3 = \frac{21470000}{18415560} = 0.116$$

$$-3631I_2 + 1326I_3 = -250$$

$$\Rightarrow I_2 = 0.111$$

$$76I_1 - 25I_2 - 50I_2 = 10$$

$$\Rightarrow I_1 = 0.2449$$

(Answer in Pg 22) Q) Find the Rank of matrix -

(i) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ $\rho(A)=3$

(ii) $\begin{bmatrix} 1 & 0 & 2 & -2 \\ 2 & -1 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 4 & -1 & 3 & -1 \end{bmatrix}$ $\rho(A)=4$

(iii) $\begin{bmatrix} 1 & 2 & 2 & 4 \\ 2 & 3 & 4 & 6 \\ 3 & 5 & 6 & 10 \\ 4 & -1 & -3 & 2 \end{bmatrix}$ $\rho(A)=3$

(ch 2) (LP 25) Q) Test the consistency of the equations & solve them if possible.

(i) $2x-3y+7z=5$
$3x+y-3z=13$
$3x+19y-47z=32$

- system of equations in matrix form $AX=B$

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 3 & 19 & -47 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$$

Augmented matrix $=[A:B]$

$$= \left[\begin{array}{ccc:c} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 3 & 19 & -47 & 32 \end{array}\right]$$

$R_2' \to 2R_2 - 3R_1$ & $R_3' \to 2R_3 - 3R_1$

$$= \left[\begin{array}{ccc:c} 2 & -3 & 7 & 5 \\ 0 & 11 & -27 & 11 \\ 0 & 47 & -115 & 49 \end{array}\right]$$

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operate $R_3' \to 11R_3 - 47R_2$

$$\approx \left[\begin{array}{ccc:c} 2 & -3 & 7 & 5 \\ 0 & 11 & -27 & 11 \\ 0 & 0 & 4 & 22 \end{array}\right]$$

$R[A:B] = 3$ and $R[A] = 3$

since $R[A:B] = R[A] = 3$

The system is consistent

and since $R[A:B] = R[A] = 3 = n$ (no. of unknowns)

System of equations have unique solutⁿ.

Writing the REF of $[A:B]$ as system of equations-

$2x_1 - 3y + 7z = 5$

$11y - 27z = 11$

$4z = 22$

$z = 5.5,\ x = 5,\ y = 14.5$

(ii) $2x + 6y + 11 = 0$

(Ans in pg. 24) $6x + 20y - 6z + 3 = 0$

$6y - 18z + 1 = 0$

system of equations in matrix form $AX = B$

$$\begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix}$$

Augmented matrix $= [A:B]$

$$= \left[\begin{array}{ccc:c} 2 & 6 & 0 & -11 \\ 6 & 20 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{array}\right]$$

operate $R_2' \to R_2 - 3R_1$

$$= \begin{bmatrix} 2 & 6 \\ 0 & 18 \\ 0 & \end{bmatrix}$$

Q) i) $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Operate $R_2 \leftrightarrow R_1$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Operate $R_2' \to R_2 - 2R_1$, $R_3' \to R_3 - 3R_1$, $R_4' \to R_4 - 6R_1$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

operate $R_3' \to 5R_3 - 4R_2$ & $R_4' \to 5R_4 - 9R_2$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33 & 22 \end{bmatrix}$$

operate $R_4 \to \cancel{R_4}\ R_4 - R_3$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are ~~tw~~ 3 non zero rows, the rank of matrix is :- $\rho(A) = \cancel{4}\ 3$

ii) $A = \begin{bmatrix} 1 & 0 & 2 & -2 \\ 2 & -1 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 4 & -1 & 3 & -1 \end{bmatrix}$

operate $R_2' \to R_2 - 2R_1$, $R_3' \to R_3 - R_1$, $R_4' \to R_4 - 4R_1$

$$A = \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -5 & 7 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$A = \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & -1 & -4 & 3 \\ 0 & -1 & -5 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

operate $R_3' \rightarrow R_3 - R_1$

$$A = \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since there are 4 non zero rows, rank of above matrix is 4.

iii) $A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 2 & 3 & 4 & 6 \\ 3 & 5 & 6 & 10 \\ 4 & -1 & -3 & 2 \end{bmatrix}$

operate $R_2' \rightarrow R_2 - 2R_1$, $R_3' \rightarrow R_3 - 3R_1$, $R_4' \rightarrow R_4 - 4R_1$

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & -1 & 0 & -2 \\ 0 & -1 & 0 & -2 \\ 0 & -9 & -11 & -14 \end{bmatrix}$$

operate $R_3' \rightarrow R_3 - R_2$ & $R_4 \rightarrow R_4 - 9R_2$

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -11 & 4 \end{bmatrix}$$

operate $R_3 \leftrightarrow R_4$

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -11 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are 3 non zero rows in REF, $\rho(A) = 3$.

Q) ii) $2x+6y+11=0$

$6x+20y-6z+3=0$

$6y-18z+1=0$

Writing system of equations as matrix $AX=B$

$$\begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix}$$

Augmented matrix :- $[A:B]$

$$= \left[\begin{array}{ccc|c} 2 & 6 & 0 & -11 \\ 6 & 20 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{array}\right]$$

operate $R_2' \to R_2 - 3R_1$

$$= \left[\begin{array}{ccc|c} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 6 & -18 & -1 \end{array}\right]$$

operate $R_3' \to R_3 - 3R_2$

$$= \left[\begin{array}{ccc|c} 2 & 6 & 0 & -1 \\ 0 & 2 & -6 & -30 \\ 0 & 0 & 0 & -91 \end{array}\right]$$

Since there are 3 non zero rows

$\rho(A) = 2$

& $\rho(A:B) = 3$

since $\rho(A) \neq \rho(A:B)$

system is inconsistent.

iii) $x+2y+z=3$, $2x+3y+2z=5$, $3x-5y+5z=2$, $3x+9y-z=4$

→ writing in $AX=B$ form

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & -5 & 5 \\ \cancel{4}3 & 9 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 4 \end{bmatrix}$$

Augmented matrix:

$$= \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 5 \\ 3 & -5 & 5 & 2 \\ 3 & 9 & -1 & 4 \end{array}\right]$$

operate $R_2' \to R_2 - 2R_1$, $R_3' \to R_3 - 3R_1$, $R_4' \to R_4 - 3R_1$

$$= \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \end{array}\right]$$

operate: $R_3' \to R_3 - 11R_2$ & $R_4' \to R_4 + 3R_2$

$$= \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -8 \end{array}\right]$$

operate $R_4' \to R_4 + 2R_2$

$$= \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho[A:B]=3$ and $\rho[A]=3$ & no. of unknown $=3$

∴ System of equations has unique solution-

$2z=4$

$-y=-1$

$x+2y+z=3$

$\Rightarrow z=2,\ y=1,\ x=-1$

iv) $2x - y + z = 4$
$3x - y + z = 6$
$4x - y + 2z = 7$
$-x + y - z = 9$

Writing system of equations in AX=B form -

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & -1 & 1 \\ 4 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 7 \\ 9 \end{bmatrix}$$

augmented matrix:-

$$= \left[\begin{array}{ccc:c} 2 & -1 & 1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \\ -1 & 1 & -1 & 9 \end{array}\right]$$

operate $R_1 \leftrightarrow R_4$

$$= \left[\begin{array}{ccc:c} -1 & 1 & -1 & 9 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \\ 2 & -1 & 1 & 4 \end{array}\right]$$

operate $R_2' \to R_2 + 3R_1$

$$= \left[\begin{array}{ccc:c} -1 & 1 & -1 & 9 \\ 0 & 2 & -2 & 33 \\ 0 & 3 & -2 & 43 \\ 0 & 1 & -1 & 22 \end{array}\right]$$

Operate $R_2 \leftrightarrow R_4$

$$= \begin{bmatrix} -1 & 1 & -1 & 9 \\ 0 & 1 & -1 & 22 \\ 0 & 3 & -2 & 43 \\ 0 & 2 & -2 & 33 \end{bmatrix}$$

operate $R_3' \to R_3 - 3R_2$ & $R_4' \to R_4 - 2R_1$

$$= \left[\begin{array}{ccc:c} -1 & 1 & -1 & 9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 1 & -23 \\ 0 & 0 & 0 & -11 \end{array}\right]$$

$\rho(A) = 3$, $\rho(A:B) = 4$

since $\rho(A) \neq \rho(A:B)$

system has no solution.

v) $2x+3y-z=1$

$3x-4y+3z=-1$

$2x-y+2z=-3$

$3x+y-2z=4$

writing in $AX=B$ form.

$$\begin{bmatrix} 2 & 3 & -z \\ 3 & -4 & 3 \\ 2 & -1 & 2 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -3 \\ 4 \end{bmatrix}$$

Augmented matrix:-

$$[A:B] = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 3 & -4 & 3 & -1 \\ 2 & -1 & 2 & -3 \\ 3 & 1 & -2 & 4 \end{bmatrix}$$

operate $R_2' \to 2R_2 - 3R_1$, $R_3' \to R_3 - R_1$, $R_4' \to 2R_4 - 3R_1$

$$= \left[\begin{array}{ccc:c} 2 & 3 & -1 & 1 \\ 0 & -17 & 9 & -5 \\ 0 & -4 & 3 & -4 \\ 0 & -7 & -1 & 5 \end{array}\right]$$

operate $R_3' \to 17R_3 - 4R_2$, $R_4' \to 17R_4 - 7R_2$

$$= \left[\begin{array}{ccc:c} 2 & 3 & -1 & 1 \\ 0 & -17 & 9 & -5 \\ 0 & 0 & 15 & -48 \\ 0 & 0 & -80 & 120 \end{array}\right]$$

~~$\rho(A)=3$, $\rho(A:B)$~~ operate $R_4' \to 15R_4 + 80R_3$

$$= \begin{bmatrix} 2 & 3 & -1 & 1 \\ 0 & -17 & 9 & -5 \\ 0 & 0 & 15 & -48 \\ 0 & 0 & 0 & 1672 \end{bmatrix}$$

$\rho(A) = 3$, $\rho(A:B) = 4$

since $\rho(A) \neq \rho(A:B)$, system has no solution.

(chapter 2)
LP Q26)

Find the values of λ & μ for which the system $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ has i) unique solution. ii) ∞ many solutions iii) no solutions.

→ Writing system of eq^n in $AX=B$ form :-

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

Augmented matrix:-

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

operate $R_2' \to R_2 - R_1$, $R_3' \to R_3 - R_1$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

operate $R_3' \to R_3 - R_2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

i) Unique solution:-

for unique solution-

$$\rho(A) = \rho(A:B) = r = n$$

we must have:

$$\rho(A) = 3 = \rho(A:B)$$

$\rho[A] = 3$ if $\lambda - 3 \neq 0 \Rightarrow \lambda \neq 3$

$\rho[A:B] = 3$ if ~~$\lambda=3$~~ irrespective of μ value.

$\therefore$ for unique solution:-

$$\lambda \neq 3$$

ii) Infinitely many solution :-
if $\rho[A] = \rho[A:B] = r<n$
we need
$\rho[A] = \rho[A:B] = r<3$
let $r=2$
$\therefore \rho[A] = \rho[A:B] = 2 < n = 2$
$\rho[A] = 2$ if $\lambda - 3 = 0 \Rightarrow \boxed{\lambda = 3}$
$\rho[A:B] = 2$ if $\mu - 10 = 0 \Rightarrow \boxed{\mu = 10}$

iii) no solution :-
if $\rho[A] \neq \rho(A:B)$
by (i) $\rho[A] = 3$ if $\lambda - 3 \neq 0$ ie $\lambda \neq 3$
If $\lambda = 3$, $\rho[A] = 2$
$\rho[A:B] = 3$ if $\mu = 10 \neq 0 \Rightarrow \mu \neq 10$
$\Rightarrow \rho(A) = 2$ & $\rho(A:B) = 3$
ie $\rho(A) \neq \rho(A:B)$
system has no solution if $\lambda = 3$ and $\mu \neq 10$

(ch 2) LP Q27) Show that the equations $-2x+y+z=a$, $x-2y+z=b$, $x+y-2z=c$ is consistent if $a+b+c=0$. Solve the system of eq^ns when $(a,b,c) = (1,1,-2)$.

→ Writing equations in matrix form:-
$AX = B$
$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Augmented matrix:-
$$A:B = \left[\begin{array}{ccc:c} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{array}\right]$$

operate $R_1 \leftrightarrow R_2$
$$A:B = \begin{bmatrix} 1 & -2 & 1 & b \\ -2 & 1 & 1 & a \\ 1 & 1 & -2 & c \end{bmatrix}$$

operate $R_2' \to R_2 + 2R_1$ $R_3' \to R_3 - R_1$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & b \\ 0 & -3 & 3 & a+2b \\ 0 & 3 & -3 & c-b \end{array}\right]$$

operate $R_3' \to R_3 + R_2$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & b \\ 0 & -3 & 3 & a+2b \\ 0 & 0 & 0 & a+b+c \end{array}\right]$$

$\rho(A) = 2$, $\rho(A:B) = 3$

$\rho(A) = 2 = \rho(A:B)$ if $a+b+c = 0$

system is consistent if $a+b+c = 0$ //

Reduced system of equations:

$x - 2y + z = b$

$-3y + 3z = a + 2b$

when $(a, b, c) = (1, 1, -2)$

$x - 2y + z = 1$

$-3y + 3z = 3$

Here z is a free variable.

Let $z = k$

$\Rightarrow -3y + 3k = 3$

$-y + k = 1$

$\boxed{y = k-1}$

and $x - 2y + z = 1$

$x = 1 - z + 2y$

$= 1 - k + 2[k-1]$

$= 1 - k + 2k + (2)$

$= -1 + k$

$= k - 1$

$\boxed{x = k-1}$

we get infinite solution:-

$x = k-1$

$y = k-1$

$z = k$ //

if considering $k = 2$

$x = 1, y = 1, z = 2$ //

Q) Find values of λ for which system is has solution-

$x+y+z=1$

$x+2y+4z=\lambda$

$x+4y+10z=\lambda^2$

→ Writing system of equations in matrix form: $AX=B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

Augmented matrix:

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{array}\right]$$

operate $R_2' \to R_2 - R_1$ & $R_3' \to R_3 - R_1$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 3 & 9 & \lambda^2-1 \end{array}\right]$$

operate $R_3' \to R_3 - 3R_2$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 0 & 0 & \lambda^2-3\lambda+2 \end{array}\right]$$

$\rho(A) = 2$

$\rho(A:B)$ should be equal to $\rho(A)$ ie 2 for the system of eq^{ns} to have solut^n.

$\rho(A:B) = \rho(A) = 2$

if $\lambda^2 - 3\lambda + 2 = 0$

$\Rightarrow \lambda^2 - 2\lambda - \lambda + 2 = 0$

$\Rightarrow \lambda(\lambda-2) - 1[\lambda-2] = 0$

$(\lambda-1)(\lambda-2) = 0$

$\Rightarrow \lambda = 1$ or $\lambda = 2$

For $\lambda=1$ or $\lambda=2$ the system has ~~no~~ solut^n {~~for any value of λ except $\lambda=1$ or $\lambda=2$, system of eq^n~~ is consistent ie ~~has solution~~}

Chapter 2. Vector space

Vector representation: $\begin{bmatrix}1\\2\\3\end{bmatrix}$ (column vector), $[1\ 2\ 3]$ row

- An ordinary no. is called as scalar
- Algebraic definit^n of vector – it's list of scalars in square brackets.
- Vector dimension is no. of scalars in the list.
- Vectors can be represented horizontally or vertically.

  Eg: $[1\ 2\ 3] \rightarrow$ row vector $\begin{bmatrix}2\\3\\4\end{bmatrix} \rightarrow$ column vector.

- zero vector is unique, it's only vector with no direct^n.
- A sequence of real no.s $x_1, \ldots\ldots x_n$ is called an ordered n-tuple.

R^n – n-dimensional space:

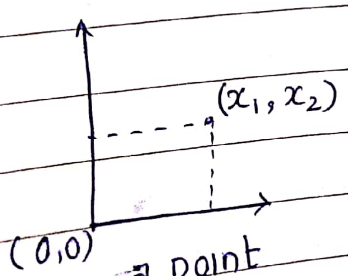
A set of all ordered n-tuple.

If $n=1$ R : 1-space $\rightarrow$ set of all real numbers

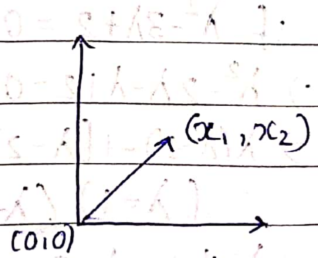
If $n=2$ R^2 : 2-space $\rightarrow$ set of all ordered pairs of real numbers (x_1, x_2)

If $n=3$ R^3 : 3-space $\rightarrow$ set of ~~of~~ all ordered triple of real numbers (x_1, x_2, x_3)

Representat^n of a point and a vector:



a point



a vector.

Scanned by CamScanner

let $U=(U_1, U_2, U_3, \dots U_n)$ and $V=(V_1, V_2, V_3, \dots V_n)$ are the vectors in R^n.

* Vectors are equal:
  Iff $U_1 = V_1,\ U_2 = V_2,\ U_3 = V_3, \dots\dots U_n = V_n$
* vector addition:- $U+V = (U_1+V_1, U_2+V_2, U_3+V_3, \dots U_n+V_n)$
* scalar multiplication: $KU = (KU_1, KU_2, KU_3, \dots\dots KU_n)$

Remark:

Sum of two vectors & scalar multiple of a vector in R^n are called standard operations in R^n.

Vector space:

Let F be a field, the elements of F are called scalars, if V is a non empty set whose elements are called as vectors the it's said to form a vector space over the field F. if the following properties holds good-

(i) Closure property: For every vectors $u, v \in V$ then $u+v$ is also a vector of V.

(ii) Associative property: For each vectors $u, v, w \in V$ then we have $(u+v)+w = u+(v+w)$

(iii) Existence of identity: There exist an element $0 \in V$ such that $0+u = u+0 = u\ \forall\ u \in V$, where 0 is called "additive identity of V.

(iv) Existence of inverse: For every element $u \in V$ there exist an ement $-u \in V$ such that $u+(-u) = 0$ where 0 being additive identity. {$-u$ is additive inverse of A}

(v) Commutative property: If vectors $u, v \in V$ then
$u+v = v+u$

(v) Scalar multiplication property: For scalar $\alpha \in F$ & vector $u \in V$ the we have $\alpha u \in V$ i.e V is closed wrt scalar multiplicatn $(\cdot)$.

(vi) Two composition$(\cdot)$ scalar multiplication & addition of vectors$(+)$ satisfy the following-

a) $\alpha(u+v) = \alpha u + \alpha v \quad \forall \alpha \in F;\ u, v \in V$

b) $(\alpha+\beta)u = \alpha u + \beta u \quad \forall\ \alpha, \beta \in F;\ u \in V$

c) $(\alpha\beta)u = \alpha(\beta u) \quad \forall\ \alpha, \beta \in F,\ u \in V$

d) $1 \cdot u = u \quad \forall u \in V$; 1 is the unit element of F.

e)

The algebraic structure $\langle V, F, +, \cdot \rangle$ ~~{vector space over~~ is said to form a vector space over field F ie V(F) is a vector ~~spays~~ space obeys 1 to 7 laws.

Subspace:

A subspace of a vector space is a non empty subset that satisfy requirements for a vector space that is linear combinations stay in the subspace.

i) If we add any two vectors u & v in the subspace then u+v is also in the subspace.

ii) If we multiply any vector u in the subspace by a scalar k such that ku is also in subspace.

iii) Subspace is a subset ie closed under additⁿ & scalar multiplicatⁿ.

NOTE :- 0 vector belongs to every subspace.

Ex :- (i) $R^n = \{(x_1, x_2, x_3, \cdots x_n) \mid x_i \in R, 1 \leq i \leq n\}$ is a vector space

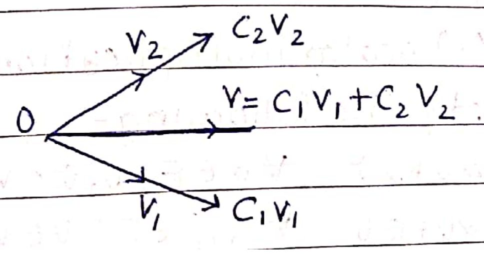
(ii) $P_n(t)$ denotes set of all polynomials $P(t)$ of degree n for $n \geq 0$.

$P(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$

$P_n(t)$ is a vector space over field F.

Linear Combinations:

Let V be a vector space over field F, a vector v in V is a linear combinatⁿ of $u_1, u_2, u_3, \cdots u_n$ in V where V is vectorspace over field F. If there exist scalars in F such that $v = c_1 u_1 + c_2 u_2 + c_3 u_3 + \cdots + c_n v_n$



v is a linear combination of u_1 to u_n if there is a soln. to the vector equation $v = c_1u_1 + c_2u_2 + \cdots + c_nu_n$ where $c_1, c_2, \cdots c_n$ are unknown scalars

Remark:

Expressing a vector v as a linear combinatn of $u_1, u_2, \cdots u_n$ in R^n is equivalent to solving $AX=B$ where v is columns of B and ~~use~~ u's are columns of coefficient matrix A. Such a system may have unique soln., many solution or no soln. Here no soln. means that v can't be written as linear combinatn of $u_1, u_2, \cdots u_n$.

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LPQ7) In the vectorspace R^3 express vector $(1,-2,5)$ as the linear combination of vectors $(1,1,1)$, $(1,2,3)$, $(2,-1,1)$.

— Let $v = (1,-2,5)$

$u_1 = (1,1,1)$

$u_2 = (1,2,3)$

$u_3 = (2,-1,1)$

We express v as linear combination of u_1, u_2, u_3 we find x, y, z such that $v = xu_1 + yu_2 + zu_3$

$$\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + z\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

i.e

$x+y+2z=1$

$x+2y-z=-2$

$5x+3y+z=5$

System of equations in matrix form:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

Augmented matrix:

$$= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array}\right]$$

operate $R_2' \to R_2 - R_1$ & $R_3' \to R_3 - R_1$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{array}\right]$$

operate $R_3' \to R_3 - 2R_2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{array}\right]$$

Since $\rho(A) = \rho(A:B) = n = 3$ system is consistent & has unique solution.

Writing system of equations from reduced matrix-

$x + y + 2z = 1 \Rightarrow x = -6$

$y - 3z = -3 \Rightarrow y = 3$

$5z = 10 \Rightarrow z = 2$

$\therefore$ Given vector v can be expressed as a linear combination of U_1, U_2, U_3 as shown below:-

$$\therefore \boxed{V = -6U_1 + 3U_2 + 2U_3}$$

PQ2) Express polynomial $V = 3t^2+5t-1$ as a linear combination of polynomial $P_1 = t^2+2t+1$, $P_2 = 2t^2+5t+4$, $P_3 = t^2+3t+6$.

— We express V as a linear combinat^n of P_1, P_2, P_3.

We find x, y, z such that $v = xP_1 + yP_2 + zP_3$

$3t^2+5t-1 = x[t^2+2t+1] + y[2t^2+5t+4] + z[t^2+3t+6]$

$\Rightarrow 3t^2+5t-1 = t^2[x+2y+z] + t[2x+5y+3z] + [x+4y+6z]$

equating coefficients of t^2, t & constants on LHS & RHS.

$x+2y+z=3$

$2x+5y+3z=5$

$x+4y+6z=-1$.

Writing system of eq^{ns} in matrix form:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$$

Augmented matrix:

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 5 & 3 & 5 \\ 1 & 4 & 6 & -1 \end{array}\right]$$

operate $R_2' \to R_2 - 2R_1$ & $R_3' \to R_3 - R_1$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 5 & -4 \end{array}\right]$$

Operate $R_3' \to R_3 - 2R_1'$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 3 & -2 \end{array}\right]$$

$\rho(A) = \rho(A:B) = 3 = n$

$\therefore$ system is consistent & has unique solutions:-

$\therefore$ V can be expressed as linear combinat^n of P_1, P_2, P_3.

$\Rightarrow \begin{cases} x+2y+z=3 \\ y+z=-1 \Rightarrow \boxed{y=-1/3} \\ \boxed{z=-2/3} \end{cases}$ {Reduced form of equations}

$\Rightarrow x = \frac{13}{3}$

$\therefore \boxed{V = \frac{13}{3}P_1 - \frac{1}{3}P_2 - \frac{2}{3}P_3}$

LP Q23) Express M as a linear combination of matrices A, B, C where

$M = \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$

→ x, y, z such that $M = Ax + By + Cz$

$$\begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix} = x\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + y\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + z\begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} x+y+z & x+2y+z \\ x+3y+4z & x+4y+5z \end{bmatrix}$$

Equating corresponding entries we get:

$x+y+z=4$

$x+2y+z=7$

$x+3y+4z=7$

$x+4y+5z=9$

Writing system of equat^{ns} in matrix form:-

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 7 \\ 9 \end{bmatrix}$$

Augmented matrix:-

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 1 & 7 \\ 1 & 3 & 4 & 7 \\ 1 & 4 & 5 & 9 \end{array}\right]$$

operate $R_2' \to R_2 - R_1$, $R_3' \to R_3 - R_1$, $R_4' \to R_4 - R_1$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 3 & 3 \\ 0 & 3 & 4 & 5 \end{array}\right]$$

operate $R_3' \to R_3 - 2R_2$ & $R_4' \to R_4 - 3R_2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 4 & -4 \end{array}\right]$$

Operate $R_4' \to R_4 - R_3$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = \rho(A:B) = n = 3$

∴ system is consistent & has unique solution.

∴ M can be expressed as linear combinat^n of A, B, C

reduced ~~for~~ equations: $x+y+z=4$; $y=3$; $3z=-3$

$\Rightarrow$ $3z=-3 \Rightarrow (z=-1)$

$y=3$

$\boxed{x=2}$

∴ $M = 2A+3B-C$

Q) Express vector $(2,-5,3)$ as a linear combination of $(1,-3,2)$, $(2,-4,-1)$, $(1,-5,7)$.

→ $V=(2,-5,3)$, $U_1=(1,-3,2)$, $U_2=(2,-4,-1)$, $U_3=(1,-5,7)$

- We express $(V = U_1x + U_2y + U_3z)$ V as linear combination of U_1, U_2, U_3 ~~such that~~ we find x, y, z such that

$V = U_1x + U_2y + U_3z$.

$$V = x\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix} + z\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

Writing system of eq^{ns}:

$x+2y+z=2$

$-3x-4y-5z=-5$

$2x-y+7z=3$

Writing system of equations in matrix form:

$$\begin{bmatrix} 1 & 2 & 1 \\ -3 & -4 & -5 \\ 2 & -1 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

A

P.T.O

Augmented matrix :-

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ -3 & -4 & -5 & -5 \\ 2 & -1 & 7 & 3 \end{array}\right]$$

operate $R_2' \to R_2 + 3R_1$ & $R_3' \to R_3 - 2R_1$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & -5 & 5 & -1 \end{array}\right]$$

operate $R_3' \to 2R_3 + 5R_2$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

$\rho(A) = 2$ and $\rho(A:B) = 3$

since $\rho(A) \neq \rho(A:B)$

The system is inconsistent.

$\therefore$ Given vector $(2,-5,3)$ can't be expressed as linear combination of U_1, U_2, U_3.

Spanning sets:

6/2/19 Let V be vector space over field F of vectors $U_1, U_2, \cdots U_m$ in V are said to span V or to form a spanning set of V if every vector v in V is a linear combinat^n of vectors $U_1, U_2, \cdots U_m$ ie if there exists scalars in F such that

$V = C_1U_1 + C_2U_2 + \cdots C_mU_m$.

Linear span:

Suppose $U_1, U_2, \cdots U_m$ are any vectors in vectorspace V any vector of form $C_1U_1 + C_2U_2 + \cdots + C_mU_m$ is said to be linear combination of $U_1, U_2, \cdots U_m$ where $C_1, C_2, C_3, \cdots C_m$ are scalars. The collect^n of all such linear combinat^{ns} is called linear span of $U_1, U_2, \cdots U_m$ and is denoted by span $(U_1, U_2, \cdots U_m)$ or span(U_i) where $i = 1, 2, 3, \cdots m$

Clearly zero vector belongs to linear span of v_i

$0 = 0v_1 + 0v_2 + 0v_3 + \cdots + 0v_m$.

Procedure to check whether the vectors $v_1, v_2, v_3, \cdots v_n$ span the vector space V is as follows:

Step 1: choose any arbitrary vector v in vector space V

Step 2: Determine whether v is a linear combinatn of given vector. If it is, then given vectors span v. If it's not, then they donot span v.

P 11 Q) Show that given vectors span R^3.

$v_1(1,1,1)$, $v_2(1,2,3)$, $v_3(1,5,8)$

— We need to show that an arbitrary vector $v(a,b,c)$ in R^3 is a linear combination of v_1, v_2, v_3.

$\therefore$ set $V = xv_1 + yv_2 + zv_3$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + z\begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix}$$

Writing the above matrix as system of eqns...

$x + y + z = a$

$x + 2y + 5z = b$

$x + 3y + 8z = c$

Write the above system of eqn in matrix form-

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Augmented matrix:

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & a \\ 1 & 2 & 5 & b \\ 1 & 3 & 8 & c \end{array}\right]$$

Operate $R_2' \to R_2 - R_1$ & $R_3' \to R_3 - R_1$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & a \\ 0 & 1 & 4 & b-a \\ 0 & 2 & 7 & c-a \end{array}\right]$$

operate $R_3' \to R_3 - 2R_2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 4 & b-a \\ 0 & 0 & -1 & a-2b+c \end{array}\right] \text{ is in REF}$$

$\rho(A) = \rho(A:B) = 3 = n$

$\Rightarrow$ unique soln. & system of eq^n is consistent.

$\therefore$ v can be expressed as linear combinat^n of u_1, u_2, u_3

$\therefore u_1, u_2, u_3$ span R^3

LP 12) Q) Which of following span R^4?

a) let $u_1 = (1,0,0,1)$, $u_2 = (0,1,0,0)$, $u_3 = (1,1,1,1)$, $u_4 = (1,1,1,0)$

We need to show that u_1, u_2, arbitrary vector $v(a,b,c,d)$ in R^4 is a linear combinat^n of u_1, u_2, u_3, u_4

— $V = U_1 x + U_2 y + U_3 z + U_4 w$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

writing system of eq^{ns}:-

$x + z + w = a$

$y + z + w = b$

$z + w = c$

$x + z = d$

writing system of eq^{ns} in matrix form:-

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Augmented matrix:

$$= \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & a \\ 0 & 1 & 1 & 1 & b \\ 0 & 0 & 1 & 1 & c \\ 1 & 0 & 1 & 0 & d \end{array}\right]$$

operate $R_4' \to R_4 - R_1$

$$= \left[\begin{array}{cccc:c} 1 & 0 & 1 & 1 & a \\ 0 & 1 & 1 & 1 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 0 & -1 & d-a \end{array}\right]$$

$\rho(A) = \rho(A:B) = n = 4$

$\therefore$ system of equations are consistent & have unique soln.

$\therefore U_1, U_2, U_3, U_4$ span $\mathbb{R}^4$.

(b) $U_1 = (1,2,1,0)$, $U_2 = (1,1,-1,0)$, $U_3 = (0,0,0,1)$

We need to show that arbitrary vector V can be expressed as linear combinat^n of U_1, U_2, U_3.

$V = U_1 x + U_2 y + U_3 z$

$$\therefore V = x \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Writing system of equations:-

$x + y = a$

$2x + y = b$

$x - y = c$

$z = d$

Writing the matrix using above equat^{ns}:-

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Augmented matrix:-

$$= \left[\begin{array}{ccc:c} 1 & 1 & 0 & a \\ 2 & 1 & 0 & b \\ 1 & -1 & 0 & c \\ 0 & 0 & 1 & d \end{array}\right]$$

Operate $R_2' \to R_2 - 2R_1$, $R_3' \to R_3 - R_1$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 0 & a \\ 0 & -1 & 0 & b-2a \\ 0 & -2 & 0 & c-a \\ 0 & 0 & 1 & d \end{array}\right]$$

operate $R_3' \to R_3 - 2R_2$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 0 & a \\ 0 & -1 & 0 & b-2a \\ 0 & 0 & 0 & c-b+a \\ 0 & 0 & 1 & d \end{array}\right]$$

operate $R_3 \longleftrightarrow R_4$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 0 & a \\ 0 & -1 & 0 & b-2a \\ 0 & 0 & 1 & \cancel{a+b+c}\ d \\ 0 & 0 & 0 & a-b+c \end{array}\right]$$

$\rho(A) = 3 \neq \rho(A:B) = 4$

$\therefore$ system of eqns are ~~incorrect~~ inconsistent.

$\therefore$ v can't be expressed in linear combinat^n of v_1, v_2, v_3

$\therefore v_1, v_2, v_3$ don't span R^4.

LPQ13) Which set of polynomials are span for P_2 {polynomial of degree 2}?

a) $\{t^2+1, t^2+t, t+1\}$

We need to express an arbitrary polynomial $V = at^2+bt+c$ in P_2 as a linear combinat^n of $P_1 = t^2+1$, $P_2 = t^2+t$, $P_3 = t+1$

$\rightarrow$ $V = xP_1 + yP_2 + zP_3$

$[at^2+bt+c] = x[t^2+1] + y[t^2+t] + (t+1)z$

$at^2+bt+c = t^2[x+y] + t[y+z] + (x+z)$

Comparing coefficients of t^2, t & constant in LHS & RHS.

$x+y = a$

$y+z = b$

$x+z = c$

writing system of eqns in matrix form:.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Augmented matrix:

$$= \left[\begin{array}{ccc:c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 1 & 0 & 1 & c \end{array}\right]$$

operate $R_3' \to R_3 - R_1$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 0 & -1 & 1 & c-a \end{array}\right]$$

operate $R_3' \to R_3 + R_2$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 2 & -a+b+c \end{array}\right]$$

$\rho(A) = \rho(A:B) = n = 3$

$\therefore$ system of eqns are consistent with unique solutions.

$\Rightarrow$ V can be expressed as linear combinatn of P_1, P_2, P_3 $\therefore$ P_1, P_2, P_3 span V.

b) $\{t^2+1, t-1, t^2+t\}$

- We need to express an arbitrary polynomial $V = at^2+bt+c$ in P_2 as a linear combinatn of $P_1 = t^2+1$, $P_2 = t-1$, $P_3 = t^2+t$

$V = xP_1 + yP_2 + zP_3$

$[at^2+bt+c] = x[t^2+1] + y[t-1] + z[t^2+t]$

$[at^2+bt+c] = t^2[x+z] + t[y+z] + [x-y]$

comparing coefficient of t^2, t & constant on LHS & RHS:-

$x+z = a$

$y+z = b$

$x-y = c$

Writing system of eq^{ns} in matrix form

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

~~operate $R_3' \to R_3 - R_1$~~ Augmented matrix:

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b \\ 1 & -1 & 0 & c \end{array}\right]$$

operate $R_3' \to R_3 - R_1$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & -1 & -1 & c-a \end{array}\right]$$

operate $R_3' \to R_3 + R_2$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & c-a-b \end{array}\right]$$

$\rho(A) = 2 \neq \rho(A:B) = 3$

$\therefore$ system ~~of~~ equations is inconsistent

$\therefore$ v can't be expressed as linear $combinat^{ns}$ of P_1, P_2, P_3

$\therefore$ P_1, P_2, P_3 don't space ~~vector~~ polynomial P_2.

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LP Q22) Find conditions on a, b, c so that $V = (a, b, c)$ in R^3 belongs to $W = \text{span}(U_1, U_2, U_3)$ where $U_1 = (1,2,0)$, $U_2 = (-1,1,2)$ & $U_3 = (3,0,-4)$

— let $v = (a,b,c)$ in $R^3 \in W = \text{span}(U_1, U_2, U_3)$ ie v is a linear combination of U_1, U_2, U_3 $\therefore V = xU_1 + yU_2 + zU_3$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + z\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

expressing above matrix in system of eq^{ns}.

$x - y + 3z = a$

$2x + y = b$

$2y - 4z = c$

Writing above system of equations in matrix form:- $AX=B$.

$$\Rightarrow \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Augmented matrix

$$= \left[\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 2 & 1 & 0 & b \\ 0 & 2 & -4 & c \end{array}\right]$$

operate $R_2' \to R_2 - 2R_1$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 0 & 3 & -6 & b-2a \\ 0 & 2 & -4 & c \end{array}\right]$$

operate $R_3' \to 3R_3 - 2R_2$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 0 & 3 & -6 & b-2a \\ 0 & 0 & 0 & 3c-2b+4a \end{array}\right]$$

$\rho(A)=2$ & $\rho(A:B)=3$, this makes system of equat^{ns} inconsist.

To make system of eq^n consistent $\rho(A:B)=2$

$\Rightarrow 3c-2b+4a=0$

$\therefore$ When $3c-2b+4a=0$, V can be expressed as linear combinat^n of U_1, U_2, U_3 & U_1, U_2, U_3 span V.

LPQ24) For what values of scalar k will the vector Y be in span. (V_1, V_2, V_3) where $V_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $V_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$, $V_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$, $Y = \begin{bmatrix} -4 \\ 3 \\ -k \end{bmatrix}$

$\rightarrow$ Given $Y \in \text{span}(V_1, V_2, V_3)$

$\Rightarrow$ Y can be expressed as linear combination of linear combinat^n of V_1, V_2 & V_3

ie $Y = xV_1 + yV_2 + zV_3$

$$\begin{bmatrix} -4 \\ 3 \\ -k \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + y\begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + z\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

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$x+5y-3z=-4$
$-x-4y+z=3$
$-2x-7y=-k$

Expressing above system of eq^{ns} in matrix form:-

$AX=B$

$$\begin{bmatrix} 1 & 5 & -3 \\ -1 & -4 & 1 \\ -2 & -7 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ -k \end{bmatrix}$$

Augmented matrix:-

$$= \left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & -k \end{array}\right]$$

operate $R_2' \to R_2+R_1$ & $R_3' \to R_3+2R_1$

$$= \left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & -k+8 \end{array}\right]$$

operate $R_3' \to R_3-3R_2$

$$= \left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & -k-5 \end{array}\right]$$

For system to be consistent

$R[A] = R[A:B] = 2$

$R[A:B] = 2$

if $-k-5=0$

i.e. $k=-5$

$\therefore$ ~~V_1, V_2~~ Y can be expressed as linear combinat^n of V_1, V_2 & V_3 if $k=-5$ & V_1, V_2, V_3 span Y if $k=-5$.

Null space:

The null space of a matrix of order $m \times n$ is written as null A and is the set of solution to the homogenous eqn $AX=0$
$Nul = \{x \mid x \in R^n \ \& \ AX=0\}$

Remark: Nul A is a subspace of R^n.
OR The soln. to $AX=0$ form of a vector space called null space of A.

Procedure to find spanning set for homogeneous system $AX=0$.

To determine a spanning set for the soln. space of homogeneous system, we have to find Reduced row echelon form of augmented matrix.

LPQ 17) Find the set of vectors spanning the null space of

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}$$

- To determine a spanning set for the sol^n space of homogeneous system, we reduce A to RREF.

operate $R_2' \to R_2 - R_1$, $R_3' \to R_3 - 2R_1$, $R_4' \to R_4 - R_1$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

operate $R_3' \to R_4 - R_3$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Operate $R_3 \leftrightarrow R_4$

$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 1 & \cancel{0} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operate $R_3' \to 2R_3 - R_2$

$$A = \begin{bmatrix} 1 & 0 & \cancel{0} & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operate $R_2' \to \frac{1}{2} R_2$

$$A = \begin{bmatrix} 1 & 0 & \cancel{0} & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operate $R_1' \to R_1 - R_3$

$$A = \begin{bmatrix} 1 & 0 & \cancel{0}1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In RREF

with 1, 2, 4 as pivot column.

Solving $AX = 0$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

Writing the system of equat^{ns}

$x_1 + x_3 = 0$

$x_2 + x_3 = 0$

$x_4 = 0$

Here x_3 is a free variale let $x_3 = k$

$\therefore x_1 = -x_3 = -k$

$x_2 = -x_3 = k$

$x_4 = 0$

$x_1 = -k$

$x_2 = -k$

$x_3 = k$

$x_4 = 0$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -k \\ -k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ vector $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ span the soln. space.

LPQ18> Find the set of vectors spanning null space of

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 3 & 6 & -2 \\ -2 & 1 & 2 & 2 \\ 0 & -2 & -4 & 0 \end{bmatrix}$$

To determine spanning set for soln. space of homogeneous system we reduce A to RREF.

operate $R_3' \to R_3 + 2R_1$ & $R_2' \to R_2 - 2R_1$

$$= \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -2 & -4 & 0 \end{bmatrix}$$

operate $R_3 \to R_3' - 3R_2$

$$= \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \cancel{0}-2 & -4 & 0 \end{bmatrix}$$

operate $R_3 \leftrightarrow R_4$

$$= \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operate $R_3' \to R_3 + 2R_2$

$$= \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operate $R_1' \to R_1 - R_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1 & 2 columns are pivot colums.

Solving for $AX=0$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Writing system of equations from matrix:

$x_1 - x_4 = 0$

$x_2 + 2x_3 = 0$

x_4 & x_3 are free variables

let $x_3 = s$ & $x_4 = r$

$x_1 = x_4 = r$

$x_2 = -2x_3 = -2s$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} r \\ -2s \\ s \\ r \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ 0 \\ r \end{bmatrix} + \begin{bmatrix} 0 \\ -2s \\ s \\ 0 \end{bmatrix}$$

$$= r\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ vectors

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ span the sol^n space & or null space.

Q) Find the set of vectors which span the solut^n space of homogeneous system $AX=0$ where

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 3 \\ 4 & 4 & -1 & 9 \end{bmatrix}$$

- to determine spanning vectors of soln. space, we reduce A to RREF

operate $R_2' \to R_2 + 2R_1$, $R_3' \to R_3 - R_1$, $R_4' \to R_4 - 4R_1$

$$= \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Operate $R_3' \to R_2 + R_3$ & $R_4' \to R_4 + R_2$

$$= \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In RREF with 1&3 as pivot columns.

~~opera~~ Solving for $AX=0$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Writing system of eq^{ns} from matrix -

$2x_4 + x_1 + x_2 = 0$

$x_3 - x_4 = 0$

x_2 & x_4 are free variable.

let $x_2 = s$ & $x_4 = r \Rightarrow x_1 = -2r - s$ & $x_3 = r$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2r-s \\ s \\ r \\ r \end{bmatrix} = \begin{bmatrix} -2r \\ 0 \\ r \\ r \end{bmatrix} + \begin{bmatrix} -s \\ s \\ 0 \\ 0 \end{bmatrix} = r \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ vectors $\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ span the sol^n space or null space.

Linear Independance & dependance:-

- Set of vectors $\{V_1, V_2, \dots V_3\}$ in V in said to be linearly independent if vectorequation ~~has on~~ $C_1V_1 + C_2V_2 + C_3V_3 + \dots + C_nV_n = 0$ has only trivial solution. ie $C_1 = 0, C_2 = 0, C_3 = 0, \dots C_n = 0$.
- Set of vectors ~~in~~ $\{V_1, V_2, \dots V_n\}$ in V is said to be linearly dependent if vector eqn $C_1V_1 + C_2V_2 + \dots C_nV_n = 0$ has a non trivial solution ie if there are some $C_1, C_2, \dots C_n$ not all are zero such that $C_1V_1 + C_2V_2 + \dots + C_nV_n = 0$ holds

Remark:-

1) In R^n, a set containing single vector V is linearly independent iff that vector $V \neq 0$.

ii) Also, a set of two vectors is linearly dependent iff one of the vectors is a multiple of other.

iii) Any set containing zero vector is linearly dependent

iv) Non zero rows of a matrix in echelon form are linearly independent & rank of matrix is num of independent rows or vectors.

PQ4) Determine whether following vectors in R^3 are linearly dependent or linearly independent.

(i) $(1,2,3), (4,5,6), (2,1,0)$

— let $U_1 = (1,2,3)$

$U_2 = (4,5,6)$

$U_3 = (2,1,0)$

set a linear combinatn of $U_1, U_2, U_3 = 0$ vector using unknowns x, y, z to obtain the homogeneous system of linear eqns & reduce system to row echelon form.

This yields

$$xU_1 + yU_2 + zU_3 = 0$$

$$x\begin{bmatrix}1\\2\\3\end{bmatrix} + y\begin{bmatrix}4\\5\\6\end{bmatrix} + z\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$x+4y+2z=0$

$2x+5y+z=0$

$3x+6y=0$

In matrix form $AX=0$

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix:

$$= \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix}$$

Operate $R_2' \to R_2 - 2R_1$, $R_3' \to R_3 - 3R_1$

$$= \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix}$$

operate $R_3' \to R_3 - 2R_2$

$$= \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

1^{st} & 2^{nd} is a pivot column & 3^{rd} column is not a pc with z as a free variable.

writing reduced eq^{ns}-

$x+4y+2z=0$

$-3y-3z=0$

System of eq^{ns} has 2 eq^n & 3 unknowns, it has a free variable meaning non zero soln.

$\therefore$ U_1, U_2, U_3 are linearly dependent.

(ii) $(1,2,-3)$, $(1,-3,6)$, $(2,1,1)$

let $U_1 = (1,2,-3)$, $U_2 = (1,-3,6)$; $U_3 = (2,1,1)$

Set a linear combinat^n of U_1, U_2, U_3 = 0 vector using unknowns x, y, z to obtain homogeneous system of linear eq^{ns} & reduce system to row echelon form.

$xU_1 + yU_2 + zU_3 = 0$

$$x\begin{bmatrix}1\\2\\-3\end{bmatrix}+y\begin{bmatrix}4\\-7\\-1\end{bmatrix}+z\begin{bmatrix}1\\2\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Writing system of eq^{ns} -

$x+4y+z=0$

$2x-7y+2z=0$

$-3x-y+2z=0$

Writing above ~~matrix~~ eq^{ns} in matrix form:-

(x)

$$\begin{bmatrix}1&4&1\\2&-7&2\\-3&-1&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Augmented matrix:-

$$=\begin{bmatrix}1&4&1\\2&-7&2\\-3&-1&2\end{bmatrix}$$

operate $R_2' \to R_2-2R_1$ & $R_3' \to R_3+3R_1$

$$=\begin{bmatrix}1&4\\0&\\0&1\end{bmatrix}$$

$$x\begin{bmatrix}1\\2\\-3\end{bmatrix}+y\begin{bmatrix}1\\-3\\6\end{bmatrix}+z\begin{bmatrix}2\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Writing matrix eq^{ns} :-

$x+y+2z=0$

$2x-3y+z=0$

$-3x+6y+z=0$

Matrix form:-

$$\begin{bmatrix}1&1&2\\2&-3&1\\-3&6&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Augmented matrix :- $\begin{bmatrix}1&1&2\\2&-3&1\\-3&6&1\end{bmatrix}$

operate $R_2' \to R_2 - 2R_1$; $R_3' \to R_3 + 3R_1$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & -3 \\ 0 & 9 & 7 \end{bmatrix}$$

operate $R_3' \to 5R_3 + 9R_2$

$$= \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & -3 \\ 0 & 0 & 8 \end{bmatrix}$$

RE system having 3 eq^{ns} & 3 unknowns thus giving trivial solns.

$x + y + 2z = 0 \Rightarrow x = 0$

$-5y - 3z = 0 \Rightarrow y = 0$

$8z = 0 \Rightarrow z = 0$

$\therefore U_1, U_2, U_3$ are linearly independent.

P.T.O

LPQ5) let $X_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, X_2 = \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix}, X_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ belong to soln. space of $AX=0$. Is (X_1, X_2, X_3) linearly independent.

Set a linear of $X_1, X_2, X_3 = 0$ vector using variables x, y, z to obtain homogeneous eqns & reduce matrix to REF.

$$x \begin{bmatrix} 2 \\ -1 \\ +1 \end{bmatrix} + y \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix} + z \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0$$

$2x + 4y + z = 0$

$-x - 7y + 2z = 0$

$+x - y + 3z = 0$

Solving for $AX = 0$

$$\begin{bmatrix} 2 & 4 & 1 \\ -1 & -7 & 2 \\ +1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix:-

$$= \begin{bmatrix} 2 & 4 & 1 \\ -1 & -7 & 2 \\ 1 & -1 & 2 \end{bmatrix}$$

operate $R_2' \to 2R_2 + R_1$ & $R_3' \to 2R_3 - R_1$

$$= \begin{bmatrix} 2 & 4 & 1 \\ 0 & -10 & 5 \\ 0 & -6 & 3 \end{bmatrix}$$

$R_2' \to \frac{-1}{5} R_2$, $R_3' \to \frac{-1}{3} R_3$

$$= \begin{bmatrix} 2 & 4 & 1 \\ 0 & +2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

operate $R_3' \to R_3 - R_2$

$$= \begin{bmatrix} 2 & 4 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Row echelon system has 2 eq^n & 3 unknown with a free variable which gives a nonzero solut^n which makes x_1, x_2, x_3 linearly independent.

LPQ6) let $X_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ 6 \\ 2 \\ 0 \end{bmatrix}$ belong to nullspace of $AX = 0$. Is (X_1, X_2, X_3) linearly independent.

- Set a linear combinat^n of $x_1, x_2, x_3 = 0$ vector using unknowns x, y & z to obtain homogeneous eq^{ns} & reduce matrix to REF.

$$xX_1 + yX_2 + zX_3 = 0$$

$$x\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + z\begin{bmatrix} 1 \\ 6 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

~~xt~~ eq^{ns} will be

$x + y + z = 0$

$2x + 6z = 0$

$-y + 2z = 0$

$x + y = 0$

Matrix equat^{ns}:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 6 \\ 0 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix :-

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 6 \\ 0 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

operate $R_2' \to R_2 - 2R_1$; $R_4' \to R_4 - R_1$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

operate $R_3' \to 2R_3 - R_2$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

operate $R_3 \leftrightarrow R_4$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$x + y + z = 0$

$-2y + 4z = 0$

$z = 0$

Row echelon system has 3 equations & 3 unknowns

$\therefore$ Giving trivial solution $x = y = z = 0$

$\therefore$ x_1, x_2, x_3 are linearly independent.

Basis & dimensions:-

Definition:-

A set $S=\{U_1, U_2, \cdots U_n\}$ is a basis of V if it has following properties-

(i) S is linearly Independent

(ii) S spans V

Remark: A vector V is said to be finite dimension n on n-dimensional written as $\dim V = n$, if V has a basis with n elements. (or) number of vectors in a basis set for the vector space is the dimension of vector space V & is denoted by $\dim(V)=n$.

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LPQ3> Show that vectors: $x_1=(1,5,2)$, $x_2=(0,0,1)$; $x_3=(1,1,0)$ forms basis for vector space R^3.

— let $S=\{x_1, x_2, x_3\}$

To show that given vectors are linearly independent:

Set the linear combination of x_1, x_2 & $x_3=0$ using x, y, z & form homogeneous equations & equate them to 0.

ie $xX_1+yX_2+zX_3=0$

$$x\begin{bmatrix}1\\5\\2\end{bmatrix}+y\begin{bmatrix}0\\0\\1\end{bmatrix}+z\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

eqns are: $x+z=0$

$5x+z=0$

$2x+y=0$

Solving $AX=0$

$$A=\begin{bmatrix}1&0&1\\5&0&1\\2&1&0\end{bmatrix}$$

operate $R_2' \to R_2-5R_1$ & $R_3' \to R_3-2R_1$

$$A=\begin{bmatrix}1&0&1\\0&0&-4\\0&-1&-2\end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -4 \end{bmatrix} \text{ is in REF}$$

$x+z=0 \Rightarrow x=0$

$y-2z=0 \Rightarrow y=0$

$-4z=0 \Rightarrow z=0$

Set S is linearly independent.

To prove S span R^3:-

~~Set a linear combin~~ To prove that S spans R^3;

let $v=(a,b,c) \in R^3$

$v = xX_1 + yX_2 + zX_3$

$$x\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + y\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + z\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

equations are: $x+z=a$

$5x+z=b$

$2x+y=c$

Writing the eq^{ns} in matrix form.

$$\begin{bmatrix} 1 & 0 & 1 \\ 5 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Augmented matrix:

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 5 & 0 & 1 & b \\ 2 & 1 & 0 & c \end{array}\right]$$

operate $R_2' \to R_2 - 5R_1$ and $R_3' \to R_3 - 2R_1$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 0 & -4 & b-5a \\ 0 & 1 & \cancel{0}-2\cancel{a} & c-2a \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & -2 & c-2a \\ 0 & 0 & -4 & b-5a \end{array}\right] \text{ in REF}$$

$R[A] = 3 = R[A:B]$

system is consistent

Span R^3

$\therefore$ S forms basis.

Remarks: Spanning set S will be linearly independent if $R[A]$ = number of unknowns.

(P Q8) Check whether vectors $x_1 = (1,0,-1)$, $x_2 = (1,2,1)$, $x_3 = (0,-3,2)$ form a basis for R^3.

— let $S = \{x_1, x_2, x_3\}$

To prove S is spanning set & linearly independent:

let v be an arbitrary vector $(a,b,c) \in R^3$

Expressing v as linear combination of x_1, x_2, x_3 using x, y, z.

$v = x x_1 + y x_2 + z x_3$.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + z \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$$

~~equs~~ equations : $x + y = a$

$2y - 3z = b$

$-x + y + 2z = c$

writing eqns in matrix form.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Augmented matrix:

$$= \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 2 & -3 & b \\ -1 & 1 & 2 & c \end{array}\right]$$

operate $R_3' \to R_1 + R_3$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 2 & -3 & b \\ 0 & 2 & 2 & c+a \end{array}\right]$$

operate $R_3' \to R_3 - R_2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 2 & -3 & b \\ 0 & 0 & 5 & c+a-b \end{array}\right]$$

$R[A] = 3$, $R[A:B] = 3$

Since $R[A] = R[A:B] = 3$

$\therefore$ system is consistent $\Rightarrow S$ spans R^3.

& $R[A] = 3 =$ number of unknown

we get trivial solution $x=0, y=0, z=0$ $\therefore S$ is linearly independent.

$\therefore$ Given set of vector form a basis.

LPQ10) Which of following set of vector are basis of R^4.

(i) $\{(1,0,0,1), (0,1,0,0), (1,1,1,1), (1,1,1,0)\}$

— let $v_1 = (1,0,0,1)$

$v_2 = (0,1,0,0)$

$v_3 = (1,1,1,1)$

$v_4 = (1,1,1,0)$

let $S = \{v_1, v_2, v_3, v_4\}$

To prove that S is a spanning set & S is linearly indepen-dent.

let $v = (a,b,c,d) \in R^4$

$v = x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

equat^{ns}: $x_1 + x_3 + x_4 = a$

$x_2 + x_3 + x_4 = b$

$x_3 + x_4 = c$

$x_1 + x_3 = d$

Writing equations in matrix form.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & a \\ 0 & 1 & 1 & 1 & b \\ 0 & 0 & 1 & 1 & c \\ 1 & 0 & 1 & 0 & d \end{array}\right]$$

operate $R_4' \to R_4 - R_1$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & a \\ 0 & 1 & 1 & 1 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 0 & -1 & d-a \end{array}\right]$$

since $\rho(A) = \rho(A:B) = 4$

system is consistent & S spans R^4

since $\rho(A) = 4 =$ no. of unknowns

These eq^{ns} on solving will give trivial solution $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$ $\therefore$ S is linearly independent and S forms basis for R^4.

(ii) $\{(1,-1,0,2), (3,-1,2,1), (1,0,0,1)\}$

$V_1 = (1,-1,0,2)$

$V_2 = (3,-1,2,1)$

$V_3 = (1,0,0,1)$

let $S = \{V_1, V_2, V_3\}$

To prove that S is a spanning set & S is linearly independ..

let $V = \{a, b, c, d\}$

$V = xV_1 + yV_2 + zV_3$

$$= x\begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} + y\begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

equat^{ns}

$x + 3y + z = a$

$-x - y = b$

$2y = c$, $2x + y + z = d$

Writing equat^{ns} in matrix form:-

$$\begin{bmatrix} 1 & 3 & 1 \\ -1 & -1 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Augmented matrix

$$= \left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ -1 & -1 & 0 & b \\ 0 & 2 & 0 & c \\ 2 & 1 & 1 & d \end{array}\right]$$

operate $R_2' \to R_2 + R_1$ & $R_4' \to R_4 - 2R_1$

$$= \left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 2 & 0 & b+a \\ 0 & 2 & 0 & c \\ 0 & -5 & -1 & d-2a \end{array}\right]$$

operate $R_3' \to R_3 - R_2$ & $R_4' \to 2R_4 + 5R_2$

$$= \left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 2 & 0 & b+a \\ 0 & 0 & 0 & c-b-a \\ 0 & 0 & -2 & 5d+3a \end{array}\right]$$

operate $R_3 \leftrightarrow R_4$

$$= \left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 2 & 0 & b+a \\ 0 & 0 & -2 & 5d+3a \\ 0 & 0 & 0 & c-b-a \end{array}\right]$$

$\rho(A) = 3$ & $\rho(A:B) = 4$

Since $\rho(A) \neq \rho(A:B)$

$\Rightarrow$ S doesn't span R^4

$\therefore$ S doesnot form basis of R^4.

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(PQ9) Which set of polynomials are span for P_3? — polynomial of degree 3

(i) $\{t^3+2t^2+3t,\ 2t^3+1,\ 6t^3+8t^2+6t+4,\ t^3+2t^2+t+1\}$

→ $P_1 = t^3+2t+3t$; $P_2 = 2t^3+1$; $P_3 = 6t^3+8t^2+6t+4$; $P_4 = t^3+2t^2+t+1$.

let $S = \{P_1, P_2, P_3, P_4\}$

To prove that S is a spanning set & linearly independent:

Let $V = at^3+bt^2+ct+d$ be an arbitrary polynomial belonging to P_3. $V = x_1P_1+x_2P_2+x_3P_3+x_4P_4$

$$at^3+bt^2+ct+d = x_1[t^3+2t^2+3t] + x_2[2t^3+1] + x_3[6t^3+8t^2+6t+4] + x_4[t^3+2t^2+t+1]$$

$$at^3+bt^2+ct+d = t^3[x_1+2x_2+6x_3+x_4] + t^2[2x_1+0x_2+8x_3+2x_4] + t[3x_1+0x_2+6x_3+x_4] + [0x_1+x_2+4x_3+x_4]$$

Equating the coefficients of t^3, t^2, t & constant on both sides.

$$x_1+2x_2+6x_3+x_4 = a$$
$$2x_1+8x_3+2x_4 = b$$
$$3x_1+6x_3+x_4 = c$$
$$x_2+4x_3+x_4 = d$$

Writing in matrix form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 & 1 \\ 2 & 0 & 8 & 2 \\ 3 & 0 & 6 & 1 \\ 0 & 1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Augmented matrix:

$$= \left[\begin{array}{cccc|c} 1 & 2 & 6 & 1 & a \\ 2 & 0 & 8 & 2 & b \\ 3 & 0 & 6 & 1 & c \\ 0 & 1 & 4 & 1 & d \end{array}\right]$$

operate $R_2' \to R_2 - 2R_1$
$R_3' \to R_3 - 3R_1$

$$= \begin{bmatrix} 1 & 2 & 6 & 1 & a \\ 0 & -4 & -4 & 0 & b-2a \\ 0 & -6 & -12 & -2 & c-3a \\ 0 & 1 & 4 & 1 & d \end{bmatrix}$$

$R_2 \longleftrightarrow R_4$

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$$= \left[\begin{array}{cccc|c} 1 & 2 & 6 & 1 & a \\ 0 & 1 & 4 & 1 & d \\ 0 & -6 & -12 & -2 & c-3a \\ 0 & -4 & -4 & 0 & b-2a \end{array}\right]$$

Operate $R_3' \to R_3 + 6R_2$ & $R_4' \to R_4 + 4R_2$

$$= \left[\begin{array}{cccc|c} 1 & 2 & 6 & 1 & a \\ 0 & 1 & 4 & 1 & d \\ 0 & 0 & 12 & 4 & c-3a+6d \\ 0 & 0 & 12 & 4 & b-2a+4d \end{array}\right]$$

Operate $R_4' \to R_4 - R_3$

$$= \left[\begin{array}{cccc|c} 1 & 2 & 6 & 1 & a \\ 0 & 1 & 4 & 1 & d \\ 0 & 0 & 12 & 4 & c-3a+6d \\ 0 & 0 & 0 & 0 & b+a-c-2d \end{array}\right]$$

$\rho[A] = 3$ & $\rho(A:B) = 4$

Since $\rho(A) \neq \rho(A:B)$

$\therefore$ System is inconsistent

$\therefore$ S doesn't span R^3.

$\therefore$ S doesn't form basis for P_3 //.

(ii) let $P_1 = \{t^3 + t^2 + 1\}$; $P_2 = t^3 - 1$; $P_3 = t^3 + t^2 + t$. & $S = \{P_1, P_2, P_3\}$

To prove that S is ~~the~~ spanning set & linearly independent-

let V be an arbitrary polynomial $at^3 + bt^2 + ct + d \in P_3$.

$V = xP_1 + yP_2 + zP_3$

$V = x[t^3 + t^2 + 1] + y[t^3 - 1] + z(t^3 + t^2 + t)$

$at^3 + bt^2 + ct + d = t^3[x+y+z] + t^2[x+z] + t[z] + [x-y]$

Comparing coefficients of t^3, t^2, t & constant

$x + y + z = a$

$x + z = b$

$z = c$

$x - y = d$

Writing equations in matrix form-

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

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Augmented matrix:

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 0 & 1 & b \\ 0 & 0 & 1 & c \\ 1 & -1 & 0 & d \end{array}\right]$$

Operate $R_2' \to R_2 - R_1$; $R_4' \to R_4 - R_1$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -1 & 0 & b-a \\ 0 & 0 & 1 & c \\ 0 & -2 & -1 & d-a \end{array}\right]$$

operate $R_4' \to R_4 - 2R_2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -1 & 0 & b-a \\ 0 & 0 & 1 & c \\ 0 & 0 & -1 & d+a-2b \end{array}\right]$$

operate $R_4' \to R_4 + R_3$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -1 & 0 & b-a \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & a-2b+c+d \end{array}\right]$$

$\rho(A) = 3$ & $\rho(A:B) = 4$

Since $\rho(A) \neq \rho(A:B)$

System is inconsistent

S doesn't span R^3

S doesn't form basis for P_3.

To find a Subset S that is a basis for W = span S

Step 1: form vector equation $x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0$

Step 2: Construct augmented matrix associated with homogeneous system of linear eq^{ns} & transform it into reduced row echelon form

Step 3: Identify pivot columns

Step 4: Vectors corresponding to pivot column of given matrix form a basis of W = span S.

LPQ14) let $S = \{v_1, v_2, v_3, v_4\}$ where $v_1 = \{1, 2, 2\}$, $v_2 = \{3, 2, 1\}$, $v_3 = \{11, 10, 7\}$, $v_4 = \{7, 6, 4\}$. Find a basis for subspace W = spans of R^3. What is dim W.

- Form a vector equation

$x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4 = 0$

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 11 \\ 10 \\ 7 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

equations:

$x_1 + 3x_2 + 11x_3 + 7x_4 = 0$

$2x_1 + 2x_2 + 10x_3 + 6x_4 = 0$

$2x_1 + x_2 + 7x_3 + 4x_4 = 0$

Writing equations in matrix form:

$$\begin{bmatrix} 1 & 3 & 11 & 7 \\ 2 & 2 & 10 & 6 \\ 2 & 1 & 7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix: solving for $AX = 0$

$$= \begin{bmatrix} 1 & 3 & 11 & 7 \\ 2 & 2 & 10 & 6 \\ 2 & 1 & 7 & 4 \end{bmatrix}$$

operate $R_2' \to R_2 - 2R_1$ & $R_3' \to R_3 - 2R_1$

$$= \begin{bmatrix} 1 & 3 & 11 & 7 \\ 0 & -4 & -12 & -8 \\ 0 & -5 & -15 & -10 \end{bmatrix}$$

$R_2' \to -\frac{1}{4} R_2$ & $R_3' \to \frac{1}{3} R_3$

$$= \begin{bmatrix} 1 & 3 & 11 & 7 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 2 \end{bmatrix}$$

Operate $R_3' \to R_3 - R_1$

$$= \begin{bmatrix} 1 & 3 & 11 & 7 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is in ~~RR~~EF

operate $R_1' \to R_1 - 3R_2$

$$= \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is in RREF with pivot columns 1 & 2 of original matrix will form basis ie $\{V_1, \& V_2\}$ will form the basis.

$\{V_1, V_2\} = \{(1,2,2), (3,2,1)\}$ is basis for W & $\dim W = 2$ {since 2 vectors}.

LP Q15) let $S = \{V_1, V_2, V_3, V_4, V_5\}$ where $V_1 = (1,1,0,-1)$, $V_2 = (0,1,2,1)$ $V_3 = (1,0,1,-1)$, $V_4 = (1,1,-6,-3)$, $V_5 = (-1,-5,1,0)$. Find a basis for subspace W = spans of R^4. What is dimension of W.

- Form a vector equation: $x_1V_1 + x_2V_2 + x_3V_3 + x_4V_4 + x_5V_5 = 0$

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ -6 \\ -3 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -5 \\ 1 \\ 0 \end{bmatrix} = 0$$

equations:

$x_1 + x_3 + x_4 - x_5 = 0$

$x_1 + x_2 + x_4 - 5x_5 = 0$

$2x_2 + x_3 - 6x_4 + x_5 = 0$

$-x_1 + x_2 - x_3 - 3x_4 = 0.$

Writing equations in matrix form:: solving for $AX = 0$

$$= \begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 & -5 \\ 0 & 2 & 1 & -6 & 1 \\ -1 & 1 & -1 & -3 & 0 \end{bmatrix}$$

operate $R_2' \to R_2 - R_1$ & $R_3' \to R_3 + R_1$

$$= \begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & +1 & -1 & 0 & -4 \\ 0 & 2 & 1 & -6 & 1 \\ 0 & 1 & 0 & -2 & -1 \end{bmatrix}$$

operate $R_3' \to R_3 - 2R_2$ & $R_4' \to R_4 - R_2$

$$= \begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 & -4 \\ 0 & 0 & 3 & -6 & 9 \\ 0 & 0 & 1 & -2 & 3 \end{bmatrix}$$

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operate $R_3' \to \frac{1}{3} R_3$

$$= \begin{bmatrix} 1 & 0 & \cancel{0}\,1 & 1 & -1 \\ 0 & 1 & -1 & 0 & -4 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 & 3 \end{bmatrix}$$

operate $R_4' \to R_4 - R_3$

$$= \begin{bmatrix} 1 & 0 & \cancel{0}\,1 & 1 & -1 \\ 0 & 1 & -1 & 0 & -4 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

operate $R_2' \to R_2 + R_3$ & $R_1' \to R_1 - R_3$

$$= \begin{bmatrix} 1 & 0 & 0 & 3 & -4 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ is in RREF}$$

$\uparrow \quad \uparrow \quad \uparrow$

with 1, 2 & 3rd columns as pivot columns. ie 1, 2, 3 columns of original matrix will form a basis for W ie $\{V_1, V_2, V_3\} = \{(1,1,0,-1), (0,1,2,1), (1,0,4,-1)\}$ will form basis for W & dim W = 3

LP Q19) Determine whether vectors $(1,1,1,1), (1,2,3,2), (2,5,6,4), (2,6,8,5)$ form a basis of R^4. If not find dimension of the subspace they span.

— let $V_1 = (1,1,1,1)$

$V_2 = (1,2,3,2)$

$V_3 = (2,5,6,4)$

$V_4 = (2,6,8,5)$

& $S = \{V_1, V_2, V_3, V_4\}$

To prove S is spanning set & linearly independent-

let v be an arbitrary vector: $v = (a,b,c,d)' \in R^4$

$v = V_1 x_1 + V_2 x_2 + V_3 x_3 + V_4 x_4$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 5 \\ 6 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 6 \\ 8 \\ 5 \end{bmatrix}.$$

equations:

$$x_1 + x_2 + 2x_3 + 2x_4 = a$$
$$x_1 + 2x_2 + 5x_3 + 6x_4 = b$$
$$x_1 + 3x_2 + 6x_3 + 8x_4 = c$$
$$x_1 + 2x_2 + 4x_3 + 5x_4 = d$$

Writing equations in matrix form:

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 5 & 6 \\ 1 & 3 & 6 & 8 \\ 1 & 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Augmented matrix:-

$$= \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & a \\ 1 & 2 & 5 & 6 & b \\ 1 & 3 & 6 & 8 & c \\ 1 & 2 & 4 & 5 & d \end{array}\right]$$

operate $R_2' \to R_2 - R_1$, $R_3' \to R_3 - R_1$, $R_4 \to R_4' - R_1$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & a \\ 0 & 1 & 3 & 4 & b-a \\ 0 & 2 & 4 & 6 & c-a \\ 0 & 1 & 2 & 3 & d-a \end{array}\right]$$

Operate $R_3' \to R_3 - 2R_1$, $R_4' \to R_4 - R_2$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & a \\ 0 & 1 & 3 & 4 & b-a \\ 0 & 0 & -2 & -2 & c+a-2b \\ 0 & 0 & -1 & -1 & d-b \end{array}\right]$$

operate $R_4' \to 2R_4 - R_3$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & a \\ 0 & 1 & 3 & 4 & b-a \\ 0 & 0 & -2 & -2 & c-2b+a \\ 0 & 0 & 0 & 0 & 2d-a-c \end{array}\right] \text{ is in REF.}$$

$\rho(A) = 3$ & $\rho(A:B) = 4$

Since $\rho(A) \neq \rho(A:B)$

system is inconsistent and S doesnot form basis of R4.

In REF, pivot columns are 1^{st}, 2^{nd} & 3^{rd} columns.

& pivot columns 1, 2, 3 in the original matrix A form basis ~~fn~~ for the subspace they span.

$\therefore \{v_1, v_2, v_3\}$ form subspace they span & dimension of subspace they span is 3.

*

P Q16)

$$A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$$

Reducing the given matrix to RREF:

operate $R_4' \to R_4 - R_1$, $R_5' \to R_5 - 2R_1$

$$= \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & -2 & -4 & -1 & 0 \\ 0 & -1 & -2 & -2 & -3 \end{bmatrix}$$

operate $R_4' \to R_4 + 2R_2$ & $R_5' \to R_5 + R_2$

$$= \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$$

operate $R_4' \to R_4 - R_3$ & $R_5' \to R_5 + R_3$

$$= \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

operate $R_1' \to R_1 - R_2$

$$= \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

operate $R_2' \to R_2 - R_3$

$$= \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ in RREF.

$\uparrow \quad \uparrow \quad \quad \uparrow$

1^{st}, 2^{nd} & 4^{th} column of original matrix will form basis

ie $\{(1,0,0,1,2), (1,1,0,-1,1), (1,1,1,0,0)\}$ will form basis.

To find dimension of solut^n space W of system:-

x_1, x_2, x_4 are basic variables & x_3, x_5 are free variable.

We have $x_1 + 2x_3 + x_5 = 0$

$x_2 + 2x_3 - x_5 = 0$

$x_4 + 2x_5 = 0$ are reduced system of eq^n.

let $x_3 = r$ & $x_5 = s$

$\Rightarrow x_4 = -2s$

$x_2 = -2x_3 + x_5 = -2r + s$

$x_1 = -2x_3 - x_5 = -2r - s$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2r-s \\ -2r+s \\ r \\ -2s \\ s \end{bmatrix} = r \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

The vectors $(-2,-2,1,0,0)$ and $(-1,1,0,-2,1)$ span the soln. space. {since 2 vectors} it's dimension is 2.

CHAPTER 3. ORTHOGONALITY

Inner product space-

Let V be a real vectorspace, suppose to each pair of vectors $u, v \in V$, then there exists a real no. assigned denoted by $\langle u,v\rangle$ is called a innerproduct on V if it satisfies following axioms-

(I) linearity property: $\langle au_1+bu_2, v\rangle = a\langle u_1,v\rangle + b\langle u_2,v\rangle$

$$\begin{cases} \langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle \\ \langle cu,v\rangle = c\langle u,v\rangle \end{cases}$$

(II) Symmetric property: $\langle u,v\rangle = \langle v,u\rangle$

(III) Positive definite property: $\langle u,u\rangle \geq 0$ and $\langle u,u\rangle = 0$ iff $u=0$.

Then vector V with an innerproduct is called inner product space.

Remark- Inner product of linear combinat^ns of vectors = linear combinat^n of the innerproduct of vectors.

Eg:

I) Let V be a real inner product space then by linearity property

$$\langle 3u_1 - 4u_2, 2v_1 - 5v_2 + 6v_3\rangle$$
$$= 3\langle u_1, 2v_1 - 5v_2 + 6v_3\rangle - 4\langle u_2, 2v_1 - 5v_2 + 6v_3\rangle$$
$$= 6\langle u_1, v_1\rangle - 15\langle u_1, v_2\rangle + 18\langle u_1, v_3\rangle - 8\langle u_2, v_1\rangle + 20\langle v_2, v_2\rangle$$
$$- 24\langle u_2, v_3\rangle$$

II) let $u = (1,3,-4,2)$

$v = (4,-2,2,1)$

$w = (5,-1,-2,6)$ in R^4

show that $\langle 3u - 2v, w\rangle = 3\langle u,w\rangle - 2\langle v,w\rangle$

– $\langle u,w\rangle = (1\times5) + (3\times-1) + (-4\times-2) + (2\times6) = 22$

$\langle v,w\rangle = 20+2-4+6 = 24$

$3\langle u,w\rangle - 2\langle v,w\rangle = 3(22) - 2(24) = 66-48 = 18$ — ①

$3u - 2v = (3-8, 9+4, -12-4, 2-2) = (-5, 13, -16, 0)$

$\langle 3u-2v, w\rangle = -25 - 13 + 32 + 24 = 56 - 38 = 18$ — ②

From ① & ② LHS = RHS

Norm of the vector :-

By the axiom (iii) $\langle u,u \rangle$ is a non-negative for $u \in V$ there exists the square root of $\langle u,v \rangle$ ie $\sqrt{\langle u,u \rangle}$ exists.
Norm of a vector u is denoted by $\|u\|$ and defined as
$\|u\| = \sqrt{\langle u,u \rangle}$

Remark: i) distance b/w two vectors u & v in vectorspace V is denoted by "$d(u-v)$" & is defined as norm $u-v$ ie $\|u-v\|$.
ii) If ~~u=1~~ $\|u\| = 1$ or $\sqrt{\langle u,u \rangle} = 1$ then u is called unit vector for R^n

$$\|u\| = \sqrt{a_1^2 + a_2^2 + \cdots a_n^2}$$

Orthogonality :-

let V be an inner product space & u is said to be orthogonal to v if their inner product is ~~said~~ zero ie $\langle u,v \rangle = 0$. The relation is symmetric ie if u is orthogonal to v then innerproduct is zero $\langle v,u \rangle = 0$

Eg:-

Q) let $u = \langle 1,1,1 \rangle$, $v = \langle 1,2,-3 \rangle$, $w = \langle 1,-4,3 \rangle$ in R^3 then
$\langle u,v \rangle = (1+2-3) = 0 \Rightarrow u$ is orthogonal to v
$\langle v,w \rangle = (1-8-9) \neq 0 \Rightarrow v$ is not orthogonal to w
$\langle u,w \rangle = (1-4+3) = 0 \Rightarrow u$ is orthogonal to w.

Orthogonal sets- $S = \{u_1, u_2, \cdots u_n\}$

Consider a set S, of nonzero vectors in an inner product space V. S is called orthogonal if each pair of vectors in S are orthogonal. And S is called orthonormal if S is orthogonal and each vector in S has unit length
ie (i) orthogonal $\langle u_i, u_j \rangle = 0$ for $i \neq j$
(ii) orthonormal $\langle u_i, u_j \rangle = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = 1 \end{cases}$

REMARK: Normalising an orthogonal set S is the process of multiplying each vector S by reciprocal of it's length in order to transform S into orthonormal set of vectors.

Orthogonal Basis:-

Consider a set $S = \{U_1, U_2, \cdots U_n\}$ of nonzero vectors in an innerproduct space V. S is called orthogonal basis if-

if (I) $V_1, V_2, \cdots U_n$ are orthogonal

(II) $U_1, U_2, \cdots U_n$ are linearly independent

REMARK: I) Suppose S is an orthogonal set of nonzero vectors then S is linearly independent.

II) let ~~B~~ $E = \{e_1, e_2, e_3, \cdots e_n\}$ be the usual basis of R^n ($e_1 = (1,0,0)$ — unit vector along x-axis)

then E is orthonormal basis.

$\langle e_1, e_2\rangle = \langle e_2, e_3\rangle = \langle \cdot, \cdot \rangle \cdots\cdots = \langle e_{n-1}, e_n\rangle = 0$

process of orthogonal → orthonormal ⇒ normalisat^n.

Grahm-Schmidt Orthogonalisat^n method -

This process is used to obtain orthogonal basis & then orthonormal basis for unit inner product space V.

Suppose $\{V_1, V_2, \cdots V_n\}$ is a basis of an inner product space V, we can use this basis to ~~contro~~ construct orthogonal basis $\{W_1, W_2, \cdots W_n\}$ of V as follows.

Set $W_1 = V_1$

$$W_2 = V_2 - \frac{\langle V_2, W_1\rangle}{\langle W_1, W_2\rangle} W_1$$

$$W_3 = V_3 - \frac{\langle V_3, W_1\rangle}{\langle W_1, W_1\rangle} W_1 - \frac{\langle V_3, W_2\rangle}{\langle W_2, W_2\rangle} W_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$W_n = V_n - \frac{\langle V_n, W_1\rangle}{\langle W_1, W_1\rangle} W_1 - \frac{\langle V_n, W_2\rangle}{\langle W_2, W_2\rangle} W_2 \cdots\cdots \frac{\langle V_n, W_{n-1}\rangle}{\langle W_{n-1}, W_{n-1}\rangle} W_{n-1}$$

In other words- for $k = 2, 3, \cdots n$

$$W_k = V_k - C_{k1} W_1 - C_{k2} W_2 - \cdots\cdots C_{k\,k-1} W_{k-1}$$

where $C_{ki} = \frac{\langle v_k, w_i \rangle}{\langle w_i, w_i \rangle}$ is a component of v_k along w_i.

Then w_k is orthogonal to preceeding w's.

$\therefore w_1, w_2, \dots w_n$ form orthogonal basis of V.

Normalising each w_i will give an orthonormal basis for V.

LPQ2)

(i) $v_1 = (1,1,1,1)$, $v_2 = (1,2,4,5)$, $v_3 = (1,-3,-4,-2)$

(ii) $v_1 = (1,1,1,1)$, $v_2 = (1,1,2,4)$, $v_3 = (1,2,-4,-3)$

i) set $w_1 = v_1 = (1,1,1,1)$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$= (1,2,4,5) - \frac{\langle (1,2,4,5); (1,1,1,1) \rangle}{\langle (1,1,1,1), (1,1,1,1) \rangle} (1,1,1,1)$$

$$= (1,2,4,5) - \frac{(1+2+4+5)}{(1+1+1+1)} (1,1,1,1)$$

$$= (1,2,4,5) - (3)(1,1,1,1) = (-2,-1,1,2)$$

$$\boxed{w_2 = (-2,-1,1,2)}$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$= (1,-3,-4,-2) - \left[\frac{(1-3-4-2)(1,1,1,1)}{4}\right] - \left[\frac{(-2+3-4-4)(-2,-1,1,2)}{(4+1+1+4)}\right]$$

$$= (1,-3,-4,-2) - [(-2,-2,-2,-2)] - [-\frac{7}{10}(-2,-1,1,2)]$$

$$= (1,-3,-4,-2) + [(2,2,2,2)] + \frac{7}{10}(-2,-1,1,2)$$

$$= \left(3 - \frac{14}{10}, -1 - \frac{7}{10}, \frac{-2+7}{10}, \frac{14}{10}\right)$$

$$= \left(\frac{8}{5}, \frac{-17}{10}, \frac{-13}{10}, \frac{7}{5}\right)$$ clear the fraction: {by multiplying by 10}

$$= (16, -17, -13, 14)$$

Thus w_1, w_2, w_3 are orthogonal basis.

Now to find orthonormal basis.

Normalise these vectors to obtain these orthonormal basis $\{u_1, u_2, u_3\}$ of V

We have $\|W_1\|^2 = \langle W_1, W_1\rangle = (1^2+1^2+1^2+1^2) = 4$

$$\|W_2\|^2 = \langle W_2, W_2\rangle = (4+4+1+1) = 10$$

$$\|W_3\|^2 = \langle W_3, W_3\rangle = 910$$

Orthonormal basis are-

$$U_1 = \frac{W_1}{\|W_1\|} = \frac{(1,1,1,1)}{2}.$$

$$U_2 = \frac{W_2}{\|W_2\|} = \frac{(-2,-1,1,2)}{\sqrt{10}}$$

$$U_3 = \frac{W_3}{\|W_3\|} = \left(\frac{16,-17,-13,14}{\sqrt{910}}\right)$$

ii) $W_1 = V_1 = (1,1,1,1)$

$$W_2 = V_2 - \frac{\langle V_2, W_1\rangle}{\langle W_1, W_1\rangle} W_1$$

$$= (1,1,2,4) - \frac{\langle (1,1,2,4),(1,1,1,1)\rangle}{1+1+1+1}(1,1,1,1)$$

$$= (1,1,2,4) - \left[\frac{(1+1+2+4)(1,1,1,1)}{4}\right]$$

$$= (1,1,2,4) - [2,2,2,2]$$

$W_2 = (-1,-1,0,2)$

$$W_3 = V_3 - \frac{\langle V_3, W_1\rangle W_1}{\langle W_1, W_1\rangle} - \frac{\langle V_3, W_2\rangle W_2}{\langle W_2, W_2\rangle}$$

$$= (1,2,-4,-3) - \frac{\langle (1,2,-4,-3)\cdot(1,1,1,1)\rangle}{4}(1,1,1,1)$$

$$- \frac{\langle (1,2,-4,-3)\cdot(-1,-1,0,2)\rangle}{1+1+0+4}(-1,-1,0,2)$$

$$= (1,2,-4,-3) - \frac{(1+2-4-3)}{4}(1,1,1,1) - \frac{(-1-2-6)}{6}(-1,-1,0,2)$$

$$= (1,2,-4,-3) - \left[\frac{-4}{4}(1,1,1,1)\right] - \left[\frac{-9}{6}(-1,-1,0,2)\right]$$

$$= (1,2,-4,-3) + (1,1,1,1) + \frac{3}{2}(-1,-1,0,2)$$

$$= \left(1+1-\frac{3}{2},\ 2+1-\frac{3}{2},\ -3+0,\ -3+3+\right)$$

$= \left(\frac{1}{2}, \frac{3}{2}, -3, 1\right)$ clearing the fraction by multiplying 2.

$= (1, 3, -6, 2)$

W_1, W_2, W_3 are orthogonal basis.
To find orthonormal basis, normalise these vectors to get orthonormal basis $\{U_1, U_2, U_3\}$.
We have $\|W_1\|^2 = 4$

$\|W_2\|^2 = 1+1+4 = 6$

$\|W_3\|^2 = 1+9+36+4 = 50$

orthonormal basis are.

$U_1 = \frac{W_1}{\|W_1\|} = \frac{1}{2}(1,1,1,1)$

$U_2 = \frac{W_2}{\|W_2\|} = \frac{1}{\sqrt{6}}(-1,-1,0,2)$

$U_3 = \frac{W_3}{\|W_3\|} = \frac{1}{\sqrt{50}}(1,3,-6,2)$