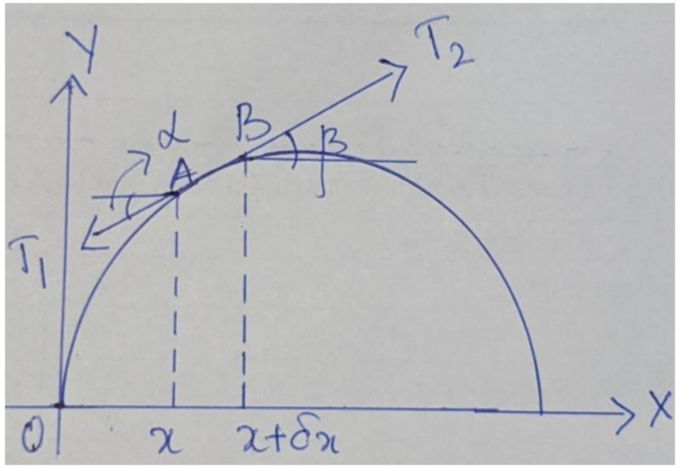
Partial differential equation

Application of Partial differential equation

Transverse vibration of the string:

One -dimensional Wave equation:

Consider a flexible elastic string tightly stretched between two fixed points at a distance l apart and displaced slightly from its equilibrium position ox Let ρ be the mass per unit length of the string. We shall obtain the equation of the motion for the string under the following assumptions.



i) The tension T of the string is same throughout.

ii) The effect of gravity can be ignored due to large tension T.

iii) The motion of the string is in small transverse vibrations.

iv) The motion takes place entirely in the xy plane and the each particle of the string of the string moves perpendicular to the equilibrium position of the string

Let us consider the forces acting on a small element AB of length δx. Let T_1 and T_2 be the tensions at the points A and B. Since there is no motion in the horizontal direction, the sum of the forces in the horizontal direction must be zero

(i.e) $-T_1 cos\alpha + T_2 cos\beta = 0$ the –ve sign is used because T_1 is directed downwards.

Therefore, $$T_1 cos\alpha = T_2 cos\beta = T \qquad \rightarrow (1)$$

where α and β are the angles made by T_1 and T_2 with the horizontal. Vertical components of tension are $-T_1 \sin\alpha$ and $T_2 sin\beta$,where the –ve sign is used because T_1 is directed downwards. Hence the resultant force acting vertically upwards is $T_2 \sin\beta - T_1 sin\alpha$.

Applying Newton's second law of motion (i.e Force = mass*acceleration),

we have, $T_2 \sin\beta - T_1 sin\alpha = (\rho\, \delta x)\frac{\partial^2 u}{\partial t^2}$

($\rho\delta x$ is the mass of the element portion AB and second derivative w.r.t 't' represents acceleration)

dividing throughout by T, we have,

$$\frac{T_2}{T}\sin\beta - \frac{T_1}{T} sin\alpha = \frac{\rho}{T}\delta x * \frac{\partial^2 u}{\partial t^2}$$

But from (1) $\frac{T_1}{T} = \frac{1}{\cos\alpha}, \quad \frac{T_2}{T} = \frac{1}{\cos\beta}$

$$\frac{\sin\beta}{\cos\beta} - \frac{\sin\alpha}{\cos\alpha} = \frac{\rho}{T}\delta x * \frac{\partial^2 u}{\partial t^2} \qquad \text{i.e } \tan\beta - \tan\alpha = \frac{\rho}{T}\delta x * \frac{\partial^2 u}{\partial t^2} \qquad \rightarrow (2)$$

But, $\tan\beta \; and \; \tan\alpha$ represents the slopes at $B(x+\delta x)$ and A(x) respectively.

$$\tan\beta = \left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \quad and \quad \tan\alpha = \left(\frac{\partial u}{\partial x}\right)_x$$

From (2), $\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$ - $\left(\frac{\partial u}{\partial x}\right)_{x} = \frac{\rho}{T}\,\delta x\,\frac{\partial^2 u}{\partial t^2}$

dividing by δx and taking limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\delta x} = \frac{\rho}{T} * \frac{\partial^2 u}{\partial t^2}$$

but, LHS is the derivative of $\frac{\partial u}{\partial x}$ w.r.t x treating t as constant.

i.e

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\rho}{T} * \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} * \frac{\partial^2 u}{\partial x^2}$$

$$u_{tt} = c^2 u_{xx}$$ (denoting $^T/_\rho$ $by\ \ c^2$)

The above equation gives transverse vibration of the string.

This is one -dimensional wave equation.

Boundary conditions :

The boundary conditions that one dimensional wave equation always satisfy are

$$\text{i)}u(0,t) = 0$$

$$ii)u(l,t) = 0$$

Solution of Wave equation by variable separable method

The one dimensional wave equation is $u_{tt} = c^2 u_{xx} \qquad \rightarrow (1)$

Let the solution of (1) be $u = XT \qquad \rightarrow (2)$

Then, substituting $u = XT$ in (1), we have $XT'' = c^2 X''T$

Separating the variables, we have $\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k \quad (say)$

We get two ODEs $X'' - kX = 0$ and $T'' - kc^2T = 0$

solving these two ODEs $X'' - kX = 0 \; and \; T'' - kc^2T = 0 \rightarrow \quad (3)$

for X and T results in the following cases:-

<u>**Case i**</u>: When k is positive, $k = p^2 \quad (say)$

.

Auxiliary Equation: $m^2 - k = 0$

Then, $m^2 - p^2 = 0 \quad \Longrightarrow m = \pm p$

Solution is $X = C_1 e^{px} + C_2 e^{-px}$

Auxiliary Equation: $m^2 - c^2 k = 0$

Then, $m^2 - c^2 p^2 = 0 \Longrightarrow m = \pm cp$

Solution is T$= C_3 e^{cpt} + C_4 e^{-cpt}$

Case ii: When k is negative, $k = -p^2 \quad (say)$

Auxiliary Equation: $m^2 - k = 0$

Then, $m^2 + p^2 = 0 \Longrightarrow m = \pm ip$

Solution is $X = C_1 \cos px + C_2 \sin px$

Auxiliary Equation: $m^2 - c^2 k = 0$

Then, $m^2 + c^2 p^2 = 0 \Longrightarrow m = \pm icp$

Solution is T$= C_3 \cos cpt + C_4 \sin cpt$

Case iii: When k is zero, i.e k $= 0$

Auxiliary Equation: $m^2 - k = 0$

Then, $m^2 = 0 \Longrightarrow m = 0, 0$

Solution is $X = C_1 x + C_2$

Auxiliary Equation: $m^2 - c^2 k = 0$

Then $m^2 = 0 \Longrightarrow m = 0, 0$

Solution is T= $C_3 t + C_4$

Various possible solutions are

$u = (C_1 e^{px} + C_2 e^{-px}) * (C_3 e^{cpt} + C_4 e^{-cpt})$

$u = (C_1 cospx + C_2 sinpx) * (C_3 \cos cpt + C_4 \sin cpt)$

$u = (C_1 x + C_2) * (C_3 t + C_4)$

Of these solutions, choose that, which is consistent with the physical nature of the problems. Since, we are dealing with the problems on vibrations, u must be a periodic function of x and t.

Therefore, the solution must involve trigonometric functions. Accordingly, the only suitable solution is (which corresponds to $k = -p^2$)

$$u(x,t) = (C_1 \cos px + C_2 \sin px) * (C_3 \cos cpt + C_4 \sin cpt) \qquad \rightarrow (4)$$

Two dimensional heat equation

- The heat flow in a metallic place which is in xy plane is given by
- $c^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = \frac{\partial u}{\partial t}$ where $c^2 = \frac{k}{\rho s}$
- The above equation gives the temperature distribution of the plane in transient state .
- In steady state ,the temperature u is independent of the time t, so that

  $\frac{\partial u}{\partial t} = 0$ and therefore the above equation reduces to $\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$ =0 which is known as Laplace equation in two dimensions

Solution of Laplace equation

- Laplace equation is given by $\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = 0 \to 1$
- Let us solve this equation by variable separable method
- Let us assume the solution for the equation 1 as $u = XY$ where X is the function of only x and Y is the function of only y
- $\frac{\partial^2 u}{\partial x^2} = X''$Y and $\frac{\partial^2 u}{\partial y^2} = XY''$
- Therefore the equation 1 becomes $X''Y + XY'' = 0$

derivation Continued

- On separating the variables we get $\frac{X''}{X} = -\frac{Y''}{Y} = k(\text{constant})$
- By taking 1 and 3, 2and 3 we get
- $X'' - kX = 0$ and $Y'' + kY = 0$
- Which are two ODEs
- Since k is constant ,depending on the value of k we will get the solution

case 1

- First let us assume that k is +ve say $k = p^2$
- For X, Auxiliary Equation is $m^2 - k = 0$
- $m^2 - p^2 = 0 \Rightarrow m = \pm p$
- The solution for X is $X = C_1 e^{px} + C_2 e^{-px}$
- For Y, Auxiliary Equation is $m^2 + k = 0$
- $m^2 + p^2 = 0 \Rightarrow m^2 = -p^2 \Rightarrow \mathrm{m} = \pm ip$
- The solution for Y is Y= $C_3 cospy + C_2 sinpy$
- Therefore the solution is $u(x, y) = (C_1 e^{px} + C_2 e^{-px}) * (C_3 cospy + C_2 sinpy)$

case 2

- second let us assume that k is -ve say $k = -p^2$
- For X, Auxiliary Equation is $m^2 - k = 0$
- $m^2 + p^2 = 0 \Rightarrow m^2 = -p^2 \Rightarrow m = \pm ip$
- The solution for X is $X = C_1 cospx + C_2 sinpx$
- For Y, Auxiliary Equation is $m^2 + k = 0$
- $m^2 - p^2 = 0 \Rightarrow m^2 = p^2 \Rightarrow \mathrm{m} = \pm p$
- The solution for Y is Y= $C_3 e^{py} + C_4 e^{-py}$
- Therefore the solution is $u(x, y) = (C_1 cospx + C_2 sinpx) * (C_3 e^{py} + C_4 e^{-py})$

case 3

- let us assume that k is zero say $k = 0$
- For X, Auxiliary Equation is $m^2 = 0 \Rightarrow m = 0,0$
- The solution for X is $X = C_1 x + C_2$
- For Y, Auxiliary Equation is $m^2 = 0 \Rightarrow m = 0,0$
- The solution for Y is Y= $C_3 y$+C_4
- Therefore the solution is $u(x, y) = (C_1 x + C_2) * (C_3 y$+$C_4)$

Various Possible Solutions are

- $u(x, y) = (C_1 e^{px} + C_2 e^{-px}) * (C_3 cospy + C_2 sinpy)$

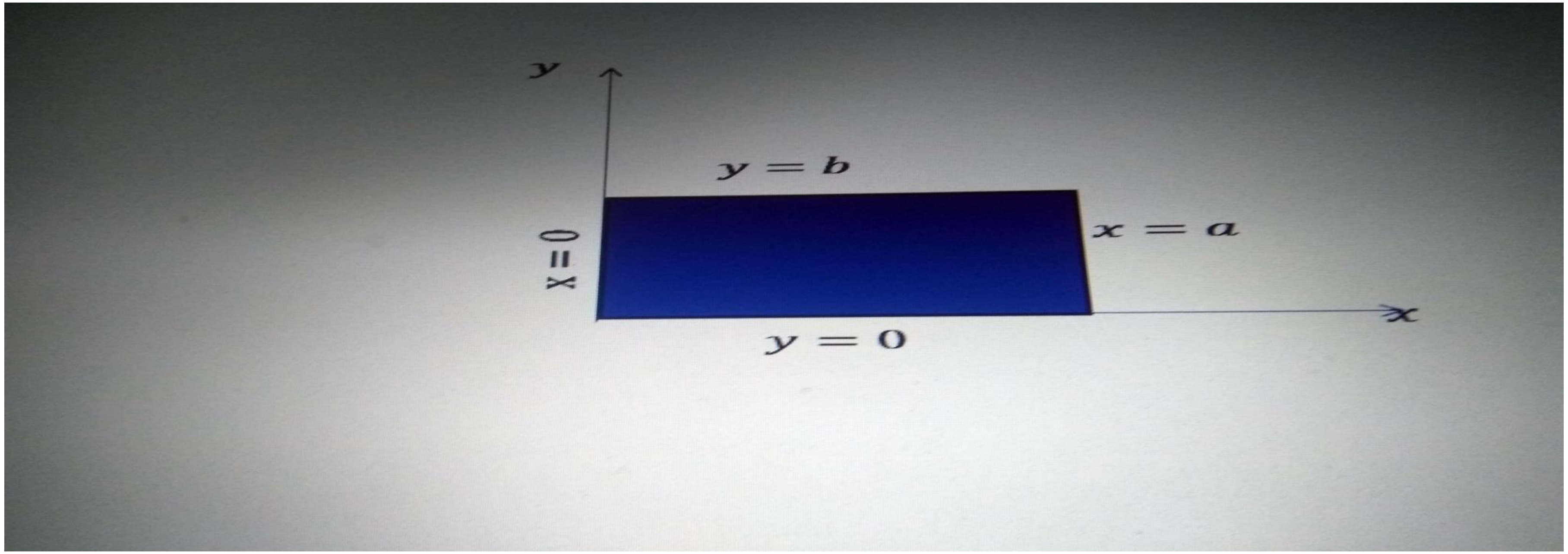
- $u(x, y) = (C_1 cospx + C_2 sinpx) * (C_3 e^{py} + C_4 e^{-py})$

- $u(x, y) = (C_1 x + C_2) * (C_3 y + C_4)$

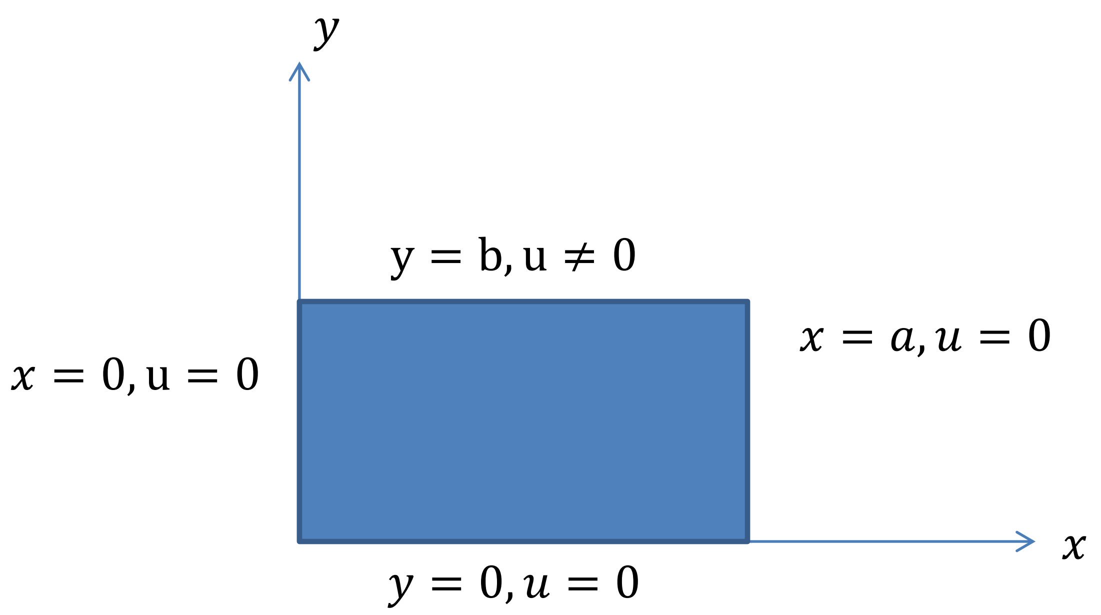
Note

- For u to be non trivial solution, atleast one of $C_1 \text{and } C_2, \text{atleast} C_3 \text{and } C_4$ are to be zero.
- The simplest regions in which the Laplace equation is solved are the regions bounded by st.lines parallel to the co-ordinates axes.

The simplest boundary condition is the one in which $'u'$ is specified to be non zero on one of the boundary lines and zero on the boundary lines

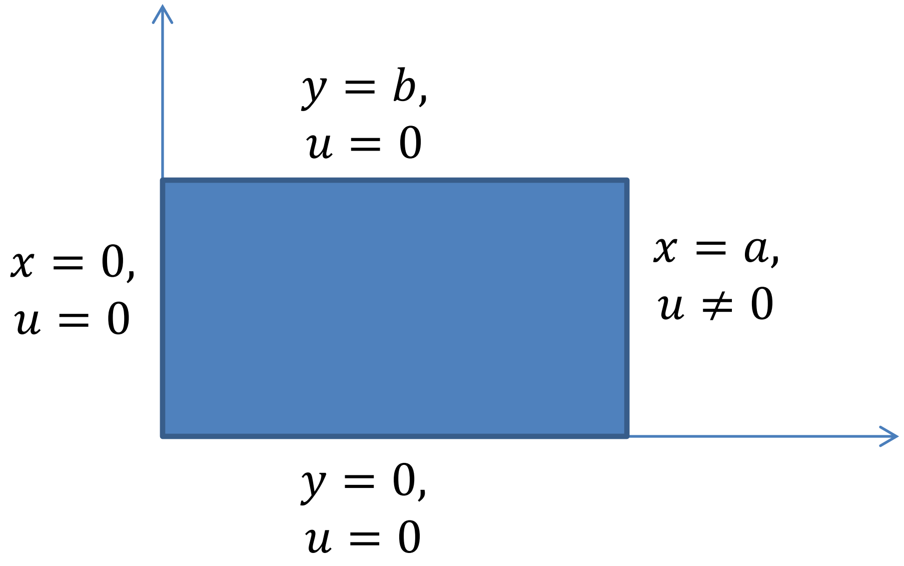


- If the boundary line on which ‘u’ is specified to be non zero,parallel to x-axis(y is constant)
  Then the solution involves cosine and sine function of x is used



- (i.e) the suitable solution is
- $u(x, y) = (C_1 cospx + C_2 sinpx) * (C_3 e^{py} + C_4 e^{-py})$

- If the boundary line on which ‘u’ is specified to be non zero,parallel to y-axis(x is constant) Then the solution involves cosine and sine function of y is used



- (i.e) the suitable solution is
- $u(x, y) = (C_1 e^{px} + C_2 e^{-px})*(C_3 cospy + C_2 sinpy)$

Classification of second order partial differential equation

The general form of a second order P.D.E. if u is a function of two independent variables x and y is given by

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=0 --- -1$$

The above equation can be classified depending on the value of the discriminant B^2-4AC

Elliptic

Eqn 1 can be classified as Elliptic if

$$B^2 - 4AC < 0$$

For example consider the following eqn

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hear B=0,A=1,C=1

$$B^2 - 4AC = -4$$

Laplace equation is Elliptic

parabolic

Eqn 1 can be classified as Parabolic if

$$B^2 - 4AC = 0$$

For example consider the following eqn

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Hear B=0,A=a^2,C=0

$$B^2 - 4AC = 0$$

Therefore the above eqn is parabolic

Hyperbolic

Eqn 1 can be classified as Hyperbolic if

$$B^2 - 4AC > 0$$

For example consider the following eqn

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Hear B=0,A=a^2,C=-1

$$B^2 - 4AC > 0$$

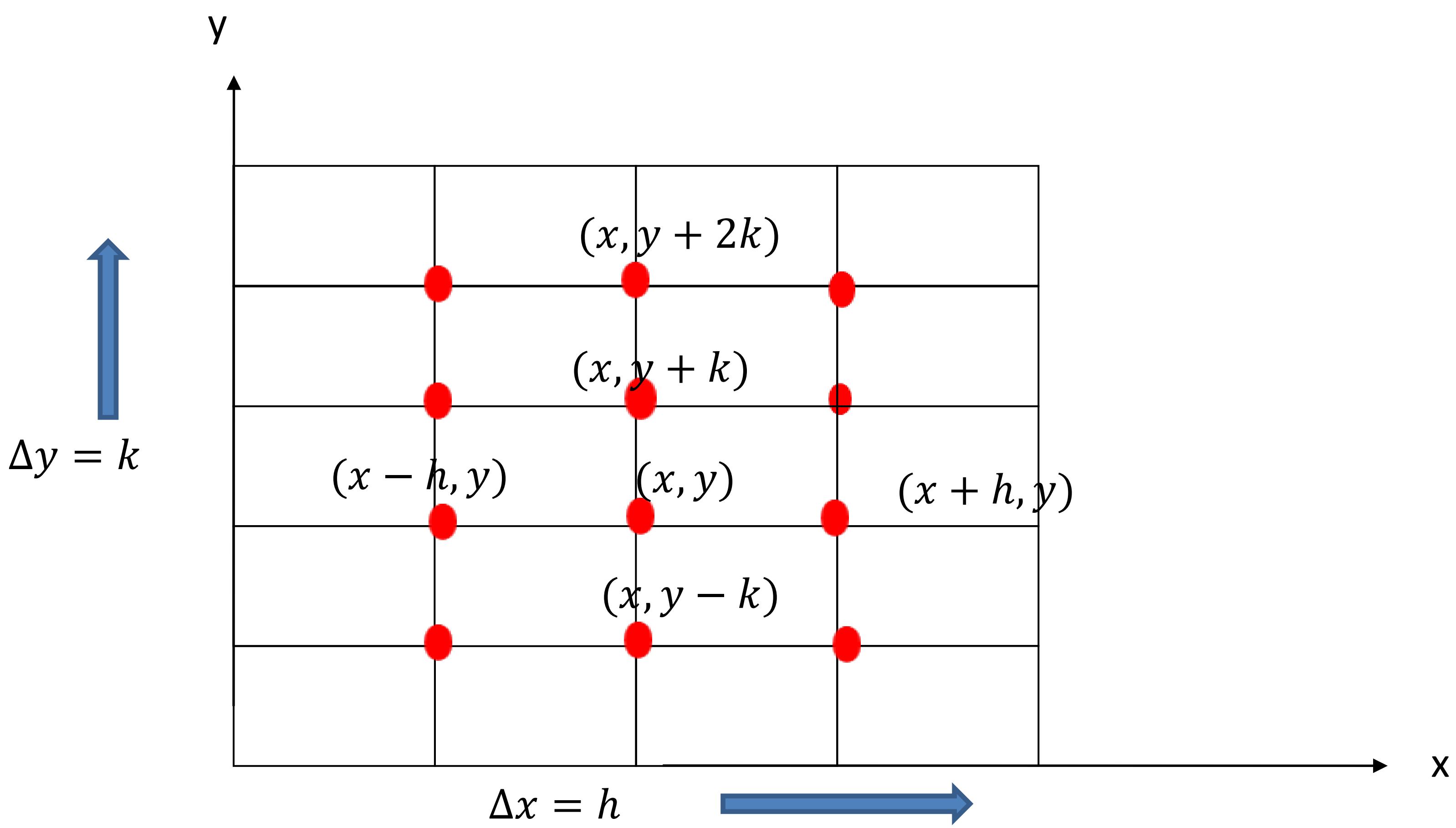
Therefore the above eqn is Hyperbolic

Finite difference method

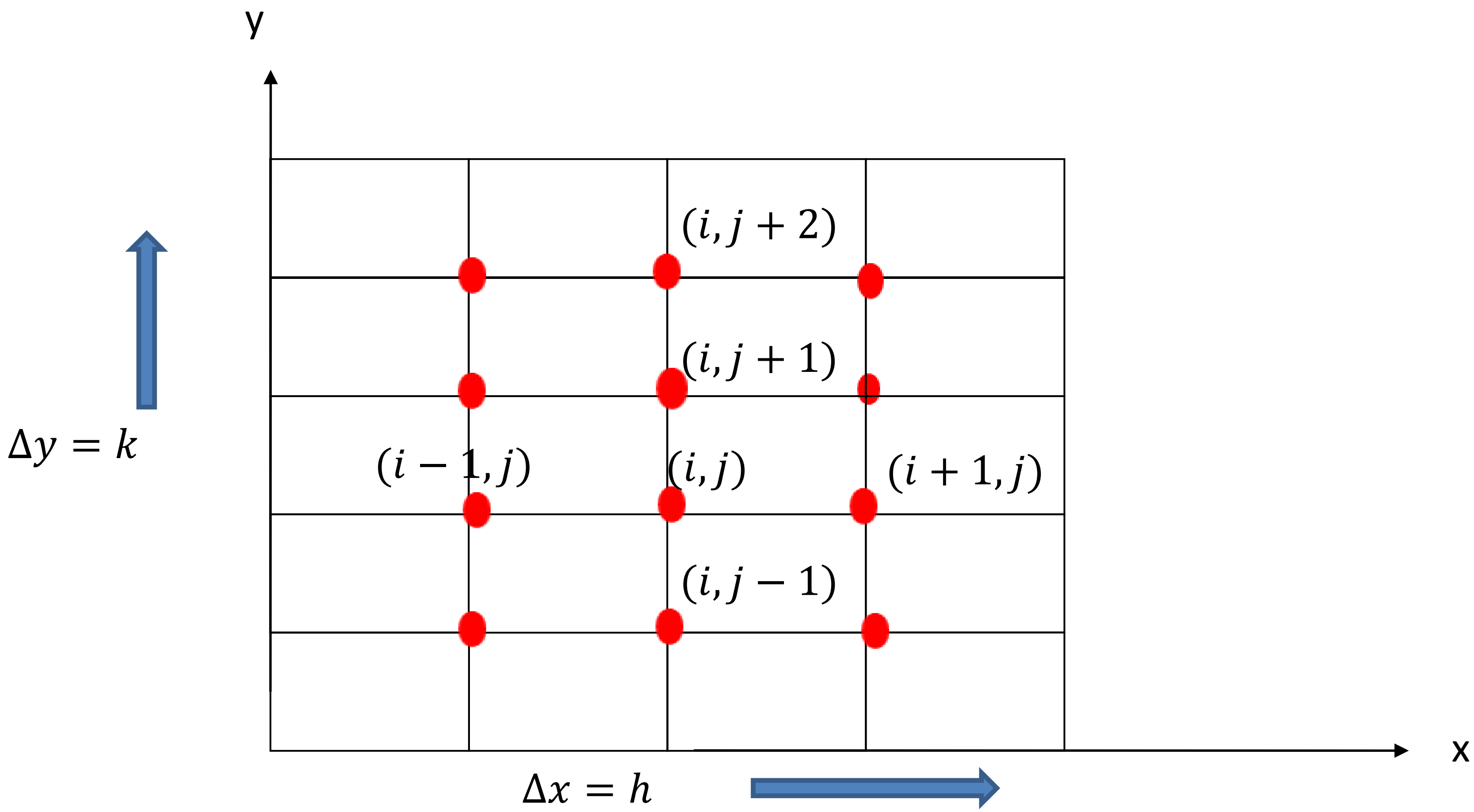
Let us consider u be a function of x and y

Here the xy plane is divided into a series of rectangles of sides $\Delta x = h$ and $\Delta y = k$ by drawing lines parallel to the coordinate axes.

Finite difference method



Finite difference method



The point of intersection of the family of these lines is called as grid point or mesh point

The value of u(x,y) at any point (x,y) is denoted by $u_{i,j}$

and here each x values and y values are given by $x = ih$ and $y = jk$

The finite difference approximation for the partial derivatives of 'u' are as follows

$$u_x(i,j) = \frac{u_{i+1,j} - u_{i,j}}{h} \text{ [Forward difference]}$$

$$u_x(i,j) = \frac{u_{i,j} - u_{i-1,j}}{h} \text{ [Backward difference]}$$

$$u_x(i,j) = \frac{u_{i+1,j} - u_{i-1,j}}{2h} \text{ [central difference]}$$

Along y direction

$$u_y(i,j) = \frac{u_{i,j+1} - u_{i,j}}{k}$$ [Forward difference]

$$u_y(i,j) = \frac{u_{i,j} - u_{i,j-1}}{k}$$ [Backward difference]

$$u_y(i,j) = \frac{u_{i,j+1} - u_{i,j-1}}{2k}$$ [central difference]

By using the above differences we get the following second order derivatives

$$u_{xx}(i,j) = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$u_{yy}(i,j) = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

Heat conduction

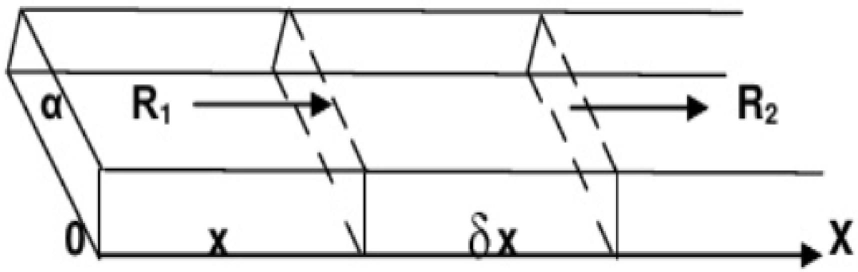
A uniform homogeneous metal bar is made of infinite number of molecules. They are interconnected by cohesive force. When heat is applied through one side of it, they tends to get expand , because of that they get more interconnected and hence heat flows from molecule to molecule.

The flow of heat in this way in a uniform rod is known as <u>Heat Conduction.</u>

One dimensional heat equation

- Consider a homogenous rod of uniform cross section (αcm^2)
- Assume the sides are covered with a material impervious to heat so that the stream lines are all parallel and perpendicular to the area α
- Let u(x,t) denotes the temperature at a distance x from origin and at time t
- Let ρ be the density, S be the thermal heat and k be the thermal conductivity.

Consider a small slab of the rod with thickness δx



- The following principles are involved in heat conduction problems
- Heat flow from a higher temperature to the lower temperature
- The rate of a heat flow across an area is proportional to the area and the rate of the change of temperature w.r.to its distance
- Let R_1 and R_2 be the rate of inflow and outflow of at x and $x + \delta x$
- $R_1 = -k\alpha \left.\frac{\partial u}{\partial x}\right]_x$ and $R_2 = -k\alpha \left.\frac{\partial u}{\partial x}\right]_{x+\delta x}$
- (-ve sign since as $x-$increases u will decrease)
- Rate of increase of heat in the slab of the thickness $\delta x = R_1 - R_2 = k\alpha \left[\left.\frac{\partial u}{\partial x}\right]_{x+\delta x} - \left.\frac{\partial u}{\partial x}\right]_x \right]$

But the quantity of the slab with thickness δx is given by $Q = S\rho\alpha\delta x(\frac{\partial u}{\partial t})$

- Therefore $S\rho\alpha\delta x\left(\frac{\partial u}{\partial t}\right) = k\alpha\left[\left[\frac{\partial u}{\partial x}\right]_{x+\delta x} - \frac{\partial u}{\partial x}\right]_x\Big]$
- $S\rho\left(\frac{\partial u}{\partial t}\right) = \frac{k\left[\left[\frac{\partial u}{\partial x}\right]_{x+\delta x} - \frac{\partial u}{\partial x}\right]_x\Big]}{\delta x}$
- as $\delta x \to 0$, $S\rho\left(\frac{\partial u}{\partial t}\right) = k \lim_{\delta x \to 0} \frac{\left[\left[\frac{\partial u}{\partial x}\right]_{x+\delta x} - \frac{\partial u}{\partial x}\right]_x\Big]}{\delta x}$
- $S\rho\left(\frac{\partial u}{\partial t}\right) = k\frac{\partial^2 u}{\partial x^2} \Rightarrow \left(\frac{\partial u}{\partial t}\right) = \frac{k}{S\rho}\left(\frac{\partial^2 u}{\partial x^2}\right)$
- $\left(\frac{\partial u}{\partial t}\right) = C^2\left(\frac{\partial^2 u}{\partial x^2}\right)$ where $C^2 = \frac{k}{S\rho}$
- This is the P.D.E which gives heat conduction
- This is called one dimensional heat equation

Solution of one dimensional heat equation

- The one dimensional wave equation is $\left(\frac{\partial u}{\partial t}\right) = C^2 \left(\frac{\partial^2 u}{\partial x^2}\right) \to 1$
- We are going to solve this by variable separable method
- Let us assume the solution be $u = XT$ where X is the function of only x and T is the function of only t
- Here $u_t = T'$ and $u_{xx} = X''$

Therefore the eqn 1 becomes

- $XT' = c^2 X''T$
- On separating the the variables we get
- $\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = k \quad (say)$
- On taking 1 and 3 , one ODEs and by taking 2 and 3
- $X'' - kX = 0$ and $\frac{T'}{T} = kc^2$
- Which can be solved as follows
- Since k is constant depending on the value of the k we will get different solution

Case 1

- First let us assume k as positive say $k = p^2$
- The above eqn become as follows
- $X'' - p^2X = 0$ and $\frac{T'}{T} = p^2c^2$
- On solving for X and T we get as follows
- For X , Auxiliary Equation: $m^2 - p^2 = 0$
- Then, $\mathrm{m}^2 = p^2 \quad \Rightarrow m = \pm p$
- Soln for X is $X = C_1e^{px}+C_2e^{-px}$
- For T, $\frac{T'}{T} = p^2c^2 \quad \Rightarrow$on integrating $\log T = p^2c^2t$
- $\Rightarrow T = C_3e^{c^2p^2t}$
- Therefore the solution is u= $(C_1e^{px}+C_2e^{-px})$*$(C_3e^{c^2p^2t})$

Case 1

- let us assume k as negative say $k = -p^2$
- The above eqn become as follows
- $X'' + p^2X = 0$ and $\frac{T'}{T} = -p^2c^2$
- On solving for X and T we get as follows
- For X , Auxiliary Equation: $m^2 + p^2 = 0$
- Then, $\mathrm{m}^2 = -p^2 \quad \Longrightarrow m = \pm ip$
- Soln for X is $X = C_1 \cos(px) + C_2 \sin(px)$
- For T, $\frac{T'}{T} = -p^2c^2$ $\Rightarrow$on integrating $\log T = -p^2c^2t$
- $\Rightarrow T = C_3e^{-c^2p^2t}$
- Therefore the solution is u= $(C_1e^{px}+C_2e^{-px})*(C_3e^{-c^2p^2t})$

Case 1

- let us assume k as zero
- The above eqn become as follows
- $X'' = 0$
- On solving for X and T we get as follows
- For X , Auxiliary Equation: $m^2 = 0$
- Then, m= 0,0
- Soln for X is $X = (C_1 x + C_2)$
- For T,⇒ $T = C_3$
- Therefore the solution is u= $(C_1 x + C_2)*(C_3)$

Various possible solutions are

- $u(x,t) = (C_1 e^{px} + C_2 e^{-px}) * (C_3 e^{c^2 p^2 t})$
- $u(x,t) = (C_1 e^{px} + C_2 e^{-px}) * (C_3 e^{-c^2 p^2 t})$
- u= $(C_1 x + C_2) * (C_3)$
- Out of these possible solution we need to choose the suitable solution according to the physical nature of the problem
- Since, u decreases as time t increases , the suitable solution is
- $u(x,t) = (C_1 e^{px} + C_2 e^{-px}) * (C_3 e^{-c^2 p^2 t})$

Classification of PDEs

Linear Second order PDEs are important sets of equations that are used to model many systems in many different fields of science and engineering.

Classification is important because:

- **Each category relates to specific engineering problems.**
- **Different approaches are used to solve these categories.**

Linear Second Order PDEs Classification

- A second order linear PDE (2−independent variables)

$$\boldsymbol{Au_{xx} + Bu_{xy} + Cu_{yy} + D = 0,}$$

A, B, and C are functions of x and y

- D is a function of x, y, u, u_x, and u_y

is classified based on $(\boldsymbol{B^2 - 4AC})$ as follows:

- $\boldsymbol{B^2 - 4AC < 0}$ ➡ Elliptic

  $B^2 - 4AC = 0$ ➡ Parabolic

  $\boldsymbol{B^2 - 4AC > 0}$ ➡ Hyperbolic

ELLIPICT EQUATION

- $A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D = 0$
- **LAPLACE EQUATION is** $u_{xx} + u_{yy} = 0$
- A =1, B =0, C =1 , D=0
- $B^2 - 4\,A\,C = -4\ < 0$ i.e. **Negative**
- Therefore it is as an **ELLIPTIC EQUATION**

PARABOLIC EQUATION

- $A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D = 0$
- **HEAT EQUATION is** $u_t = C^2\,u_{xx}$
- A = C^2, B =0, C =0 , D=-1
- $B^2 - 4\,A\,C = 0$
- Therefore it is **PARABOLIC EQUATION**

HYPERBOLIC EQUATION

- $A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D = 0$
- **WAVE EQUATION** $C^2\ u_{xx} = u_{tt}$
- A = C^2, B =0, C =-1 , D=0
- $B^2 - 4\,A\,C > 0$ i.e. **Positive**
- Therefore it is **HYPERBOLIC EQUATION**

Numerical Methods

- Most PDEs cannot be solved analytically.
- Variable separation works only for some simple cases and in particular usually not for **in-homogenous** and/or **nonlinear PDEs.**
- Numerical methods require that the PDE become discretized on a grid.

- **Finite difference methods** are popular/most commonly used in science.

➢ **They replace differential equation by difference equations.**

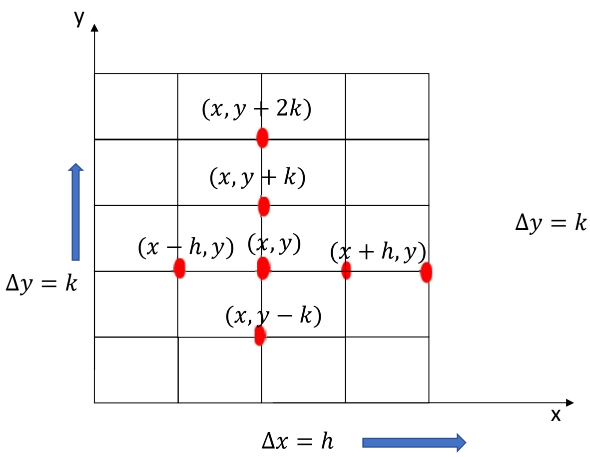
- Engineers (and a growing number of scientists too) often use **Finite Elements.**

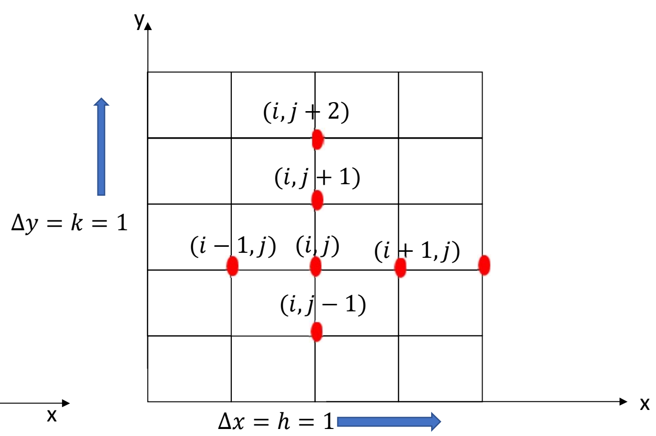
METHODS

- **Finite Difference (FD) Approaches**
- Based on approximating solution at a finite INTERSECTION of points, usually arranged in a regular grid.
- **Finite Element (FE) Method**
- Based on approximating solution on an assemblage of simply shaped(triangular, quadrilateral) finite pieces or "elements" which together make up (perhaps complexly shaped) domain.

Finite Difference Approximations to Derivatives

- Consider a rectangular region R in xy-plane.
- Divide the region into rectangular network of sides $\boldsymbol{\Delta x} = \boldsymbol{h}$ and $\boldsymbol{\Delta y} = \boldsymbol{k}$ as shown.
- The points of intersection of the dividing the Lines are called **MESH POINTS** or **NODAL POINTS** or **GRID POINTS.**

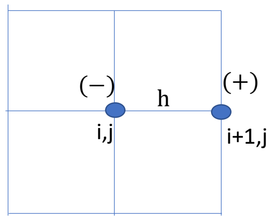




- We can interpret the above idea in different notation by drawing two sets of Parallel Lines $\boldsymbol{x = ih}$ and $\boldsymbol{y = jk}$
- Where i,j=0,1,2,3----
- The point $(\boldsymbol{i, j})$ is called a Grid Point and is surrounded by the neighbouring points as shown in the above diagram.
- If u is a function of two variables x, y then the value of $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y})$ is denoted by $\boldsymbol{u_{i,j}}$
- Therefore $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{u}(\boldsymbol{ih}, \boldsymbol{jk}) = \boldsymbol{u_{i,j}}$

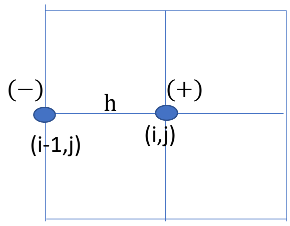
First order Forward Difference w.r.t x

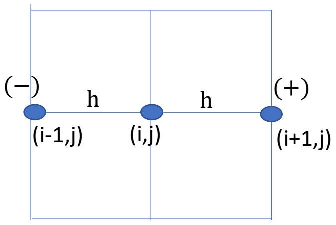
$$\frac{\partial u}{\partial x} = u_x = \frac{u_{i+1,j} - u_{i,j}}{h}$$



First order Backward Difference w.r.t.x

$$\frac{\partial u}{\partial x} = u_x = \frac{u_{i,j} - u_{i-1,j}}{h}$$





First order Central Difference w.r.t. x

$$\frac{\partial u}{\partial x} = u_x = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

First order Forward Difference w.r.t y

$$\frac{\partial u}{\partial y} = u_y = \frac{u_{i,j+1} - u_{i,j}}{k}$$

- **First order Backward Difference w.r.t. y**

$$\frac{\partial u}{\partial y} = u_y = \frac{u_{i,j-1} - u_{i,j}}{k}$$

First order Central Difference w.r.t. y

$$\frac{\partial u}{\partial y} = u_y = \frac{u_{i,j+1} - u_{i,j-1}}{2k}$$

Second order Forward Difference w.r.t. x

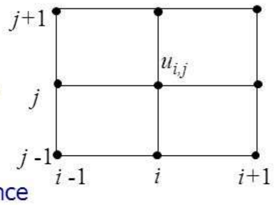
$$\frac{\partial^2 u}{\partial x^2} = u_{xx} = \frac{u_{i-1,j} - 2\, u_{i,j} + u_{i-1,j}}{h^2}$$

Second order Backward Difference w.r.t. y

$$\frac{\partial^2 u}{\partial y^2} = u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

Finite Difference Method

- Method based on truncated Taylor series



$$\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{h}$$ - Forward difference

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{h}$$ - Backward difference

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$ - Central difference

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$ - Central difference

- Above approximations can be substituted for differentials in the PDE.

EXAMPLES ON HEAT EQUATION

Ex.1: **LP-16:** Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with conditions $u(x, 0) = 3\sin n\pi x$, $u(0, t) = 0, u(1, t) = 0$, where 0< x < 1 , t >0

Solution: Heat Equation is $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

3-possible solutions of Heat Equation are,

$u = (c_1 e^{px} + c_2 e^{-px})\ c_3 e^{p^2 c^2 t}$

u= $(c_1 cospx + c_2 sinpx)\ c_3 e^{-p^2 c^2 t}$

u = $(c_1 x + c_2)\ c_3$

Comparing $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, we get c=1

We have the Solution of heat equation as

u(x, t)= $(c_1 cospx + c_2 sinpx)\ e^{-p^2 c^2 t}$

Which implies **u(x, t)= $(c_1 cospx + c_2 sinpx)\ e^{-p^2 t}$** ----→(1)

Now, using the Boundary Conditions,

$u(0, t) = 0$ put x=0 in equation (1),

we get $u(0, t) = c_1 e^{-p^2 t} = 0$ implies $\boldsymbol{c_1 = 0}$

u(x, t)= $(c_1 cospx + c_2 sinpx)\, e^{-p^2 t}$ ----→(1)

Thus, eqn (1) becomes **u(x, t)= $(\boldsymbol{c_2 sinpx})\, \boldsymbol{e^{-p^2 t}}$** → (2)

Given, u(1,t)=0 using in eqn(2), we get u(1,t)=$(c_2 sinp)\, e^{-p^2 t} = 0$

Which implies, **since $\boldsymbol{c_2 \neq 0}$** we have $\boldsymbol{sinp = 0}$

$$\mathbf{sinp} = \boldsymbol{sinn\pi} \Longrightarrow \boldsymbol{p = n\pi}$$

Thus, the solution becomes $u(x,t) = c_2 \sin(n\pi x) \,.\, e^{-\pi^2 n^2 t}$ →(3)

Adding all the solutions, we get the General Solution of heat eqn as

$$\boldsymbol{u(x,t)} = \sum_{n=1}^{\infty} \boldsymbol{b_n\, sin(n\pi x)\,.\, e^{-\pi^2 n^2 t}} \_\_\_(4)$$

Given, IC $\Longrightarrow u(x,0) = 3\sin(n\pi x)$, using this in eqn (4), we get

$$\boldsymbol{u(x,0) = f(x)} = \sum_{n=1}^{\infty} \boldsymbol{b_n\, sin(n\pi x)}$$

which is Half-Range Fourier Sine Series

$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$

then $\boldsymbol{b_n = \frac{2}{l}\int_0^l f(x) * \sin(n\pi x)\, dx}$

$b_n = \frac{2}{1}\int_0^1 \mathbf{3\sin(n\pi x)\,.\,\sin(n\pi x)}\, dx$

$= 6\int_0^1 sin^2(n\pi x)dx$

$=6\int_0^1 \frac{1-\cos(2n\pi x)}{2}dx = 3\left\{x - \frac{\sin(2n\pi x)}{2n\pi}\right\}$ 0 to 1

= 3

Thus, the eqn (4) becomes

$\boldsymbol{u(x,t) = \sum_{n=1}^{\infty} b_n\, sin(n\pi x)\,.\,e^{-\pi^2 n^2 t}}$ ___(4)

$$\mathrm{u}(\mathrm{x},t) = \sum_{n=1}^{\infty} 3\sin(n\pi x)\,.\,e^{-\pi^2 n^2 t}$$

This is the required solution of heat equation.

Ex.2:LP-19: Solve $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ subject to the conditions $u(x,0)=x-x^2$, $u(0,t)=0, u(1,t)=0$, where 0< x <1.

Solution: Heat Equation is $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$

3-possible solutions of Heat Equation are,

$u=(c_1e^{px}+c_2e^{-px})\ c_3e^{p^2C^2t}$

u= $(c_1cospx+c_2sinpx\ )\ c_3e^{-p^2C^2t}$

u = $(c_1x+c_2)\ c_3$

Comparing $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ with $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$, we get c=1

We have the Solution of heat equation as

u(x, t)= $(c_1cospx+c_2sinpx\ )\ e^{-p^2c^2t}$

Which implies **u(x, t)= $(c_1cospx+c_2sinpx\ )\ e^{-p^2t}$** ----→(1)

Now, using the given conditions,

$u(0,t)=0$ put x=0 in eqn (1),

we get $u(0,t)=c_1e^{-p^2t}=0$

implies $\boldsymbol{c_1=0}$

u(x, t)= $(c_1 cospx + c_2 sinpx)\ e^{-p^2t}$ ----→(1)

Thus, eqn (1) becomes **u(x, t)= $(c_2 sinpx)\ e^{-p^2t}$** → (2)

Given, u(1,t)=0 using in eqn(2), we get u(1,t)=$(c_2 sinp)\ e^{-p^2t} = 0$

Which implies, since $c_2 \neq 0$ we have, $\boldsymbol{sinp = 0}$

Thus, $\boldsymbol{sinp = sinn\pi}$

$\Longrightarrow \boldsymbol{p = n\pi}$

Thus, the solution becomes $u(x,t) = c_2 \sin(n\pi x) . e^{-\pi^2 n^2 t}$ →(3)

Adding all the solutions, we get the general solution of heat eqn as

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) . e^{-\pi^2 n^2 t} \quad \rightarrow (4)$$

Given, $u(x,0) = f(x) = x - x^2$, using this in eqn (4),

we get

$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$

$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$

Which is Fourier Sine series

Where $\boldsymbol{b_n = \frac{2}{l}\int_0^l f(x) * \sin(n\pi x)\, dx}$

$$\boldsymbol{b_n = \frac{2}{1}\int_0^1 (x - x^2).\sin(n\pi x)\, dx}$$

Integrating by using Bernouli's Rule, we get

$$b_n = 2\left[(x - x^2).\left(\frac{-cosn\pi x}{n\pi}\right) - (1-2x)\left(\frac{-sinn\pi x}{(n\pi)^2}\right) + (-2)\left(\frac{cosn\pi x}{(n\pi)^3}\right)\right]$$

limits from 0 to 1,

$$b_n = 2\{[0 - 0 - \frac{2cosn\pi}{n^3\pi^3}] - [0\text{-}0\text{-}\frac{2}{n^3\pi^3}]\}$$

$$= \frac{2}{n^3\pi^3}\{\frac{-2cos}{n^3\pi^3} + \frac{2}{n^3\pi^3}\}$$

$$= \boldsymbol{\frac{4}{(n\pi)^3}[1 - (-1)^n]}$$ as $sinn\pi = 0$ and $cosn\pi = (-1)^n$

Thus, the solution of heat eqn becomes , from eqn(4)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^3}[1 - (-1)^n].\sin(n\pi x).e^{-\pi^2 n^2 t}$$

Ex.3: Find the temperature in a bar of length 2 whose ends are kept at zero and lateral surface insulated if the initial temperature is $\sin\left(\frac{\pi x}{2}\right) + 3\sin\left(\frac{5\pi x}{2}\right)$

Solution: Heat Equation is $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

3-possible solutions of Heat Equation are,

$u = (c_1 e^{px} + c_2 e^{-px})\, c_3 e^{p^2 c^2 t}$

u= $(c_1 cospx + c_2 sinpx)\, c_3 e^{-p^2 c^2 t}$

u = $(c_1 x + c_2)\, c_3$

We have the solution of heat eqn as

u(x, t)= $(c_1 cospx + c_2 sinpx)\, e^{-p^2 c^2 t}$ → (1)

Boundary Conditions are $u(0, t) = 0, u(2,0) = 0$ *and*
Initial Condition is $u(x, 0) = sin\left(\frac{\pi x}{2}\right) + 3sin\left(\frac{5\pi x}{2}\right)$

Now, using $u(0, t) = 0$ put x=0 in eqn (1),
we get $u(0, t) = c_1 e^{-c^2 p^2 t} = 0$
implies $\boldsymbol{c_1 = 0}$

Thus, eqn (1) becomes **u(x, t)= $(c_2 sinpx)\, e^{-c^2 p^2 t}$** → (2)

Given, u(2, t)=0 using in eqn(2),
we get, u(2, t) = $(c_2 sin2p)\, e^{-c^2 p^2 t} = 0$

Which implies, since $c_2 \neq 0$ we have, $\boldsymbol{sin2p = 0}$

Thus, $\mathbf{sin2}\boldsymbol{p = sinn\pi \Longrightarrow 2p = n\pi \Longrightarrow p = \frac{n\pi}{2}}$

Thus, the solution becomes $u(x,t) = c_2 \sin(n\pi x)\,.\,e^{-\frac{\pi^2 n^2 c^2 t}{4}}$ →(3)

Adding all the solutions, we get the general solution of heat eqn as

$$u(x,t) = \sum_{n=1}^{\infty} c_2 \sin(n\pi x)\,.\,e^{-\frac{\pi^2 n^2 c^2 t}{4}}$$

$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\,.\,e^{-\frac{\pi^2 n^2 c^2 t}{4}}$ →(4)

Given, $u(x,0) = \sin\left(\frac{\pi x}{2}\right) + 3\sin\left(\frac{5\pi x}{2}\right)$ using this in eqn (4),
we get
$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$ $=\sin\left(\frac{\pi x}{2}\right) + 3\sin\left(\frac{5\pi x}{2}\right)$

Which implies, $\sin\left(\frac{\pi x}{2}\right) + 3\sin\left(\frac{5\pi x}{2}\right)$

$=b_1 sin\left(\frac{\pi x}{2}\right) + b_2 sin\left(\frac{2\pi x}{2}\right) + b_3 sin\left(\frac{3\pi x}{2}\right) + b_4 sin\left(\frac{4\pi x}{2}\right) + b_5 sin\left(\frac{5\pi x}{2}\right) + \ldots$

Comparing the coefficients of Like terms on both sides, we get

$$b_1 = 1, b_2 = b_3 = b_4 = 0, b_5 = \frac{3}{2}, b_6 = \cdots \ldots . = 0$$

Thus , the Required Solution is

$$u(x,t) = sin\left(\frac{\pi x}{2}\right) e^{-\frac{\pi^2 c^2 t}{4}} + \frac{3}{2} sin\left(\frac{5\pi x}{2}\right) e^{-\frac{\pi^2 5^2 c^2 t}{4}}$$

Examples on heat equations

Ex.1: Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with boundary conditions

$u(x,0) = 3 sin n\pi x, u(0,t) = 0 = u(1,t)$, where $0 < x < 1, t > 0$

- Solution:
- By comparing the given equation with the std. heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$
- We have c=1
- The suitable solution for the heat equation is given by
- $u(x,t) = (c_1 cospx + c_2 sinpx)\, c_3 e^{-p^2 c^2 t}$
- The solution for the given equation is
- $u(x,t) = (c_1 cospx + c_2 sinpx)\, c_3 e^{-p^2 t}$ ----→(1)(put c=1)
- Boundary conditions are $i) u(0,t) = 0 \quad ii) u(1,t) = 0$
- Initial condition is $u(x,0) = 3 sin n\pi x$

We have the solution as $u(x,t) = (c_1 cospx + c_2 sinpx)\, c_3 e^{-p^2 t}$

- by using boundary condition $u(0,t) = 0$ (i.e)put x=0 and u=0 in the above eqn
- We get $u(0,t) = c_1 c_3 e^{-p^2 t} = 0$ implies $c_1 = 0$
- Therefore the solution becomes $u(x,t) = (c_2 sinpx)\, c_3 e^{-p^2 t}$
- By using the secondary boundary condition $u(1,t) = 0$ (i.e put x=1 and u=0)
- We get $u(1,t)$=($c_2 sinp$) $c_3 e^{-p^2 t} = 0 \Rightarrow sinp = 0$ (since $c_2 \neq 0$) $\Rightarrow p = n\pi$
- therefore , the solution becomes $u(x,t) = c_2 c_3 \sin(n\pi x) . e^{-\pi^2 n^2 t}$ →(2)
- We write this as $u(x,t) = a_n \sin(n\pi x) . e^{-\pi^2 n^2 t}$ where $\mathrm{a_n} = C_2 C_3$
- Adding all the solutions, we get the general solution of heat eqn as

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) . e^{-\pi^2 n^2 t} \rightarrow (4)$$

Given $u(x,0) = 3\sin(n\pi x)$, using this in eqn (4), we get

$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) = 3\sin(n\pi x)$

Using Fourier Sine series, $b_n = \frac{2}{1}\int_0^1 3\sin(n\pi x).\sin(n\pi x)\,dx = 6\int_0^1 \frac{1+\cos(2n\pi x)}{2}\,dx$

$$= 3[\int_0^1 dx + \int_0^1 \cos(2n\pi x)\,dx] = 3\left[x + \frac{\sin(2\pi n x)}{2n\pi}\right]_{x\to 0 \; to \; 1}$$

Hence, $b_n = 3$

Thus, the eqn (4) becomes$u(x,t) = \sum_{n=1}^{\infty} 3\sin(n\pi x).e^{-\pi^2 n^2 t}$

This is the required solution of heat equation.

Note:if $f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$ then b_n=$\frac{2}{l}\int_0^l f(x).\sin(n\pi x)dx$

Ex.2: Find the temperature in a bar of length 2 whose ends are kept at zero and lateral surface insulated if the initial temperature is $\sin\left(\frac{\pi x}{2}\right)+3\sin\left(\frac{5\pi x}{2}\right)$

Soln: Let the heat eqn be in std. form as $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$

We have the solution of heat eqn as u(x,t)= $(c_1 cospx + c_2 sinpx\ )\ c_3 e^{-p^2 c^2 t}$ →(1)

Boundary conditions are $u(0,t)=0, u(2,t)=0$ &

initial condition is $u(x,0)=\sin\left(\frac{\pi x}{2}\right)+3\sin\left(\frac{5\pi x}{2}\right)$

Now, using the given conditions,

$u(0,t)=0$ put x=0 in eqn (1), we get $u(0,t)=c_1 c_3 e^{-c^2 p^2 t}=0$ implies $c_1=0$

Thus, eqn (1) becomes u(x,t)= $(c_2 sinpx\ )c_3 e^{-c^2p^2t}$→ (2)

Given, u(2,t)=0 using in eqn(2), we get u(2,t)=$(c_2 sin2p\ )c_3 e^{-c^2p^2t} = 0$

Which implies, since $c_2 \neq 0$ we have $sin2p = 0$ thus $p = \frac{n\pi}{2}$

Thus, the solution becomes $u(x,t) = c_2 c_3 \sin\left(\frac{n\pi}{2}x\right).e^{-\frac{\pi^2 n^2 c^2 t}{4}}$ →(3)

Adding all the solutions, we get the general solution of heat eqn as

$$u(x,t) = \sum_{n=1}^{\infty} C_2 C_3 \sin\left(\frac{n\pi}{2}x\right) e^{-\frac{\pi^2 n^2 c^2 t}{4}} = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\, e^{-\frac{\pi^2 n^2 c^2 t}{4}} \rightarrow 4$$

$where\ b_n = C_2 C_3$

Given, $u(x,0) =$, using this in eqn (4), we get $\sin\left(\frac{\pi x}{2}\right) + 3\sin\left(\frac{5\pi x}{2}\right)$

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{2}x\right) = \sin\left(\frac{\pi x}{2}\right) + 3\sin\left(\frac{5\pi x}{2}\right)$$

Which implies, $\sin\left(\frac{\pi x}{2}\right) + 3\sin\left(\frac{5\pi x}{2}\right)$

$$= b_1 sin\left(\frac{\pi x}{2}\right) + b_2 sin\left(\frac{2\pi x}{2}\right) + b_3 sin\left(\frac{3\pi x}{2}\right) + b_4 sin\left(\frac{4\pi x}{2}\right) + b_5 sin\left(\frac{5\pi x}{2}\right) + \ldots$$

Comparing the co-efficients on both sides, we get

$$b_1 = 1, b_2 = b_3 = b_4 = 0, b_5 = \frac{3}{2}, b_6 = \cdots \ldots . = 0$$

Thus , the solution is $u(x,t) = sin\left(\frac{\pi x}{2}\right) e^{-\frac{\pi^2 c^2 t}{4}} + \frac{3}{2} sin\left(\frac{5\pi x}{2}\right) e^{-\frac{\pi^2 5^2 c^2 t}{4}}$

Ex.3: Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to the conditions $u(x,0) = x - x^2$, $u(0,t) = 0, u(1,t) = 0$, where 0<x<1.

- Solution:
- By comparing the given equation with the std. heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$
- We have c=1
- The suitable solution for the heat equation is given by
- $u(x,t) = (c_1 cospx + c_2 sinpx)\, c_3 e^{-p^2 c^2 t}$
- The solution for the given equation is
- $u(x,t) = (c_1 cospx + c_2 sinpx)\, c_3 e^{-p^2 t}$ ----→(1)(put c=1)
- Boundary conditions are $i) u(0,t) = 0 \quad ii) u(1,t) = 0$
- Initial condition is $u(x,0) = x - x^2$

Thus, eqn (1) becomes u(x,t)= ($c_2 sinpx$) $c_3 e^{-p^2 t}$ → (2)

Given, u(1,t)=0 using in eqn(2), we get u(1,t)=($c_2 sinp$) $c_3 e^{-p^2 t} = 0$

Which implies, since $c_2 \neq 0$ we have $sinp = 0$ thus $p = n\pi$

Thus, the solution becomes $u(x,t) = c_2 c_3 \sin(n\pi x) . e^{-\pi^2 n^2 t}$ →(3)

Adding all the solutions, we get the general solution of heat eqn as

$u(x,t) = \sum_{n=1}^{\infty} c_2 c_3 \sin(n\pi x) . e^{-\pi^2 n^2 t}$=$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) . e^{-\pi^2 n^2 t}$→(4)

where $b_n = C_2 C_3$

Given, $u(x,0) = x - x^2$, using this in eqn (4), we get

$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$ = $x - x^2$

Using Fourier Sine series, $b_n = \frac{2}{1}\int_0^1 (x - x^2).\sin(n\pi x)\,dx$

Integrating by parts, we get

$$b_n = 2\left[(x - x^2).\left(\frac{-\cos n\pi x}{n\pi}\right) - (1 - 2x)\left(\frac{-\sin n\pi x}{(n\pi)^2}\right) + (-2)\left(\frac{\cos n\pi x}{(n\pi)^3}\right)\right] \text{ limits from 0 to 1,}$$

$$b_n = 2\left[\frac{-2}{(n\pi)^3}(\cos n\pi - 1)\right] = \frac{4}{(n\pi)^3}[1 - (-1)^n]$$

Thus, the solution of heat eqn becomes , from eqn(4)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^3}[1 - (-1)^n]\ \sin(n\pi x)\,.\,e^{-\pi^2 n^2 t} \rightarrow (4)$$

Ex.4: A rod of length 1 with insulated sides is initially at a uniform temperature x^2. It ends are suddenly cooled to $0°C$ and are kept at that temperature. Find the temperature function $u(x,t)$.

Soln: one dimensional heat equation is given by $\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial t^2}\right)$

We have the solution of heat eqn as $u(x,t) = (c_1 cospx + c_2 sinpx\ )\, c_3 e^{-p^2 C^2 t}$ →(1)

boundary conditions are $u(0,t) = 0, u(1,0) = 0$ &

initial condition is $u(x,0) = x^2$

Now, using the given conditions,

$u(0,t) = 0$ put $x = 0$ in eqn (1), we get $u(0,t) = c_1 c_3 e^{-c^2 p^2 t} = 0$ implies $c_1 = 0$

Thus, eqn (1) becomes $u(x,t) = (c_2 sinpx\ )c_3 e^{-c^2p^2t}$ → (2)

Given, $u(1,t) = 0$ using in eqn(2), we get $u(1,t)$ =$(c_2 sinp\ )c_3 e^{-c^2p^2t} = 0$

Which implies, since $c_2 \neq 0$ we have $sinp = 0$ thus $p = n\pi$

Thus, the solution becomes $u(x,t) = c_2 c_3 \sin(n\pi x)\, e^{-n^2\pi^2c^2t}$ →(3)

Adding all the solutions, we get the general solution of heat eqn as

$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\, e^{-n^2\pi^2c^2t}$→(4) where$b_n = C_2 C_3$

By using initial condition $u(x,0) = x^2$,the above equation becomes

$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\, e^{-n^2\pi^2c^2 0}$

$x^2 = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$

- Which is half range sine series where

  - b_n is given by $b_n = 2\int_0^1 x^2 \sin n\pi x \, dx$ where length = 1(given)
  - Applying integration by parts
  - $b_n = 2\left[x^2\left(-\frac{\cos(n\pi x)}{n\pi}\right) - ((2x) * \left(-\frac{\sin(n\pi x)}{n^2\pi^2}\right) + ((2) * (\frac{\cos(n\pi x)}{n^3\pi^3})\right]_{x\to 0 to 1}$
  - $b_n = 2\left[-\left(\frac{\cos(n\pi)}{n\pi}\right) + (2) * (\frac{1}{n^3\pi^3})(\cos(n\pi) - \cos(0))\right]$
  - $b_n = 2\left[\frac{(-1)^{n+1}}{n\pi} + 2\left(\frac{1}{n^3\pi^3}\right) * ((-1)^n - 1)\right]$
  - There fore the required solution is
  - $u(x,t) = \sum_{n=1}^{\infty} 2\left[\frac{(-1)^{n+1}}{n\pi} + 2\left(\frac{1}{n^3\pi^3}\right) * ((-1)^n - 1)\right] \sin(n\pi x)\, e^{-n^2\pi^2c^2t}$

Example1: Solve Laplace equation given that $u(0, y) = u(l, y) = u(x, 0) = 0$ and $u(x, a) = \sin\left(\frac{m\pi x}{l}\right)$.

Solution:

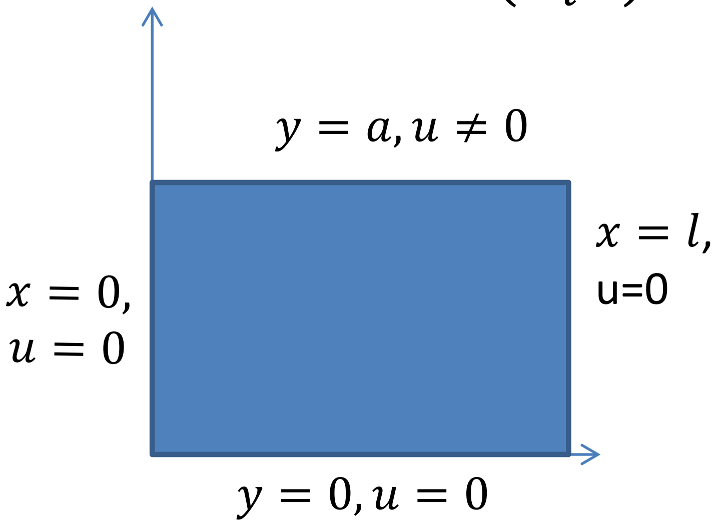
- Laplace Equation is $u_{xx} + u_{yy} = 0$
- 3-possible solutions are
- $u = (C_1 e^{px} + C_2 e^{-px})(C_3\ cospy + C_4\ sinpy)$
- $u = (C_1 cospx + C_2 sinpx)(C_3 e^{py} + C_4 e^{-py})$
- $u = (C_1 + C_2 x)(C_3 + C_4 y)$

Since the non zero boundary condition is parallel to x-axis , the suitable solution is

Therefore suitable solution involves cosine and sine functions of x.

- $u = (C_1 cospx + C_2 sinpx)(C_3 e^{py} + C_4 e^{-py})$---(1)
- Since $u(0, y) = 0$, from (1)
- $0 = C_1(C_3 e^{py} + C_4 e^{-py}) \implies C_1 = 0$. From (1), we get
- $u(x, y) = C_2 sinpx\,(C_3 e^{py} + C_4 e^{-py})$---------(2)

$u(0, y) = u(l, y) = u(x, 0) = 0$ and $u(x, a) = \sin\left(\frac{m\pi x}{l}\right)$.



$$u(x,y) = C_2 sinpx\ (C_3 e^{py} + C_4 e^{-py})\text{---------}(2)$$

- Since $u(l,y) = 0$, From (2), we get
- 0= $C_2 sinpl\ (C_3 e^{py} + C_4 e^{-py}) \Longrightarrow sinpl = 0 \ \Longrightarrow pl = n\pi$.
- $p = \frac{n\pi}{l}$. Therefore, equation(2) becomes
- $u(x,y) = C_2 \sin(\frac{n\pi x}{l})\ (C_3 e^{\frac{n\pi y}{l}} + C_4 e^{\frac{-n\pi y}{l}})$------------(3)
- Also $u(x,0) = 0,$ using this in equation (3), we get
- 0= $C_2 \sin(\frac{n\pi x}{l})\ (C_3 + C_4) \Longrightarrow C_3 + C_4 = 0, \Longrightarrow C_4 = -C_3$

$$u(x,y) = C_2 \sin(\frac{n\pi x}{l})\,(C_3 e^{\frac{n\pi y}{l}} + C_4 e^{\frac{-n\pi y}{l}})\text{-------------}(3)$$

- Equation (3) becomes,
- $u(x,y) = C_2 \sin\left(\frac{n\pi x}{l}\right)(C_3 e^{\frac{n\pi y}{l}} - C_3 e^{\frac{-n\pi y}{l}})$
- $u(x,y) = C_2\, C_3 \sin\left(\frac{n\pi x}{l}\right)(e^{\frac{n\pi y}{l}} - e^{\frac{-n\pi y}{l}})$
- $u(x,y) = 2C_2\, C_3 \sin\left(\frac{n\pi x}{l}\right)\{\frac{e^{\frac{n\pi y}{l}} - e^{\frac{-n\pi y}{l}}}{2}\}$
- We know $sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$
- $u(x,y) = C_n\ \sin\left(\frac{n\pi x}{l}\right) \sinh[\frac{n\pi y}{l}]$--------------(4) where $C_n = 2C_2C_3$

Adding up of such solutions for all values of n, we get General Solution as

- $u(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right) \sinh[\frac{n\pi y}{l}]$--------------(5)

- Using $u(x,a) = \sin(\frac{m\pi x}{l})$ in (5) we get

- Sin$(\frac{m\pi x}{l}) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)$ sinh$[\frac{n\pi a}{l}]$, where m is +ve integer.

- Equating the coefficients on both sides, we get

- $1 = C_m \sinh(\frac{m\pi a}{l})$ and $0 = C_n$ for $n \neq m$.

Therefore,

- $C_m = \frac{1}{\sinh(\frac{m\pi a}{l})}$ using this in equation in (5), we get

- $u(x, y) = \frac{\sin\left(\frac{m\pi x}{l}\right)}{\sinh\left(\frac{m\pi a}{l}\right)} \sinh(\frac{m\pi y}{l})$

- Which is the required solution.

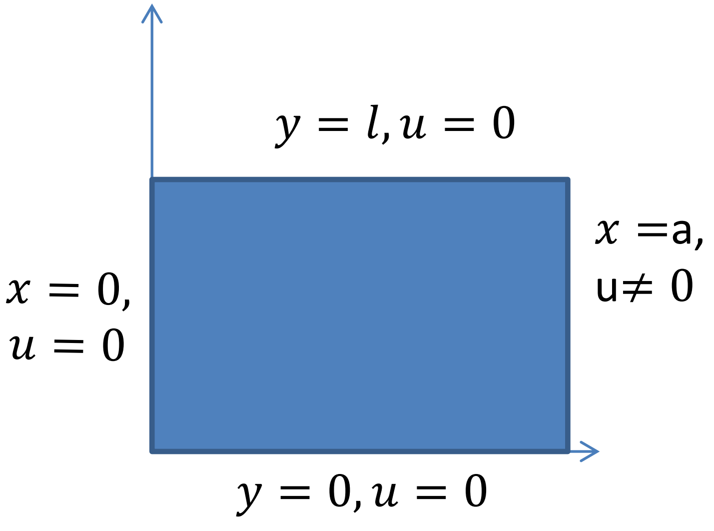
Example2: Solve Laplace equation for $0< x < a, 0 < y < l$, given that $u(0,y) = u(x,l) = u(x,0) = 0$ and $u(a,y) = y$.

Solution:

- Laplace Equation is $u_{xx} + u_{yy} = 0$
- 3-possible solutions are
- $u = (C_1 e^{px} + C_2 e^{-px})(C_3\ cospy + C_4\ sinpy)$
- $u = (C_1 cospx + C_2 sinpx)(C_3 e^{py} + C_4 e^{-py})$
- $u = (C_1 + C_2 x)(C_3 + C_4 y)$

Since the non zero boundary condition is parallel to y-axis , the suitable solution is

$u(0,y) = u(x,l) = u(x,0) = 0$ and $u(a,y) = y$.



Therefore suitable solution involves cosine and sine functions of y.

- $u = (C_1e^{px} + C_2e^{-px})(C_3\, cospy + C_4\, sinpy)$--(1)

using $u(x,0) = 0$, in (1), we get,

0= $(C_1e^{px} + C_2e^{-px})C_3 \quad \Longrightarrow \; C_3 = 0$, From (1)

$u = (C_1e^{px} + C_2e^{-px})\; C_4$ ---------(2)

Using $u(x,l) = 0$ in (2), we get

0= $(C_1e^{px} + C_2e^{-px})C_4\, sinpl \Longrightarrow sinpl = 0$ as $C_4 \neq 0$

$pl = n\pi \quad \Longrightarrow \quad p = \frac{n\pi}{l}$.

Therefore, equation(2) becomes $u(x,y) = (C_1 e^{\frac{n\pi x}{l}} + C_2 e^{\frac{-n\pi x}{l}})\, C_4 \sin(\frac{n\pi y}{l})$-----3

- Using u(0,y)=0 in equation(3), we get
- 0= $(C_1 + C_2)\, C_4 \sin(\frac{n\pi y}{l}) \implies C_1 + C_2 = 0 \Rightarrow C_2 = -C_1.$
- Equation (3) becomes
- $u(x,y) = (C_1 e^{\frac{n\pi x}{l}} - C_1 e^{\frac{-n\pi x}{l}})\, C_4 \sin(\frac{n\pi y}{l})$.
- $u(x,y) = C_1 C_4 (e^{\frac{n\pi x}{l}} - C_1 e^{\frac{-n\pi x}{l}}) \sin(\frac{n\pi y}{l})$.
- $u(x,y) = 2C_1\, C_4 \sin\left(\frac{n\pi y}{l}\right) \{\frac{e^{\frac{n\pi x}{l}} - e^{\frac{-n\pi x}{l}}}{2}\}$ we know $sinh\theta = (e^{\theta} - e^{-\theta})/2$

$$u(x,y) = C_n \sin\left(\frac{n\pi y}{l}\right) \text{sinh}[\frac{n\pi x}{l}]\text{--------------}(4) \text{ where } C_n = 2C_2C_3$$

Adding up of such solutions for all values of n, we get General Solution as

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{l}\right) \text{sin}[\frac{n\pi y}{l}]\text{--------------}(5)$$

Using $u(a,y) = f(y) = y$ in (5) we get,

$$y = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi a}{l}\right) \text{sin}[\frac{n\pi y}{l}]\text{-------------}(6)$$

Which is half range sine series where

$$\mathrm{C_n} \sinh\left(\frac{n\pi a}{l}\right) = \frac{2}{l}\int_0^l y \sin\left(\frac{n\pi y}{l}\right) dy$$

By applying integration by parts

$$C_n \sinh\left(\frac{n\pi a}{l}\right) = \frac{2}{l}\left[(y)\left(-\frac{\cos\left(\frac{n\pi y}{l}\right)}{\frac{n\pi}{l}}\right) - (1)\,)(\frac{\sin\left(\frac{n\pi y}{l}\right)}{\frac{n^2\pi^2}{l^2}})\right]_{y\to 0\to to 1}$$

$$\mathrm{C_n} \sinh\left(\frac{n\pi a}{l}\right) = \frac{2}{l}(-l\left(\frac{l}{n\pi}\right)(\cos(n\pi)) = \ (\frac{-2l}{n\pi}(-1)^n)$$

$$C_n = \frac{-2l(-1)^n}{n\pi(\frac{sinhn\pi a}{l})}$$

Therefore the required solution is $u(x,y) = \sum_{n=1}^{\infty} \frac{-2l(-1)^n}{\mathrm{n\pi sinh}(\frac{n\pi a}{l})}\ \sinh\left(\frac{n\pi x}{l}\right) \sin[\frac{n\pi y}{l}]$

Example 3: Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$,for 0< $x < a, 0 < y < l,$

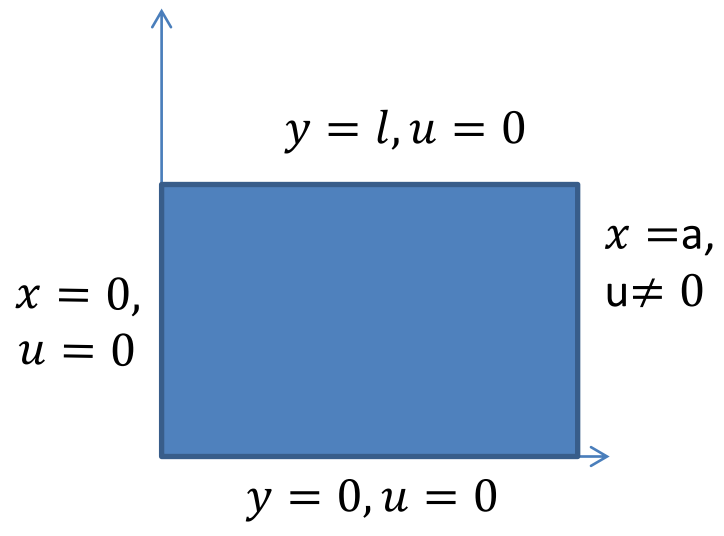
given that $u(0, y) = u(x, l) = u(x, 0) = 0$ and $u(a, y) = \begin{cases} y & 0 \leq y \leq \frac{l}{2} \\ l - y & \frac{l}{2} \leq y \leq l \end{cases}$

Solution:

- The given equation is $u_{xx} + u_{yy} = 0$
- 3-possible solutions are
- $u = (C_1 e^{px} + C_2 e^{-px})(C_3\, cospy + C_4\, sinpy)$
- $u = (C_1 cospx + C_2 sinpx)(C_3 e^{py} + C_4 e^{-py})$
- $u = (C_1 + C_2 x)(C_3 + C_4 y)$

Since the non zero boundary condition is parallel to y-axis , the suitable solution is

$u(0,y) = u(x,l) = u(x,0) = 0$ and $u(a,y) = f(y)$.



Therefore suitable solution involves cosine and sine functions of y.

- $u = (C_1 e^{px} + C_2 e^{-px})(C_3\, cospy + C_4\, sinpy)$--(1)

using $u(x,0) = 0$, in (1), we get,

0$= (C_1 e^{px} + C_2 e^{-px})C_3 \implies C_3 = 0$, From (1)

$u = (C_1 e^{px} + C_2 e^{-px})\, C_4$ ---------(2)

Using $u(x,l) = 0$ in (2), we get

0$= (C_1 e^{px} + C_2 e^{-px})C_4\, sinpl \implies sinpl = 0$ as $C_4 \neq 0$

$pl = n\pi \implies p = \frac{n\pi}{l}$.

Therefore, equation(2) becomes $u(x,y) = (C_1 e^{\frac{n\pi x}{l}} + C_2 e^{\frac{-n\pi x}{l}})\ C_4 \sin(\frac{n\pi y}{l})$-----3

- Using u(0,y)=0 in equation(3), we get
- 0= $(C_1 + C_2)\ C_4 \sin(\frac{n\pi y}{l}) \Rightarrow C_1 + C_2 = 0 \Rightarrow C_2 = -C_1.$
- Equation (3) becomes
- $u(x,y) = (C_1 e^{\frac{n\pi x}{l}} - C_1 e^{\frac{-n\pi x}{l}})\ C_4 \sin(\frac{n\pi y}{l})$.
- $u(x,y) = C_1 C_4 (e^{\frac{n\pi x}{l}} - C_1 e^{\frac{-n\pi x}{l}}) \sin(\frac{n\pi y}{l})$.
- $u(x,y) = 2C_1\ C_4 \sin\left(\frac{n\pi y}{l}\right) \{\frac{e^{\frac{n\pi x}{l}} - e^{\frac{-n\pi x}{l}}}{2}\}$ we know $sinh\theta = (e^{\theta} - e^{-\theta})/2$

$u(x, y) = C_n \sin\left(\frac{n\pi y}{l}\right) \sinh[\frac{n\pi x}{l}]$--------------(4) where $C_n = 2C_2C_3$

Adding up of such solutions for all values of n, we get General

Solution as $u(x, y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{l}\right) \sin[\frac{n\pi y}{l}]$--------------(5)

Using $u(a, y) = f(y) = \begin{cases} y & 0 \le y \le \frac{l}{2} \\ l - y & \frac{l}{2} \le y \le l \end{cases}$ in (5) we get,

$y = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi a}{l}\right) \sin[\frac{n\pi y}{l}]$-------------(6)

Which is half range sine series where

$$C_n \sinh\left(\frac{n\pi a}{l}\right) = \frac{2}{l} \int_0^l f(y) \sin\left(\frac{n\pi y}{l}\right) dy$$

$$\mathrm{C_n}\sinh\left(\frac{n\pi a}{l}\right)=\frac{2}{\mathrm{l}}\int_0^l f(y)sin\left(\frac{n\pi y}{l}\right)dy$$

- By applying integration by parts
- $\mathrm{C_n}\sinh\left(\frac{n\pi a}{l}\right)=\frac{2}{\mathrm{l}}\int_0^{\frac{l}{2}} ysin\left(\frac{n\pi y}{l}\right)dy+\int_{\frac{l}{2}}^{l}(l-y)\sin\left(\frac{n\pi y}{l}\right)dy$
- $\mathrm{C_n}\sinh\left(\frac{n\pi a}{l}\right)=\left[\frac{2}{l}\left[(y)\left(-\frac{\cos\left(\frac{n\pi y}{l}\right)}{\frac{n\pi}{l}}\right)-(1)\,)\left(\frac{\sin\left(\frac{n\pi y}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right)\right]_{y\to 0\to\frac{l}{2}}\right]+$

$$\left[(l-y)\left(-\frac{\cos\left(\frac{n\pi y}{l}\right)}{\frac{n\pi}{l}}\right)-(-1)\,)\left(\frac{\sin\left(\frac{n\pi y}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right)\right]_{y\to\frac{l}{2}\to l}$$

Applying the limits we get

$$C_n \sinh\left(\frac{n\pi a}{l}\right) = \frac{2}{l}\left[-\frac{l}{2}\left(\frac{\cos\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{l}}\right) + \left(\frac{sin\left(\frac{n\pi}{2}\right)}{\frac{n^2\pi^2}{l^2}}\right) + \frac{l}{2}\left(\frac{\cos\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{l}}\right) + \left(\frac{\sin\left(\frac{n\pi}{2}\right)}{\frac{n^2\pi^2}{l^2}}\right)\right]$$

- $C_n \sin h(\frac{n\pi a}{l}) = \frac{2}{l} * \left(\frac{l^2}{n^2\pi^2}\right)\left(2\sin\left(\frac{n\pi}{2}\right)\right)$
- $C_n \sin h\left(\frac{n\pi a}{l}\right) = \frac{4l}{(n^2\pi^2)} * \left(\sin\left(\frac{n\pi}{2}\right)\right)$
- $C_n = \frac{4l}{(n^2\pi^2)\sinh(\frac{n\pi a}{l})} * \left(\sin\left(\frac{n\pi}{2}\right)\right)$

Therefore the required solution is

$$u(x,y) =$$

$$\sum_{n=1}^{\infty} \frac{4l\left(\sin\left(\frac{n\pi}{2}\right)\right)}{(n^2\pi^2)\sinh(\frac{n\pi a}{l})} \sinh\left(\frac{n\pi x}{l}\right) \sin[\frac{n\pi y}{l}]$$

One dimensional wave equation

Eg 1: A string is stretched and fastened to two points '*l*' apart. Motion is started by displacing the string in the form u $= Asin\left(\frac{\pi x}{l}\right)$ from which it is released at time t=0. Show that the displacement of any point at a distance x from one end at time t is given by

u $= Asin\left(\frac{\pi x}{l}\right) cos\left(\frac{\pi ct}{l}\right)$.

- Solution:
  one dimensional wave eqn is $u_{tt} = c^2 u_{xx}$ $\rightarrow$ (1)
- **Boundary conditions**
- u(0, t) = 0 = u(*l*, t)
- **initial conditions**
- $u(x,0) = Asin\left(\frac{\pi x}{l}\right), \left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$ (initially velocity is zero)

Various possible solutions are

$$u = (C_1 e^{px} + C_2 e^{-px}) * (C_3 e^{cpt} + C_4 e^{-cpt})$$

$$u = (C_1 cospx + C_2 sinpx) * (C_3 \cos cpt + C_4 \sin cpt)$$

$$u = (C_1 x + C_2) * (C_3 t + C_4)$$

- Of these solutions, choose that, which is consistent with the physical nature of the problems. Since, we are dealing with the problems on vibrations, u must be a periodic function of x and t.

  Therefore, the solution must involve trigonometric functions. Accordingly, the only suitable solution is (which corresponds to $k = -p^2$)

$$u(x, t) = (C_1 \cos px + C_2 \sin px) * (C_3 \cos cpt + C_4 \sin cpt) \quad \rightarrow 2$$

By using $u(0,t) = 0$ in eqn 2

$$0 = (C_3 \cos cpt + C_4 \sin cpt) * C_1 \quad \Longrightarrow \quad \boldsymbol{C_1 = 0}$$

By using u(l, t) = 0 (i.e) Put x = l and u= 0 in (2)

$$0 = [C_3 \cos cpt + C_4 \sin cpt] * C_2 \sin pl \qquad (\, bcoz\ C_1 = 0)$$

$$\Longrightarrow \sin pl = 0 = \sin n\pi \Longrightarrow pl = n\pi \Longrightarrow \boldsymbol{p = \frac{n\pi}{l}}, \text{ n=1,2,3}$$

(2) reduces to $u(x,t) = \left(C_3 cos\left(\frac{n\pi ct}{l}\right) + C_4 sin\left(\frac{n\pi ct}{l}\right)\right) * C_2 sin\left(\frac{n\pi x}{l}\right)$

we write this

$$u(x,t) = \left(a_n cos\left(\frac{n\pi ct}{l}\right) + b_n sin\left(\frac{n\pi ct}{l}\right)\right) * sin\left(\frac{n\pi x}{l}\right) \to 3$$

Where $a_n = C_2 C_3\ and\ b_n = C_2 C_4$

<u>**Therefore the solution satisfying boundary conditions is**</u>

$$u(x,t) = \left(a_n cos\left(\frac{n\pi ct}{l}\right) + b_n sin\left(\frac{n\pi ct}{l}\right)\right) * sin\left(\frac{n\pi x}{l}\right) \to 3$$

Initial condition:

i) $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$: Put t = 0, $\frac{\partial u}{\partial t} = 0$

Differentating eqn 3 w.r.t. t

$$\frac{\partial u}{\partial t} = (\frac{n\pi c}{l})\left(a_n(-sin\left(\frac{n\pi ct}{l}\right)) + b_n cos\left(\frac{n\pi ct}{l}\right)\right) * sin\left(\frac{n\pi x}{l}\right)$$

At t=0:

$$0 = \frac{(b_n)n\pi c}{l} * sin\left(\frac{n\pi x}{l}\right) \Longrightarrow \qquad \boldsymbol{b_n = 0}$$

therefore the solution becomes

$$u(x,t) = a_n \cos\left(\frac{n\pi ct}{l}\right) * \sin\left(\frac{n\pi x}{l}\right)$$

Adding for different values of n

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n cos\left(\frac{n\pi ct}{l}\right)\right) * sin\left(\frac{n\pi x}{l}\right) \qquad \rightarrow (4)$$

By $u(x,0) = Asin\ \left(\frac{\pi x}{l}\right)$ (i.e) put t=0 in the above equation

equation 4 becomes

$$Asin\ \left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} a_n * sin\left(\frac{n\pi x}{l}\right) \qquad \rightarrow (5)$$

On expanding R.H.S we get

$$Asin\ \left(\frac{\pi x}{l}\right) = a_1 \sin\left(\frac{\pi x}{l}\right) + a_2 \sin\left(\frac{2\pi x}{l}\right) + a_3 \sin\left(\frac{3\pi x}{l}\right) + a_4 \sin\left(\frac{4\pi x}{l}\right) + \ldots$$

On comparing both sides

$$a_n = \begin{cases} A, & n = 1 \\ 0, & n \neq 1 \end{cases}$$

Therefore, required solution is, $u(x,t) = A\cos\left(\frac{\pi ct}{l}\right)\sin\left(\frac{\pi x}{l}\right)$

Example 2: A tightly stretched string with fixed end points x=0 and x=*l* is initially in a position given by $u = u_0 sin^3 \frac{\pi x}{l}$. If it is released from rest from this position, find the displacement $u(x,t)$.

<u>Solution:</u> one dimensional wave eqn is $u_{tt} = c^2 u_{xx}$ $\rightarrow (1)$

- **<u>Boundary conditions</u>**
- u(0, t) = 0 = u(*l*, t)
- **<u>initial conditions</u>**
- $u(x,0) = u_0 sin^3 \frac{\pi x}{l}$, $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$(initially velocity is zero)

Various possible solutions are

$$u = (C_1 e^{px} + C_2 e^{-px}) * (C_3 e^{cpt} + C_4 e^{-cpt})$$

$$u = (C_1 cospx + C_2 sinpx) * (C_3 \cos cpt + C_4 \sin cpt)$$

$$u = (C_1 x + C_2) * (C_3 t + C_4)$$

Of these solutions, choose that, which is consistent with the physical nature of the problems. Since, we are dealing with the problems on vibrations, u must be a periodic function of x and t.

Therefore, the solution must involve trigonometric functions. Accordingly, the only suitable solution is (which corresponds to $k = -p^2$)

$$u(x,t) = (C_1 \cos px + C_2 \sin px) * (C_3 \cos cpt + C_4 \sin cpt) \quad \rightarrow 2$$

By using $u(0,t) = 0$ in eqn 2

$$0 = (C_3 \cos cpt + C_4 \sin cpt) * C_1 \qquad \Longrightarrow \qquad \boldsymbol{C_1 = 0}$$

By using u(l, t) = 0 (i.e) Put x = l and u= 0 in (2)

$$0 = [C_3 \cos cpt + C_4 \sin cpt] * C_2 \sin pl \qquad\qquad (\,bcoz\ C_1 = 0)$$

$$\Longrightarrow \sin pl = 0 = \sin n\pi \Longrightarrow pl = n\pi \Longrightarrow \boldsymbol{p = \frac{n\pi}{l}},\ n=1,2,3$$

(2) reduces to $u(x,t) = \left(C_3 cos\left(\frac{n\pi ct}{l}\right) + C_4 sin\left(\frac{n\pi ct}{l}\right)\right) * C_2 sin\left(\frac{n\pi x}{l}\right)$

we write this

$$u(x,t) = \left(a_n cos\left(\frac{n\pi ct}{l}\right) + b_n sin\left(\frac{n\pi ct}{l}\right)\right) * sin\left(\frac{n\pi x}{l}\right) \to 3$$

Where $a_n = C_2C_3\ and\ b_n = C_2C_4$

Therefore the solution satisfying boundary conditions is

$$u(x,t) = \left(a_n cos\left(\frac{n\pi ct}{l}\right) + b_n sin\left(\frac{n\pi ct}{l}\right)\right) * sin\left(\frac{n\pi x}{l}\right) \to 3$$

Initial condition:

i) $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$: Put t = 0, $\frac{\partial u}{\partial t} = 0$

Differentating eqn 3 w.r.t. t

$$\frac{\partial u}{\partial t} = (\frac{n\pi c}{l})\left(a_n(-sin\left(\frac{n\pi ct}{l}\right)) + b_n cos\left(\frac{n\pi ct}{l}\right)\right) * sin\left(\frac{n\pi x}{l}\right)$$

At t=0:

$$0 = \frac{(b_n)n\pi c}{l} * sin\left(\frac{n\pi x}{l}\right) \Longrightarrow \qquad \boldsymbol{b_n = 0}$$

therefore the solution becomes

$$u(x,t) = a_n \cos\left(\frac{n\pi ct}{l}\right) * \sin\left(\frac{n\pi x}{l}\right)$$

Adding for different values of n

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n cos\left(\frac{n\pi ct}{l}\right)\right) * sin\left(\frac{n\pi x}{l}\right) \qquad \rightarrow (4)$$

by $u(x,0) = u_0 sin^3 \frac{\pi x}{l}$ Put t = 0 and $u(x,t) = u_0 sin^3 \frac{\pi x}{l}$ in (4)

$$u_0 sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} a_n * sin\left(\frac{n\pi x}{l}\right) \qquad \rightarrow (5)$$

we know that $sin3\theta = 3sin\theta - 4\sin^3\theta \quad \therefore \sin^3\theta = \frac{3}{4}\sin(\theta) - \frac{1}{4}\sin(3\theta)$

Eqn 5 becomes $u_0(\frac{3}{4}\sin\left(\frac{\pi x}{l}\right) - \frac{1}{4}\sin\left(\frac{3\pi x}{l}\right)) = \sum_{n=1}^{\infty} a_n * sin\left(\frac{n\pi x}{l}\right)$

on expanding the R.H.S we get

$u_0(\frac{3}{4}\sin\left(\frac{\pi x}{l}\right) - \frac{1}{4}\sin\left(\frac{3\pi x}{l}\right)) = a_1 \sin\left(\frac{\pi x}{l}\right) + a_2 \sin\left(\frac{2\pi x}{l}\right) + a_3 \sin\left(\frac{3\pi x}{l}\right) + \cdots$

on comparing both sides we get

$$a_1 = \frac{3}{4}u_0, a_2 = 0, a_3 = -\frac{1}{4}u_0, a_4 = a_5 = a_6 = \cdots = 0$$

therefore the required solution is

$$u(x,t) = \frac{3}{4}u_0 \sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi ct}{l}\right) - \frac{1}{4}u_0 \sin\left(\frac{3\pi x}{l}\right)\cos(\frac{3\pi ct}{l})$$

Eg 3: A tightly stretched string with fixed end points x=0 and x=*l* is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points a velocity $\lambda x(l-x)$, find the displacement of the string at any distance x from one end at any time t.

- <u>Solution</u>:

  Equation of the string is $u_{tt} = c^2 u_{xx}$ $\rightarrow$ (1)
- **<u>Boundary conditions</u>**
- u(0, t) = 0 = u(*l*, t)
- **<u>initial conditions</u>**
- $u(x,0)$ =0 , $\left(\frac{\partial u}{\partial t}\right)_{t=0} = \lambda x(l-x)$ (since the string is initially at rest in equilibrium position)

Various possible solutions are

$$u = (C_1 e^{px} + C_2 e^{-px}) * (C_3 e^{cpt} + C_4 e^{-cpt})$$

$$u = (C_1 cospx + C_2 sinpx) * (C_3 \cos cpt + C_4 \sin cpt)$$

$$u = (C_1 x + C_2) * (C_3 t + C_4)$$

Of these solutions, choose that, which is consistent with the physical nature of the problems. Since, we are dealing with the problems on vibrations, u must be a periodic function of x and t.

Therefore, the solution must involve trigonometric functions. Accordingly, the only suitable solution is (which corresponds to $k = -p^2$)

$$u(x,t) = (C_1 \cos px + C_2 \sin px) * (C_3 \cos cpt + C_4 \sin cpt) \quad \rightarrow 2$$

By using $u(0,t) = 0$ in eqn 2

$$0 = (C_3 \cos cpt + C_4 \sin cpt) * C_1 \quad \Longrightarrow \quad \boldsymbol{C_1 = 0}$$

By using u(l, t) = 0 (i.e) Put x = l and u= 0 in (2)

$$0 = [C_3 \cos cpt + C_4 \sin cpt] * C_2 \sin pl \qquad (\, bcoz\ C_1 = 0)$$

$$\Longrightarrow \sin pl = 0 = \sin n\pi \Longrightarrow pl = n\pi \Longrightarrow \boldsymbol{p = \frac{n\pi}{l}},\ n=1,2,3$$

(2) reduces to $u(x,t) = \left(C_3 cos\left(\frac{n\pi ct}{l}\right) + C_4 sin\left(\frac{n\pi ct}{l}\right)\right) * C_2 sin\left(\frac{n\pi x}{l}\right)$

we write this

$$u(x,t) = \left(a_n cos\left(\frac{n\pi ct}{l}\right) + b_n sin\left(\frac{n\pi ct}{l}\right)\right) * sin\left(\frac{n\pi x}{l}\right) \rightarrow 3$$

Where $a_n = C_2 C_3\ and\ b_n = C_2 C_4$

Therefore the solution satisfying boundary conditions is

$$u(x,t) = \left(a_n cos\left(\frac{n\pi ct}{l}\right) + b_n sin\left(\frac{n\pi ct}{l}\right)\right) * sin\left(\frac{n\pi x}{l}\right) \rightarrow 3$$

Initial condition:

i) $u(x,0) = 0$ Since the string was initially at rest $\Longrightarrow$ put t = 0 in (3)

$0 = (\mathrm{x},\mathrm{t}) = (a_n) * sin\left(\frac{n\pi x}{l}\right) \Longrightarrow a_n = 0$

therefore eqn 4 becomes $\mathrm{u}(\mathrm{x},\mathrm{t}) = \left(b_n sin\left(\frac{n\pi ct}{l}\right)\right) * sin\left(\frac{n\pi x}{l}\right)$

Adding for different values of n

The solution becomes $u(x,t) = \sum_{n=1}^{\infty}\left(b_n sin\left(\frac{n\pi ct}{l}\right)\right) * sin\left(\frac{n\pi x}{l}\right)$ $\rightarrow (4)$

by $\left(\frac{\partial u}{\partial t}\right)_{t=0} = \lambda x(l-x)$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi ct}{l}\right) * sin\left(\frac{n\pi x}{l}\right)\left(\frac{n\pi c}{l}\right)$$ (differentiate (4) w.r.to t)

when t=0, $\lambda x(l-x) = \sum_{n=1}^{\infty} b_n * sin\left(\frac{n\pi x}{l}\right)\left(\frac{n\pi c}{l}\right)$

$$\frac{l}{\pi c}\lambda x(l-x)=\sum_{n=1}^{\infty} nb_n * sin\left(\frac{n\pi x}{l}\right)$$

Which is half range Fourier series

[half range Fourier series for the function f(x)is given by

$f(x) = \sum_{n=1}^{\infty} b_n * \sin(\frac{n\pi x}{l})$ where $b_n = \frac{2}{l}(\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx)$ and l is the length)]

here coefficient of sin is nb_n and $f(x) = \frac{l}{\pi c}\lambda x(l-x)$

therefore $nb_n = \frac{2}{l} * \frac{l}{\pi c}\lambda \int_0^l x(l-x)\sin\left(\frac{n\pi x}{l}\right)dx$

applying integration by parts

$$nb_n = \frac{2\lambda}{\pi c}\left[\big(x(l-x)\big) * \frac{\frac{-\cos\left(\frac{n\pi x}{l}\right)}{n\pi}}{l} - (l-2x) * \left(-\frac{\sin\left(\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right) + (-2) * \left(\frac{\cos\left(\frac{n\pi x}{l}\right)}{\frac{n^3\pi^3}{l^3}}\right)\right]_{x\to 0 \text{ to } l}$$

$$nb_n = \frac{2\lambda}{\pi c}\left[(-2) * (\frac{l^3}{n^3\pi^3})(\cos(n\pi) - \cos(0))\right] = \frac{4\lambda l^3}{n^3\pi^4 c}[(-1)((-1)^n - 1)]$$

$$nb_{\mathrm{n}} = \frac{4\lambda l^3}{n^3\pi^4 c}[(1-(-1)^n)] \Rightarrow b_n = \frac{4\lambda l^3}{n^4\pi^4 c}[(1-(-1)^n)]$$

therefore the required solution is

$$u(x,t) = \sum_{n=1}^{\infty}\frac{4\lambda l^3}{n^4\pi^4 c}[(1-(-1)^n)]\sin(\frac{n\pi ct}{l}) * sin\left(\frac{n\pi x}{l}\right)$$