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RANDOM PROCESS

JOINT PROBABILITY

TWO DIMENSIONAL RANDOM VARIABLE

Let x and y be 2 random variables defined on the same sample space 'S' then a function (x,y) that assigns a point in two dimension R^2 is called 2 dimensional Random variable

i.e $\boxed{(X,Y) \longrightarrow R^2}$

DISCRETE 2-D RANDOM VARIABLE

A 2 dimensional Random variable is said to be discrete if it when ~~is~~ utmost accountable number of points in 2D.

CONTINUOUS 2-D RANDOM VARIABLE

A 2D random variable is said to be continuous if it takes all the values b/w a certain limits.

JOINT PROBABILITY ~~DISTRIBUTION~~ FUNCTION.

Let x and y be the RV on a sample space S the joint probability function of X and Y is denoted and defined as follows:

$$\boxed{P_{ij} = P(X=x_i, Y=y_j) = P(x_i, y_j)}$$

1] discrete RV

For discrete RV the JPF is usually represented in the form of table as follows:

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X \ Y	y_1	y_2	y_3	------	y_m	Total
x_1	P_{11}	P_{12}	P_{13}	------	P_{1m}	$P_{1\cdot}$
x_2	P_{21}	P_{22}	P_{23}	------	P_{2m}	$P_{2\cdot}$
x_3	P_{31}	P_{32}	P_{33}	------	P_{3m}	$P_{3\cdot}$
⋮	----	----	----	------	⋮	⋮
⋮	----	----	----	------	⋮	⋮
x_n	P_{n1}	P_{n2}	P_{n3}	------	P_{nm}	$P_{n\cdot}$
	$P_{\cdot 1}$	$P_{\cdot 2}$	$P_{\cdot 3}$	------	$P_{\cdot m}$	1

JOINT PROBABILITY MASS FUNCTION

The JPF P_{ij} is said to be joint prob. mass function if all $\boxed{P_{ij} \geq 0}$ and $\boxed{\sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) = 1}$

MARGINAL DISTRIBUTION

Let (X,Y) be a 2-D discrete RV then the marginal distribution of X is denoted by $f(x)$ and it is the probability distribution of X alone, which is determined as follows:

$$\boxed{f(x) = P_X(x_i) = P(X = x_i \cap Y_1) + P(x_i \cap Y_2) + \dots + P(x_i \cap Y_m)}$$

i.e

X	x_1	x_2	x_3	--------	x_n
$f(x)$	$P_{1\cdot}$	$P_{2\cdot}$	$P_{3\cdot}$	--------	$P_{n\cdot}$

llly. marginal distribution of y is the probability distribution of the variable Y alone, which is determined as follows.

$$\boxed{f(y) = P_y(y_i) = P(y = y_i \cap X_1) + P(y_i \cap X_2) + P(y_i \cap X_3) --- P(y_i \cap X_n)}$$

i.e

Y	y_1	y_2	y_3	--------	y_m
$f(y)$	$P_{\cdot 1}$	$P_{\cdot 2}$	$P_{\cdot 3}$	--------	$P_{\cdot m}$

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Q A fair coin is tossed 3 times, let X denote $= 0$, OR $= 1$ accordingly as a head or a tail occurs on the first toss.

Let Y denote the number of heads which occured

→ $X = 0, 1$

$Y = 0, 1, 2, 3.$

Find the marginal distribution of X and Y. Also determine the joint distribution of X and Y.

→ Let $S = \{(HHH), (HHT), (HTH), (THH), (TTH), (THT), (HTT), (TTT)\}$.

Marginal distribution of X

X	0	1
$P(X)$	$4/8 = 1/2$	$4/8 = 1/2$

Marginal distribution of Y

Y	0	1	2	3
$P(Y)$	$1/8$	$3/8$	$3/8$	$1/8$

Joint Probability distribution of X and Y

$X \backslash Y$	$y_1 = 0$	$y_2 = 1$	$y_3 = 2$	$y_3 = 3$	
$x_1 = 0$	0	$1/8$	$2/8 = 1/4$	$1/8$	$= 1/2$
$x_2 = 1$	$1/8$	$2/8 = 1/4$	$1/8$	0	$= 1/2$
					1
	$= 1/8$	$= 3/8$	$= 3/8$	$= 1/8$	1

ii) Continuous RV.

For continuous RV the JPF is usually represented as follows.

JOINT PROBABILITY MASS FUNCTION

Let $P_{ij} = f_{XY}(x, y)$ be JPF for the continuous RV (X, Y).

We say that P_{ij} is probability mass function if $\boxed{P_{ij} \geq 0}$

and $$\boxed{\int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} f_{XY}(x, y)\, dy\, dx = 1}$$

MARGINAL DENSITY FUNCTION

Marginal density function of X is given by

$$\boxed{f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x, y)\, dy}$$

Similarly, marginal density function Y is given by

$$\boxed{f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x, y)\, dx.}$$

MEAN AND VARIANCE

For discrete.

$$\boxed{E(x, y) = \sum_{j=1}^{m} \sum_{i=1}^{n} x_i y_i P_{ij}}$$

For Continuous.

$$\boxed{E(x, y) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} x y f_{XY}(x, y)\, dy\, dx.}$$

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CO-VARIANCE

$$\boxed{Cov(X,Y) = E(xy) - E(x)E(y)}$$

CORRELATION COEFFICIENTS.

denoted by $\gamma(x,y)$ or $\rho(x,y)$

$$\boxed{\gamma(x,y) = \rho(x,y) = \frac{Cov(x,y)}{\sigma_x \cdot \sigma_y}}$$

STOCHASTIC INDEPENDENT

Two random variable with JPMF $f_{xy}(x,y)$ and MDF $f_x(x)$ and $f_y(y)$ respectively are said to be stochastically independent if and only if

$$\boxed{f_{xy}(x,y) = f_x(x) * f_y(y)}$$ (multiplication)

Q2> $f(x,y) = K(2x+y)$.

$0 < x \leq 2, 0 \leq y \leq 3$.

i> find K

ii> Obtain Marginal distribution of x and y

iii> Compute coeff of correlation and interpret your result

iv> $P(x=2, y=1)$

v> $P(x \geq 1, y \leq 2)$ and $P(x+y \leq 1)$

i> J.P distribution.

X \ Y	0	1	2	3
0	0	K	2K	3K
1	2K	3K	4K	5K
2	4K	5K	6K	7K

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Since it is a joint probability distribution $\sum\sum P_{ij} = 1$

$$42K = 1 \Rightarrow \boxed{K = 1/42}$$

ii> Marginal distribution

X	0	1	2
P(X)	$6/42 = \frac{1}{7}$	$14/42$	$22/42$
		$1/3$	$11/21$

← for X

Y	0	1	2	3
P(Y)	$6/42$	$9/42$	$12/42$	$15/42$
	$1/7$	$3/14$	$2/7$	$5/14$.

← for Y

iv> $\cancel{P(X=2, Y=1) = P(2,1) = 6K = 6/42 = 1/7}$. $5K = 5/42$

v> $P(X \geq 1, Y \leq 2) = P(X=1; Y \leq 2) + P(X=2, Y \leq 2)$

$= P(1,0) + P(1,1) + P(1,2) + P(2,0) + P(2,1) + P(2,2)$

$= 2K + 3K + 4K + 4K + 5K + 6K$

$= 24K = 24/42 = 4/7$.

$P(X+Y \leq 1) = P(0,0) + P(0,1) + P(1,0)$

$= 0 + 2K + K = 3K = 3/42 = 1/14$.

iii> $\rho = \dfrac{\mathrm{cov}(x,y)}{\sigma x \cdot \sigma y}$

$\mathrm{Cov}(x,y) = E(xy) - E(x)\cdot E(y)$; $E(x) = P(x)\Sigma x$, $E(y) = \Sigma y P(y)$

$E(xy) = \Sigma\Sigma x_i y_i P_{ij} = 0 + 0 + 0 + 0 + 0 + 3K + 8K + 15K + 0 + 10K$
$+ 24K + 42K$

$= 102K = 102/42 = 17/7$

$\mathrm{Cov}(x,y) = \dfrac{17}{7} - 29/21\,(13/7) = \dfrac{17}{7} - \dfrac{377}{147} = -0.13605$.

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$\sigma x^2 = \Sigma x^2 P(X=x) - \mu x^2 = 0.5215 \Rightarrow \sigma x = 0.7221$

$\sigma y = \Sigma y^2 P(Y=y) - \mu y^2 = 1.3254 \Rightarrow \sigma y = 1.1513$

$$\rho = \frac{-0.13605}{\cancel{0.5215}\ (1.1513)} = -0.1636$$
0.7221

Thus, x and y are negatively correlated.

Q If the joint PF of 2 variables x and y is given by

$$f(x,y) = \begin{cases} Kxy & 0<x<4,\ 1<y<5 \\ 0 & \text{otherwise} \end{cases}$$

i> Find the value of K

ii> $E(x)$, $E(y)$, MDF of x and y, covariance and find $P(x<1, y>3)$, $P(0<x<4)$, $P(y<4)$.

i> Since $f(x,y)$ is JPDF

$$\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_{y=1}^{5}\int_{x=0}^{4} Kxy\,dx\,dy = 1$$

$96K = 1 \Rightarrow K = 1/96$.

ii> Marginal density function →

$$f_X(x) = \int_{y=-\infty}^{\infty} f(x,y)\,dy = \int_{1}^{5} Kxy\,dy = \frac{x}{8}$$

$$f_Y(y) = \int_{x=-\infty}^{\infty} f(x,y)\,dx = \int_{0}^{4} Kxy\,dx = y/12.$$

$$E(x) = \int_{0}^{4} x f_X(x)\,dx = \int_{0}^{4} \frac{x^2}{8}\,dx = \frac{8}{3}$$

$$E(y) = \int_{1}^{5} y f_Y(y)\,dy = \int_{1}^{5} \frac{y^2}{12}\,dy = 31/9$$

Covariance $\rightarrow$ $Cov = E(xy) - E(x) \cdot E(y)$.

$E(xy) = \int_1^5 \int_0^4 xy f_{xy}(x,y)\, dy\, dx = \frac{248}{27}$

$Cov = \frac{248}{27} - \frac{8}{3}\left(\frac{31}{9}\right) = \frac{248}{27} - \frac{248}{27} = 0$

$P(X<1, y>3) \rightarrow \int_0^1 \int_3^5 f(x,y)\, dy\, dx =$

$P(x>0, x<4) \rightarrow \int_0^4 \int_1^5 f(x,y)\, dy\, dx =$

$P(y<4) \rightarrow \int_0^4 \int_1^4 f(x,y)\, dy\, dx =$

LP3> The joint probability distribution of two random variable X, & y ------

y \ x	-2	-1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

$y(x,y) = \frac{Cov(x,y)}{\sigma x \sigma y}$

$Cov(x,y) = E(xy) - E(x)E(y)$

$E(xy) = \sum\sum xy P_{ij}$

$= (1\times 2)0.1 + (-1\times 0.2) + 0 + (5\times 0.3) + (-4\times 0.2) - (2\times 0.1) + 8(0.1) + 10(0)$

$= 0.9$

Marginal distribution of x.

x	-2	-1	4	5
P(x)	0.3	0.3	0.1	0.3

Marginal distribution of Y

Y	1	2
P(y)	0.6	0.4

Mean of X and y

$E(x) = \Sigma x \cdot P(x) = 1$

$E(y) = \Sigma y \cdot P(y) = 1.4$

$conv = E(xy) - E(x)E(y)$

$= 0.9 - 1(1.4) = -0.5$

$\sigma x = \sqrt{\Sigma x^2 p(x) \cdot \mu x^2} \Rightarrow \sigma x^2 = 1.2 + 0.3 = 1.6 + 7.5 - 1 \Rightarrow \sigma x = 3.09$

$\sigma y = \sqrt{\Sigma y^2 p(y) - \mu y^2} \Rightarrow \sigma y^2 = 0.6 + 1.6 - (1.4)^2 = \cancel{0.489} \Rightarrow \sigma y = 0.489$

$\boxed{\rho = -0.329}$

$f_{xy}(-2, 1) = 0.1 \neq P(x)\,P(y)$.

The 2 random variables x and y are not independent.

Q The joint probability distribution of two Random variables X and y is given by

x \ y	2	3	4	
1	0.06	0.15	0.09	P_1
2	0.14	0.35	0.21	P_2

i) Determine marginal distribution of x & y ii) Verify that X and Y are sarcastically independent and also find ρ.

Marginal distribution of X

X	1	2
P(x)	0.3	0.7

Marginal distribution of Y

Y	2	3	4
P(y)	0.20	0.50	0.30

$F_{xy}(1,2) = 0.06 = P_x(1) + P_y(2)$

$F_{xy}(1,3) = 0.15$

$F_{xy}(1,4) = 0.09$

$F_{xy}(2,2) = 0.14$

$F_{xy}(2,3) = 0.35$

$F_{xy}(2,4) = 0.21$

Given 2 are sariastically independent

$\sigma = \frac{Cov(x,y)}{\sigma x \sigma y}$

$Cov = E(x,y) - E(x)E(y)$

$E(xy) = (2\times0.06 + 3\times0.15 + 4\times0.09 + 4\times0.14 + 6\times0.35 + 8\times0.21) = 5.27$

$E(x) = 1.7 \; ; \; E(y) = 3.1$

$Cov = E(xy) - E(x)E(y) = 0$

$\boxed{\rho = 0}$

Note: If two variables are sarcastically independent the correlation coefficient is zero.

Q Suppose that two dimensional continuous random variable has joint probability mass functions as follows

$$F(x,y) = \begin{cases} 6x^2y & ; 0<x<1 \text{ and } 0<y<1 \\ 0 & ; \cancel{0<y<1} \text{ otherwise} \end{cases}$$

i) Verify that $\int_0^1\int_0^1 F(x,y)\,dx\,dy = 1$ ii) Find Marginal density function of x and Y and verify that X and Y are independent

$$\int_0^1\int_0^1 6x^2y\,dx\,dy = 6\cdot \frac{x^3}{3}\Big|_0^1 \cdot \frac{y^2}{2}\Big|_0^1 = 6(1/3)(1/2) = 1$$

$$F_X(x) = \int_{-\infty}^{\infty} F(x,y)\,dy = \int_0^1 6x^2y\,dy = 6x^2\cdot\frac{y^2}{2}\Big|_0^1 = 3x^2$$

$$F_Y(y) = \int_{-\infty}^{\infty} F(x,y)\,dx = \int_0^1 6x^2y\,dx = 6y\frac{x^3}{3}\Big|_0^1 = 2y$$

$$F(x,y) = 6x^2y = 3x^2\cdot 2y = F_X(x)\cdot F_Y(y)$$

They are sarcastically independent

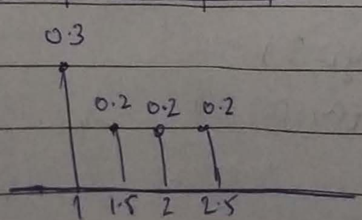
∴ Given Random Variables are SI.

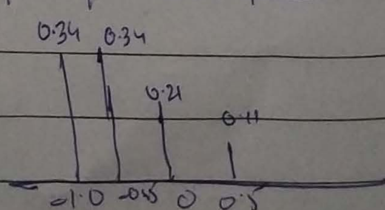
Q4)

y \ x	1.0	1.5	2.0	2.5
-1.0	0.15	0.08	0.06	0.05
-0.5	0.1	0.13	0.06	0.05
0.0	0.04	0.07	0.05	0.05
0.5	0.01	0.02	0.03	0.05

X	1.0	1.5	2.0	2.5
F(x)	0.30	0.20	0.20	0.20

Y	-1.0	-0.5	0	0.5
F(y)	0.34	0.34	0.21	0.11





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Q5) The joint probability density function of random ----

$f_{xy}(x,y) = \frac{1}{4} e^{-|x|-|y|} \quad -8<x<\infty, \ -\infty<y<\infty$

$\int_{-\infty}^{\infty}\int_{\infty}^{\infty} \frac{1}{4} e^{-|x|-|y|} \, dx\,dy = \frac{1}{4} e^{-|x|}$

$F_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-\infty}^{\infty} \frac{1}{4} e^{-|x|-|y|} dy$

$= \frac{1}{4} e^{-|x|} \ e^{-|y|}\Big|_{-\infty}^{\infty} = \frac{1}{4} e^{-|x|} \int_{-\infty}^{0} e^{+y} + \int_{0}^{\infty} e^{-y} dy$

$= \frac{1}{4} e^{-|x|} (1-0)+(0+1)$

$= \frac{1}{4} e^{-|x|} \times 2 = \frac{1}{2} e^{-|x|}$

$F_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{-\infty}^{\infty} \frac{1}{4} e^{-|x|-|y|} dx = \frac{1}{2} e^{-|y|}$

The given variables X and Y are sarcastically independent

$P(X \le 0, Y \le 0)$

$= \int_{-\infty}^{0}\int_{-\infty}^{0} f(x,y)\,dy\,dx$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{0} e^{x} e^{y} \, dy\,dx = \frac{1}{4}$ //