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Laplace Transform

Laplace Transform - It is the mathematical tool for converting well behaved time domain function into frequency domain expressions.

- It is a powerfull technique and it replaces operations of calculus by operations of algebra.

Def^n:- Laplace Transforms of a real value function $f(t)$ where $t \geq 0$ is denoted by $F(s)$ and is defined by an improper integral as

$$L\{f(t)\} = \int_{0^-}^{\infty} f(t) \cdot e^{-st} dt$$

or

$$= \lim_{b \to \infty} \int_{0}^{b} f(t) \cdot e^{-st} dt$$

provided it exists.

NOTE:-

1) Region in s-plane in which Laplace transform is finite or convergent is called region of convergence. where s is parameter either real or complex ($s = \sigma + j\omega$)

Sufficient condition for the existence of Laplace Transform:-

Laplace Transform of $f(t)$ converges

$\int e^{ax}\sin bx\,dx = \frac{e}{a^2+b^2}(a\sin bx + \dots$

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$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

when the following conditions are satisfied

i) $f(t)$ is a piecewise continuous funⁿ.

ii) $f(t)$ is exponential order

($i.e \lim_{t\to\infty} e^{-st} f(t) = \text{finite}$).

iii)

Examples:-

L.P I

i) $f(t) = \sin t, \quad 0 < t < \pi$

$\quad = 0 \quad t > \pi$

Sol:- By defⁿ:- $L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$

$$= \int_0^\pi e^{-st}\sin t\,dt + \int_\pi^\infty e^{-st}(0)\,dt$$

$$= \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_0^\pi$$

$$= \frac{e^{-s\pi}}{s^2+1}(-s\sin\pi - \cos\pi) - \frac{e^0}{s^2+1}(-s\sin 0 - \cos 0)$$

$$= \frac{e^{-s\pi}}{s^2+1}(1) - \frac{1}{s^2+1}(-1)$$

$$= \frac{e^{-s\pi}}{s^2+1} + \frac{1}{s^2+1}$$

$$= \frac{1}{s^2+1}(e^{-s\pi}+1).$$

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4$

(where either u or v should be polynomial, then consider u as polynomial. $v_1 = \int v$ $v_2 = \int v_1$)

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iii) $f(t) = t^2 \qquad 0 < t < 2$
$= t-1 \qquad 2 < t < 3$
$= 7 \qquad t > 3$

Sol:- By defn:-

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^2 f(t)e^{-st}\,dt + \int_2^3 f(t).e^{-st}\,dt + \int_3^\infty f(t)e^{-st}\,dt$$

$$= \int_0^2 t^2.e^{-st}\,dt + \int_2^3 (t-1)e^{-st}\,dt + \int_3^\infty 7e^{-st}\,dt$$

$$= \left[t^2\left(\frac{e^{-st}}{-s}\right) - 2t\left(\frac{e^{-st}}{s^2}\right) + 2\left(\frac{e^{-st}}{-s^3}\right)\right]_0^2$$

$$+ \left[(t-1)\frac{e^{-st}}{-s} - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_2^3 + \left[\frac{7e^{-st}}{(-s)}\right]_3^\infty$$

$$= \left[\frac{4e^{-s(2)}}{-s} - 2(2)\left(\frac{e^{-2s}}{s^2}\right) + \frac{2e^{-2s}}{-s^3} + \frac{2}{s^3}\right]$$

$$+ \left[\frac{2e^{-3s}}{-s} - \frac{e^{-3s}}{s^2} - \frac{e^{-2s}}{-s} + \frac{e^{-2s}}{s^2}\right]$$

$$+ \frac{7e^{-3s}}{-s} - \frac{7e^{-\infty}}{-s}$$

$$= \frac{4e^{-2s}}{-s} - \frac{4e^{-2s}}{s^2} - \frac{2e^{-2s}}{s^3} + \frac{2}{s^3} + \frac{2e^{-3s}}{-s} - \frac{e^{-3s}}{s^2}$$

$$+ \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s^2} + \frac{7e^{-3s}}{s} + 0$$

$$= \frac{-3e^{-2s}}{s} - \frac{3e^{-2s}}{s^2} - \frac{2e^{-2s}}{s^3} + \frac{5e^{-3s}}{s} + \frac{2}{+s^3} - \frac{e^{-3s}}{s^2}$$

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Properties of Laplace Transform:-

1) Linearity property :-

If

$$x(t) \overset{LT}{\rightleftharpoons} X(s)$$

$$y(t) \overset{LT}{\rightleftharpoons} Y(s)$$

Then $a\,x(t) + b\,y(t) \overset{LT}{\rightleftharpoons} a\,X(s) + b\,Y(s)$

2) Time scaling property -

If

$$x(t) \overset{LT}{\rightleftharpoons} X(s)$$

then

$$x(at) \overset{LT}{\rightleftharpoons} \frac{1}{a} X\left(\frac{s}{a}\right)$$

3) Frequency shifting property -

If

$$x(t) \overset{LT}{\rightleftharpoons} X(s)$$

then

$$e^{s_0 t} x(t) \overset{LT}{\rightleftharpoons} X(s - s_0)$$

and

$$e^{-s_0 t} x(t) \overset{LT}{\rightleftharpoons} X(s + s_0)$$

PROOF :-

$$L\{x(t)\} = \int_0^\infty e^{-st} f(t) \, dt$$

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ii)

Sol:

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ii) $f(t) = \frac{t}{\pi} \qquad 0 < t < \pi$

$= 1 \qquad t > \pi$

Sol:- By defⁿ:- $L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$

$$L\{f(t)\} = \int_0^\pi \frac{t}{\pi} e^{-st}\,dt + \int_\pi^\infty e^{-st}dt$$

$$= \frac{1}{\pi}\left[\frac{te^{-st}}{-s} - (1)\frac{e^{-st}}{s^2}\right]_0^\pi + \left(\frac{e^{-st}}{-s}\right)_\pi^\infty$$

$$= \frac{1}{\pi}\left[\frac{\pi e^{-s\pi}}{-s} - \frac{e^{-s\pi}}{s^2} - 0 + \frac{e^0}{s^2}\right] + \frac{\pi^{-s\pi}}{s}$$

$$= \frac{-e^{-s\pi}}{s} - \frac{e^{-s\pi}}{s^2\pi} + \frac{1}{\pi s^2} - \frac{\pi^{-s\pi}}{s}$$

PROOF OF PROPERTIES

1) Linearity property :-

PROOF :-

given $f(t) = ax(t) + by(t)$

$$L\{x(t)\} = \int_0^\infty e^{-st}\left(ax(t) + by(t)\right)dt$$

$$= \int_0^\infty e^{-st}ax(t) + e^{-st}by(t)\,dt$$

$$= \int_0^\infty e^{-st}ax(t)dt + \int_0^\infty e^{-st}by(t)\,dt$$

$$= a\int_0^\infty e^{-st}x(t)dt + b\int_0^\infty e^{-st}y(t)\,dt$$

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$$= aX(s) + bY(s)$$

2) Time scaling property

PROOF

Put $ta = u$ $adt = du$

$$L\{x(at)\} = \int_0^\infty e^{-st} x(at)\, dt$$

$$= \int_0^\infty e^{-s\frac{u}{a}} x(u) \frac{du}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-s\frac{u}{a}} x(u)\, du$$

$$= \frac{1}{a} X\left(\frac{s}{a}\right)$$ (where s and a are constants)

Properties

4) Differentiation in freq - (If we multiply ~~po~~ by powers of t to input signal then output results interms of the differentiation)

If $x(t) \overset{LT}{\rightleftharpoons} X(s)$

then

$$t^n x(t) \overset{LT}{\rightleftharpoons} (-1)^n \frac{d^n}{ds^n} X(s)$$

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5) Integration in freq - (It is obtained by division of t to input signal)

If $x(t) \overset{LT}{\rightleftharpoons} X(s)$

then $\frac{x(t)}{t^n} \rightleftharpoons \int_s^{\infty} X(s)\, ds$

6) Differentiation in time - (It gives formula for derivative of i/p function)

If $x(t) \overset{LT}{\rightleftharpoons} X(s)$
then

$$x'(t) \rightleftharpoons sX(s) - x(0)$$

$$x''(t) \rightleftharpoons s^2X(s) - sx(0) - x'(0)$$

7) Integration in time - (If $x(t) \rightleftharpoons X(s)$
then

$$\int_0^t x(t)\, dt \rightleftharpoons \frac{X(s)}{s}$$

Laplace transform of standard funn:-

1) $L\{1\} = \frac{1}{s}$ $s > 0$

or $1 \overset{LT}{\rightleftharpoons} 1/s$

By defn $L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$

$$L(1) = \int_0^{\infty} e^{-st} (1)\, dt.$$

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$$= \left. \frac{e^{-st}}{-s} \right]_0^\infty$$

$$= \frac{e^{-\infty}}{-s} - \frac{e^{-s(0)}}{-s}$$

$$= 0 + \frac{1}{s}$$

$$\boxed{L(1) = \frac{1}{s}}$$

ii) $L(e^{+at}) = \int_0^\infty e^{-st} e^{+at} dt$

$$= \int_0^\infty e^{-t(s-a)} dt$$

$$= \left. \frac{e^{-t(s-a)}}{s-a} \right]_0^\infty$$

$$\boxed{L(e^{+at}) = \frac{1}{s-a}} \quad s > a$$

iii) $L\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$

WKT

$e^{j\omega x} = \cos\omega x + i\sin\omega x$

$e^{-j\omega x} = \cos\omega x - i\sin\omega x$

Adding

$\frac{e^{j\omega x} + e^{-j\omega x}}{2} = \cos\omega x$

$$L\{\cos\omega t\} = L\left\{\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right\}$$

$$= \frac{1}{2}\left[L\{e^{j\omega t}\} + L\{e^{-j\omega t}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2-(j\omega)^2}\right]$$

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$$L\{\cos(\omega t)\} = \frac{s}{s^2 - \omega^2(-1)}$$

$$\boxed{L\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}}$$

iv) $L\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$

WKT $\sin\omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$

$$L\{\sin(\omega t)\} = L\left\{\frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)\right\}$$

$$= \frac{1}{2j}\left[L\{e^{j\omega t}\} - L\{e^{-j\omega t}\}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right]$$

$$= \frac{1}{2j}\frac{2j\omega}{s^2-(-1)(\omega)^2}$$

$$\boxed{L\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}}$$

(v) $L\{\sinh\omega t\} =$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$$L\{\sin h(\omega t)\} = L\left\{\frac{1}{2}\left(e^{\omega t} - e^{-\omega t}\right)\right\}$$

$$= \frac{1}{2}\left\{L(e^{\omega t}) - L(e^{-\omega t})\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s-\omega} - \frac{1}{s+\omega}\right]$$

$$= \frac{1}{2}\left[\frac{2\omega}{s^2-\omega^2}\right]$$

$$\boxed{L\{\sinh\omega t\} = \frac{\omega}{s^2-\omega^2}}$$

WKT $\Gamma n = \int_0^\infty e^{-t} t^{n-1}\, dt$

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$$\boxed{L\{\cosh \omega t\} = \frac{s}{s^2-\omega^2}}$$

vi) $L\{t^n\} = \frac{\Gamma n+1}{s^{n+1}}$

By def^n $L\{t^n\} = \int_0^\infty e^{-st} t^n\, dt$

Put $st = u$ $\quad u \to 0 \Rightarrow t \to 0$

$s\,dt = du$ $\quad u \to \infty \Rightarrow t \to \infty$

$$L\{t^n\} = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n\, du$$

$$\boxed{L\{t^n\} = \frac{1}{s^{n+1}} \Gamma n+1}$$

Solve :-

L.P II

i) $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$

Sol:- $L\{e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t\}$

$$\frac{1}{s-2} + 4\frac{\Gamma 3+1}{s^4} - 2\left(\frac{3}{s^2+9}\right) + 3\left(\frac{s}{s^2+9}\right)$$

$$\frac{1}{s-2} + \frac{4\Gamma 4}{s^4} - \frac{6}{s^2+9} + \frac{3s}{s^2+9}$$

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→ $L\{2^t\}$.

Sol:- $L\{e^{\log 2^t}\}$

$L\{e^{t\log 2}\}$

$\frac{1}{s-\log 2}$.

Trigonometric properties

1) $\sin 2\theta = 2\sin\theta \cdot \cos\theta$

2) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 2\cos^2\theta - 1$

$= 1 - 2\sin^2\theta$

3) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

4) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

5) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

6) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$.

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L.P II

2. $1 + 2\sqrt{t} + \frac{3}{\sqrt{t}}$

Sol:- $L\{x(t)\} = L\left\{1 + 2\sqrt{t} + \frac{3}{\sqrt{t}}\right\}$

$= L\{1\} + L\{2\sqrt{t}\} + L\left\{\frac{3}{\sqrt{t}}\right\}$

$= \cancel{\frac{\Gamma 1+1}{s^{1+1}}}\ \frac{1}{s} + \frac{2\,\Gamma\frac{1}{2}+1}{s^{\frac{1}{2}+1}} + \frac{3\,\Gamma -\frac{1}{2}+1}{s^{-\frac{1}{2}+1}}$

$= \frac{1}{s} + \frac{2\,\Gamma \frac{3}{2}}{s^{3/2}} + \frac{3\,\Gamma\frac{1}{2}}{s^{1/2}}$

$= \frac{1}{s} + \frac{2\cdot\frac{1}{2}\,\Gamma\frac{1}{2}}{s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}}$

$= \frac{1}{s} + \frac{\sqrt{\pi}}{s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}}$

3) $\left(t + \frac{1}{\sqrt{t}}\right)^3$

Sol:- ~~Lbeck~~ $\left(t + \frac{1}{\sqrt{t}}\right)^3 = t^3 + (t)^{-3/2} + 3t^2\frac{1}{\sqrt{t}} + 3t\left(\frac{1}{t}\right)$

$= t^3 + (t)^{-3/2} + 3t^{3/2} + 3$

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for positive fractions $\Gamma n = (n-1)\,\Gamma(n-1)$

$$L\{x(t)\} = L\left\{ t^3 + (t)^{-3/2} + 3t^{3/2} + 3 \right\}$$

$$= \frac{\Gamma(3+1)}{s^4} + \frac{\Gamma(-3/2+1)}{s^{-1/2}} + \frac{3\,\Gamma(3/2+1)}{s^{5/2}} + \frac{3}{s}$$

$$= \frac{6}{s^4} + \frac{\Gamma(-1/2)}{s^{-1/2}} + \frac{3\,\Gamma(5/2)}{s^{5/2}} + \frac{3}{s}$$

$$= \frac{6}{s^4} + \frac{\Gamma(-1/2)}{s^{-1/2}} + \frac{3 \times 3/2 \times 1/2 \times \sqrt{\pi}}{s^{5/2}} + \frac{3}{s}$$

$$= \frac{6}{s^4} + \frac{9\sqrt{\pi}}{4s^{5/2}} + \frac{3}{s}$$

For -ve fraction, find Γn value we use reduction formula $\boxed{\Gamma n = \frac{\Gamma(n+1)}{n}}$

$$L\{x(t)\} = \frac{6}{s^4} + \frac{\Gamma(-1/2+1)}{-1/2\, s^{-1/2}} + \frac{9/4\sqrt{\pi}}{s^{5/2}} + \frac{3}{s}$$

$$= \frac{6}{s^4} - \frac{2\sqrt{\pi}}{s^{-1/2}} + \frac{9/4\sqrt{\pi}}{s^{5/2}} + \frac{3}{s}$$

vi) $\sin 2t \sin 3t$

sol. [illegible]

$$\sin 2t \sin 3t = \frac{1}{2}\left[\cos(3t-2t) - \cos(2t+3t)\right]$$

$$= \frac{1}{2}\left[\cos(t) - \cos(5t)\right]$$

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$$\mathcal{L}\{x(t)\} = \frac{1}{2}\mathcal{L}\{\cos t - \cos 5t\}$$

$$= \frac{1}{2}\mathcal{L}\{\cos t\} - \mathcal{L}\{\cos 5t\}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+\omega^2} - \frac{s}{s^2+25}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right]$$

L.P $\overline{III}$

i) $e^{-3t}(2\cos 5t \sin 5t)$.

Sol:-

$$\mathcal{L}\{e^{-3t}(2\cos 5t \sin 5t) = \mathcal{L}\{e^{-3t}x(t)\}$$

Since $\mathcal{L}\{e^{-3t}x(t)\} = X(s-s_0)$

where

$$X(s) \overset{LT}{\rightleftharpoons} x(t).$$

$$\mathcal{L}\{x(t)\} = \mathcal{L}\{2\cos 5t . \sin 5t\}$$

$$= \mathcal{L}\{\sin 10t\}$$

$$= \frac{\cancel{s}10}{s^2+100}.$$

$$\therefore \mathcal{L}\{e^{-3t}x(t)\} = X(s-s_0)$$ where $s_0 = -3$.

$$= \frac{10}{(s-s_0)^2+100}$$

$$= \frac{10}{(s+3)^2+100}$$

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iii) $\sqrt{t}e^{3t}$

sol:- $L\{\sqrt{t}e^{3t}\} = L\{e^{3t}x(t)\}$

$x(t) = \sqrt{t}$

$L\{x(t)\} = L\{\sqrt{t}\}$

$= \frac{\Gamma(1/2+1)}{s^{3/2}}$

$= \frac{1/2\sqrt{\pi}}{s^{3/2}}$

By shifting property

$L\{e^{3t}x(t)\} = X(s-s_0)$ where

$= \frac{1\sqrt{\pi}}{2(s-3)^{3/2}}$

v) t^3e^{-3t}

sol:- By differentiation in frequency property :-

$L\{t^3e^{-3t}\} = (-1)^3\frac{d^3}{ds^3}L\{e^{-3t}\}$

$L\{e^{-3t}\} = \frac{1}{s+3}$

$L\{t^3e^{-3t}\} = -1\frac{d^3}{ds^3}\frac{1}{s+3}$

$= -1\frac{d^2}{ds^2}\frac{(-1)}{(s+3)^2}$

$= -1\frac{d}{ds}\frac{(+2)}{(s+3)^3}$

$= \frac{(-1)(-6)}{(s+3)^4} = \frac{6}{(s+3)^4}$

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By shifting property

$x(t) = t^3$

$X(s) = \frac{\Gamma 3+1}{s^4} = \frac{3!}{s^4} = \frac{6}{s^4}$

$L\{t^3 e^{-3t}\} = X(s-s_0)$

$= \frac{6}{(s+3)^4}$.

ii) $e^{2t}\cos^2 t$

Sol:- $x(t) = \cos^2 t$

$= \frac{1+\cos 2t}{2}$

$L\{x(t)\} = L\left\{\frac{1}{2} + \frac{\cos 2t}{2}\right\}$

$= \frac{1}{2s} + \frac{1}{2} L\{\cos 2t\}$

$= \frac{1}{2s} + \frac{1}{2}\frac{s}{(s^2+4)}$

$= \frac{1}{2s} + \frac{s}{2s^2+8}$

$L\{e^{2t}\cos^2 t\} = X(s-s_0)$

$= \frac{1}{2(s-s_0)} + \frac{s-s_0}{2(s-s_0)^2+8}$ $\quad s_0 = 2$

$= \frac{1}{2(s-2)} + \frac{s-2}{2(s-2)^2+8}$

iv) $t\sin^2 3t$

Sol:-

By differentiation in frequency property

$$L\{t\sin^2 3t\} = (-1)^1 \frac{d}{ds} L\{\sin^2 3t\}$$

$$\sin^2 3t = \frac{1-\cos 6t}{2}$$

$$L\{\sin^2 3t\} = L\left\{\frac{1}{2} - \frac{\cos 6t}{2}\right\}$$

$$= \frac{1}{2s} - \frac{1}{2}\left(\frac{s}{s^2+36}\right)$$

$$= \frac{1}{2s} - \frac{b}{2(s^2+36)}$$

$$L\{t\sin^2 3t\} = \frac{-1}{2} \frac{d}{ds}\left\{\frac{1}{s} - \frac{s}{s^2+36}\right\}.$$

$$= \frac{-1}{2}\left(\frac{-1}{s^2} - \frac{(s^2+36) - (s)(2s)}{(s^2+36)^2}\right.$$

$$= \frac{1}{2}\left(\frac{1}{s^2} + \frac{s^2+36-2s^2}{(s^2+36)^2}\right).$$

L.P II

iv) $\cos^2 2t$

Sol:- $x(t) = \cos^2 2t$

$$= \frac{1+\cos 4t}{2}$$

$$L\{x(t)\} = \frac{1}{2} L\{1+\cos 4t\}$$

$$= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right)$$

v) $\sin^3 2t$

Sol:- $x(t) = \sin^3 2t$

$$= \frac{-\sin 6t + 3\sin 2t}{4}$$

$$x(t) = \frac{3\sin 2t - \sin 6t}{4}$$

$$L\{x(t)\} = L\left\{\frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t\right\}$$

$$= \frac{3}{4}\left(\frac{2}{s^2+4}\right) - \frac{1}{4}\left(\frac{6}{s^2+36}\right)$$

Find

i) $f(t) =$

Sol:-

$x(t)$

$x(t$

$L\{x$

By

$L\{$

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Find the Laplace transform of :-

i) $f(t) = t\sqrt{1-\sin t}$

Sol:

$$x(t) = \sqrt{1-\sin t}$$

$$= \sqrt{\cos^2 t/2 + \sin^2 t/2 - 2\sin t/2 \cdot \cos t/2}$$

$$= \sqrt{(-\cos t/2 + \sin t/2)^2}$$

$$x(t) = \sin t/2 - \cos t/2$$

$$L\{x(t)\} = L\{\sin t/2 - \cos t/2\}$$

$$= \frac{1/2}{s^2+1/4} - \frac{s}{s^2+1/4}$$

By differentiation in frequency property

$$L\{f(t)\} = (-1)^1 \frac{d}{ds} L\{x(t)\}$$

$$= (-1)\frac{d}{ds}\left[\frac{2-4s}{4s^2+1}\right]$$

$$= (-1)\left[\frac{(4s^2+1)(-4) - (-4s+2)(8s)}{(4s^2+1)^2}\right]$$

$$= \frac{16s^2+4-32s^2+16s}{(4s^2+1)^2}$$

$$L\{f(t)\} = \frac{-16s^2+16s+4}{(4s^2+1)^2}$$

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ii) $te^{-t}\sin 2t$

sol:- $f(t) = e^{-t}(t\sin 2t)$

Let $x(t) = t\sin 2t$

$f(t) = e^{-t}x(t)$

By shifting property

$$L\{f(t)\} = X(s-s_0)$$

$$X(s) = L\{x(t)\} = L\{t\sin 2t\}$$

$$= (-1)\frac{d}{ds}L\{\sin 2t\}$$

$$= (-1)\frac{d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$= (+1)\,2(s^2+4)^{-2}(2s)$$

$$L\{x(t)\} = \frac{4s}{(s^2+4)^2}$$

$$\therefore L\{f(t)\} = X(s-s_0)$$

$$= \frac{4s}{(s^2+4)^2}$$

$$= \frac{4(s-s_0)}{((s-s_0)^2+4)^2} \qquad s_0 = (-1)$$

$$= \frac{4(s+1)}{((s+1)^2+4)^2}$$

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vi) $te^{-2t}\cos 2t$

sol:- $f(t) = e^{-2t}(t\cos 2t)$

$x(t) = t\cos 2t$

$L\{x(t)\} = L\{t\cos 2t\}$

$= (-1)\frac{d}{ds} L\{\cos 2t\}$

$L\{\cos 2t\} = \frac{s}{s^2+4}$

$L\{x(t)\} = (-1)\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$

$= (-1)\left[\frac{(s^2+4) - s(2s)}{(s^2+4)^2}\right]$

$= \frac{-s^2-4+2s^2}{(s^2+4)^2}$

$= \frac{s^2-4}{(s^2+4)^2}$

By frequency shifting property

$L\{f(t)\} = L\left\{e^{-2t}(t\cos 2t)\right\}$

$= X(s)\Big|_{s \to s_0}$

$= \frac{s^2-4}{(s^2+4)^2}$

$= \frac{(s+2)^2-4}{((s+2)^2+4)^2}$

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L.P II

vii) $\frac{\cos at - \cos bt}{t}$

Sol: $f(t) = \frac{1}{t}[\cos at - \cos bt]$

By integration in frequency

$$L\left\{\frac{x(t)}{t}\right\} = \int_s^\infty X(s)\,ds \quad \text{—— ①}$$

$$L\{x(t)\} = L\{\cos at - \cos bt\}$$

$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

Substituting in eq^n 1.

$$L\{f(t)\} = \int_s^\infty \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\,ds$$

$$= \frac{1}{2}\int_s^\infty \frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\,ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{1+a^2/s^2}{1+b^2/s^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log_e 1 - \log \frac{1+a^2/s^2}{1+b^2/s^2}\right]$$

$$L\{f(t)\} = -\frac{1}{2}\log \frac{s^2+a^2}{s^2+b^2}$$

viii) $f(t) = \frac{e^{-t}\sin t}{t}$

Sol:- $f(t) = e^{-t}\left[\frac{\sin t}{t}\right]$ $\qquad x(t) = \frac{\sin t}{t}$

By frequency shifting method.

$$L\{f(t)\} = X(s)_{s \to s_0}$$

$$L\{x(t)\} = L\left\{\frac{\sin t}{t}\right\} \qquad \text{Integration}$$

$$= \int_s^{\infty} L\{\sin t\} ds$$

$$= \int_s^{\infty} \frac{1}{s^2+1}\, ds$$

$$= \cancel{+(s^2+1)^{-2}}\Big]_s^{\infty} \qquad \tan^{-1} s\Big]_s^{\infty}$$

$$= \frac{-1}{(s^2+1)^2}\Big]_s^{\infty} \qquad \frac{\pi}{2} - \tan^{-1} s$$

$$\cancel{L\{s(t)\}} = \frac{+}{\cancel{(s^2+1)}^2} \qquad L\{x(t)\} = \frac{\pi}{2} - \tan^{-1} s$$

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$$\cancel{L\{f(t)\}} = \cancel{\frac{1}{((s+1)^2+1)^2}}$$

$$L\{f(t)\} = \frac{\pi}{2} - \tan^{-1}(s+1)$$

$$= \cot^{-1}(s+1)$$

1) If $L\{\sin t\} = \frac{1}{s^2+1}$ Find $L\{\sin 4t\}$.

Sol: Given

$$L\{\sin t\} = \frac{1}{s^2+1}$$

By time scaling property

$$L\{x(at)\} = \frac{1}{a} X\left(\frac{s}{a}\right)$$

$$L\{\sin 4t\} = \frac{1}{4}\left(\frac{1}{(s/4)^2+1}\right)$$

$$= \frac{1}{4} \frac{16}{s^2+4}$$

$$= \frac{4}{s^2+4}$$

2) If $L\{f(t)\} = \frac{1}{s(s^2+1)}$ Find $L\{e^{-t} f(3t)\}$.

Sol:-

$$L\{e^{-t} f(3t)\} = F(s)\Big|_{s \to s_0}$$

$$F(s) = L\{f(3t)\}$$

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By time scaling property

$$L\{f(3t)\} = \frac{1}{3} \frac{1}{s/3\left(\left(\frac{s}{3}\right)^2+1\right)}$$

$$= \frac{1}{3} \times \frac{3}{s}\left(\frac{9}{s^2+9}\right)$$

$$= \frac{9}{s(s^2+9)}.$$

$$L\{e^{-t}f(3t\} = \frac{9}{(s+1)\left((s+1)^2+9\right)}$$

L.P6

iii) $L\left\{\int_0^t \frac{e^{-t}\sin t}{t}\, dt\right\}$

Sol: It is of form

$$L\left\{\int_0^t x(t)\, dt = \frac{X(s)}{s}\right.$$ (By integration in time)

$$X(s) = L\{x(t)\}$$

Let $x(t) = \frac{e^{-t}\sin t}{t}$

$$L\{x(t)\} = L\left\{\frac{e^{-t}\sin t}{t}\right\}$$

$$= H(s)_{s \to s-s_0}$$

$$L\{h(s)\} = L\left\{\frac{\sin t}{t}\right\}$$

$$= \int_s^\infty L\{\sin t\}\, ds$$

$$= \int_s^\infty \frac{1}{s^2+1}\, ds = \left.\tan^{-1}s\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}s$$

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$$L\{x(t)\} = \frac{\pi}{2} - \tan^{-1}s \Big|_{s \to s-s_0}$$

$$= \frac{\pi}{2} - \tan^{-1}(s-s_0)$$

$$L\left\{\int_0^t e^{-t}\frac{\sin t}{t}\,dt\right\} = \frac{\pi/2 - \tan^{-1}(s-s_0)}{s}$$

$$= \frac{\cot^{-1}(s-s_0)}{s}$$

$$= \frac{\cot^{-1}(s+1)}{s} \qquad s_0 = -1$$

iv) $L\left\{\int_0^t e^t \sin^2 t\,dt\right\}$

Sol:- $x(t) = e^t \sin^2 t$.

By integration in time property

$$L\left\{\int_0^t e^t \sin^2 t\,dt\right\} = \frac{X(s)}{s}$$

$$L\{x(t)\} = L\{e^t \sin^2 t\}$$

$$= L\left\{e^t\left(\frac{1-\cos 2t}{2}\right)\right\}$$

Let $h(t) = \frac{1-\cos 2t}{2}$

$$L\{x(t)\} = H(s)_{s \to s-s_0}$$

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$$L\{h(t)\} = L\left\{\frac{1-\cos 2t}{2}\right\}$$

$$= \frac{1}{2s} - \frac{1}{2}\left(\frac{s}{s^2+4}\right).$$

$$L\{x(t)\} = \frac{1}{2(s-1)} - \frac{1}{2}\left(\frac{s-1}{(s-1)^2+4}\right).$$

$$L\left\{\int_0^t e^t \sin^2 t\, dt\right\} = \frac{\frac{1}{2}\left(\frac{1}{s-1} - \frac{(s-1)}{(s-1)^2+4}\right)}{s}.$$

XII)

P 6

1) $L\left\{\int_0^t e^t \sin t\, dt\right\}$

Sol:- By integration in time property

$$L\left\{\int_0^t e^t \sin t\, dt\right\} = \frac{X(s)}{s}$$

$$x(t) = e^t \sin t$$

$$L\{x(t)\} = L\{\sin t\}\Big|_{s \to s-s_0}$$

$$= \frac{1}{s^2+1}\Big|_{s \to s-1}$$

$$= \frac{1}{(s-1)^2+1}.$$

$$L\left\{\int_0^t e^t \sin t\, dt\right\} = \frac{\frac{1}{(s-1)^2+1}}{s}.$$

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→ time scaling, whenever input t is changing from t...
→ otherwise use differentiation in time papergrid

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L.P V

i) $L(t\sin\omega t) = \frac{2\omega s}{(s^2+\omega^2)^2}$ find $L\{\omega t\cos\omega t + \sin\omega t\}$

Sol:- $x(t) = t\sin\omega t$

$x'(t) = \omega t\cos\omega t + \sin\omega t.$

$L\{x(t)\} = \frac{2\omega s}{(s^2+\omega^2)^2} = X(s)$

$x(0) = 0.$

By differention of time property

$L\{x'(t)\} = sX(s) - x(0)$

$= s\left(\frac{2\omega s}{(s^2+\omega^2)^2}\right) - 0$

$L\{x'(t)\} = \frac{2\omega s^2}{(s^2+\omega^2)^2}$

$L\{\omega t\cos\omega t + \sin\omega t\} = \frac{2\omega s^2}{(s^2+\omega^2)^2}$

iii) If $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4\pi}$. Prove that

$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \left(\sqrt{\frac{\pi}{s}}\right) e^{-1/4\pi}$

Sol: Let $x(t) = \sin\sqrt{t}$

$x'(t) = \cos\sqrt{t} \times \frac{1}{2\sqrt{t}}.$

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$x(0) = 0.$

By differentiation in time

$$L\{x'(t)\} = sX(s) - x(0)$$

$$= s\,L\{\sin\sqrt{t}\} - 0$$

$$= s\left(\frac{\sqrt{\pi}}{2s^{3/2}}\,e^{-1/4\pi}\right)$$

$$L\left\{\frac{1}{2}\,\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = s\left(\frac{\sqrt{\pi}}{2s^{3/2}}\,e^{-1/4\pi}\right)$$

$$\frac{1}{2}\,L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{1}{2}\,s\left(\frac{\sqrt{\pi}}{2s^{3/2}}\,e^{-1/4\pi}\right)$$

$$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\,e^{-1/4\pi}.$$

L.P IV. Evaluation of integrals using Laplace transform

i) $\int_0^{\infty} te^{-3t}\sin t\,dt.$

sol: We have $\int_0^{\infty} e^{-st} f(t)\,dt = L\{f(t)\}.$

Put $s = 3$, $f(t) = t\sin t$

$$\therefore \int_0^{\infty} te^{-3t}\sin t\,dt = L\{t\sin t\}\Big|_{s=3}$$

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$$= (-1)\frac{d}{ds} L\{\sin t\}\Big|_{s=3}$$

$$= (-1)\frac{d}{ds}\frac{1}{s^2+1}\Big|_{s=3}$$

$$= (-1)\left(\frac{-2s}{(s^2+1)^2}\right)\Big|_{s=3}$$

$$= \frac{2(3)}{(9+1)^2}$$

$$= \frac{6}{100} = \frac{3}{50}.$$

ii) $\int_0^\infty e^{-t}\frac{\sin^2 t}{t}\,dt$

Sol:- $s = 1$ $\qquad x(t) = \frac{\sin^2 t}{t}$

$$\int_0^\infty e^{-t}\frac{\sin^2 t}{t} = L\left\{\frac{\sin^2 t}{t}\right\}\Big|_{s=1} \quad \text{--- } (1)$$

$$L\left\{\frac{\sin^2 t}{t}\right\} = \int_s^\infty L\{\sin^2 t\}\,ds$$

$$= \int_0^\infty L\left\{\frac{1-\cos 2t}{2}\right\} ds$$

$$= \int_0^\infty \frac{1}{2s} - \frac{s}{2(s^2+4)}\,ds$$

$$= \frac{1}{2}\left(\left[\log s\right]_s^\infty - \int_s^\infty \frac{2s}{2(s^2+4)}\right) ds$$

$$= \frac{1}{2}\left(\log s\right]_s^\infty - \frac{1}{2}\log(s^2+4)\Big)_s^\infty$$

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$$= \frac{1}{4}\left(\log \frac{s^2}{s^2+4}\right)_s^\infty$$

$$= \frac{1}{4}\left(\log \frac{1}{1+4/s^2}\right)_s^\infty$$

$$= \frac{1}{4}\left(\log 1 - \log \frac{s^2}{s^2+4}\right)$$

$$\int_0^\infty e^{-t}\frac{\sin^2 t}{t} = \frac{1}{4}\left(\log 1 - \log \frac{1}{5}\right)$$

$$= \frac{1}{4}\log\frac{1}{1/5}$$

$$= \frac{1}{4}\log 5$$

iii) $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt$

sol:- $\int_0^\infty e^{0t}\left(\frac{e^{-at}-e^{-bt}}{t}\right)dt$

$s=0$ $\qquad x(t) = \frac{e^{-at}-e^{-bt}}{t}$

$$\int_0^\infty e^{0t}\left(\frac{e^{-at}-e^{-bt}}{t}\right)dt = L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}\Bigg|_{s=0} \quad (1)$$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^\infty L\left\{e^{-at}-e^{-bt}\right\}ds$$

$$= \int_s^\infty \frac{1}{s+a} - \frac{1}{s+b}\,ds$$

$$= \log(s+a) - \log(s+b)\Big|_s^\infty$$

$$= \log\frac{s+a}{s+b}\Bigg|_s^\infty$$

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$$= \log \frac{1 + a/s}{1 + b/s} \Big|_s^{\infty}$$

$$L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\} = \log 1 - \log \frac{s+a}{s+b} \quad \text{— (2)}$$

Putting eq^n 2 in eq^n 1

$$\int_0^{\infty} \frac{e^{-at} - e^{-bt}}{t}\, dt = \log 1 - \log \frac{s+a}{s+b} \Big|_{s=0}$$

$$= \log 1 - \log \frac{a}{b}$$

$$= \log \frac{1}{a/b}$$

$$= \log \frac{b}{a}.$$

L.P 5)

ii) $L\left(2\sqrt{\frac{t}{\pi}}\right) = \frac{1}{s^{3/2}}$

Sol:- $x(t) = 2\sqrt{\frac{t}{\sqrt{\pi}}}$

$$x'(t) = 2 \times \frac{1}{2} \frac{1}{\sqrt{\pi t}}$$

$$x(0) = 0.$$

$$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = s X(s) - x(0)$$

$$= s \cancel{L}\left\{2\sqrt{\frac{t}{\pi}}\right) - 0$$

$$= \frac{s \cdot 2 \cdot 1}{s^{3/2}}$$

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$$= \frac{1}{\sqrt{s}}$$

L.P 6)

ii) $L\left\{\int_0^t e^{-t}(1+t+t^2)\,dt\right\}$

Sol: $x(t) = e^{-t}(1+t+t^2)$

$$L\left\{\int_0^t e^{-t}(1+t+t^2)\,dt\right\} = \frac{X(s)}{s}$$

$$L\{x(t)\} = L\{e^{-t}(1+t+t^2)\}$$

$$= L\{1+t+t^2\}\Big|_{s \to s+1}$$

$$= \frac{1}{s} + \frac{|\overline{1+1}}{s^2} + \frac{|\overline{2+1}}{s^3}\Bigg|_{s \to s+1}$$

$$= \frac{1}{s} + \frac{1}{s^2} + \frac{2}{s^3}\Bigg|_{s \to s+1}$$

$$L\{x(t)\} = \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3}$$

$$L\left\{\int_0^t e^{-t}(1+t+t^2)\,dt\right\} = \left(\frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3}\right)\times\frac{1}{s}$$

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Solution of Laplace Transform for periodic function

A real valued function $f(t)$ defined for all values of t is said to be periodic function with fundamental period '$T > 0$'. If

$$f(t) = f(t+T) \quad \forall\, T$$

NOTE:- If f is a periodic function with period T then

$$f(t) = f(t+T) = f(t+2T) = f(t+3T) \ldots$$

imp (with proof)

THEOREM:- Find the laplace transform of periodic funⁿ of period T

or

Show that laplace transform of periodic funⁿ of period T is given by

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

Sol:- Given $f(t)$ is periodic funⁿ with period T

$$f(t+T) = f(t)$$

$$f(t) = f(t+2T) = \ldots$$

Now $$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt$$

$$\ldots$$

Put $t = u$, $t = u+T$, $t = u+2T$ resp terms of the above eqⁿ.

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In each case we have $dt = du$ and limits of integration are changing between 0 to T.

$$L\{f(t)\} = \int_0^T e^{-su} f(u)\,du + \int_0^T e^{-s(u+T)} f(u+T)\,du + \int_0^T e^{-s(u+2T)} f(u+2T)\,du \text{ ---}$$

$$= \int_0^T e^{-su} f(u)\,du + e^{-sT}\int_0^T e^{-su} f(u+T)\,du$$

$$+ e^{-2sT}\int_0^T e^{-su} f(u+2T)\,du + \text{-----}$$

Since $f(t)$ is periodic funⁿ.

$$L\{f(t)\} = \int_0^T e^{-su} f(u)\,du + e^{-sT}\int_0^T e^{-su} f(u)\,du$$

$$+ e^{-2sT}\int_0^T e^{-su} f(u)\,du + \text{---}$$

$$= \int_0^T e^{-su} f(u)\,du \; \big[\underbrace{1 + e^{-sT} + e^{-2sT} + \text{---}}_{\text{G.M.S} \quad |r| < 1}\big]$$

$$= \int_0^T e^{-su} f(u)\,du \left[\frac{1}{1 - e^{-st}}\right]$$

$$\boxed{L\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\,dt}$$

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L.P

8. Find the Laplace Transform of periodic function of period 2c given $f(t) = t \quad 0<t<c$
$= 2c-t \quad c<t<2c$

Sol:- Given :- $f(t) = t \quad 0<t<c$
$= 2c-t \quad c<t<2c$
$T = 2c$

By defⁿ:-
$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$\therefore L\{f(t)\} = \frac{1}{1-e^{-s(2c)}}\int_0^{2c} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2cs}}\left[\int_0^c e^{-st}.t\,dt + \int_c^{2c} e^{-st}(2c-t)\,dt\right]$$

$$= \frac{1}{1-e^{-2cs}}\left[\frac{te^{-st}}{-s} - (1)\frac{e^{-st}}{s^2}\Big|_0^c + \int_c^{2c} e^{-st}2c - e^{-st}t\right]$$

$$= \frac{1}{1-e^{-2cs}}\left[\frac{ce^{-sc}}{-s} - \frac{e^{-sc}}{s^2} + \frac{1}{s^2}\right] + \frac{e^{-st}}{-s}2c\Big]_c^{2c}$$

$$- \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_c^{2c}$$

$$= \frac{1}{1-e^{-2cs}}\left[\frac{ce^{-sc}}{-s} - \frac{e^{-sc}}{s^2} + \frac{1}{s^2}\right] + \frac{e^{-2sc}}{-s}2c - \frac{e^{-sc}}{-s}2c$$
$$- \left[\frac{2ce^{-2cs}}{-s} - \frac{e^{-2sc}}{s^2} - \frac{ce^{-sc}}{-s} + \frac{e^{-sc}}{s^2}\right]$$

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$$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

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$$= \frac{1}{1-e^{-2cs}}\left[\frac{c\cancel{e^{-sc}}}{-s} - \frac{e^{-sc}}{s^2} + \frac{1}{s^2} - \frac{2c\cancel{e^{-2sc}}}{s} + \frac{2c\cancel{e^{-sc}}}{s} + \frac{2c\cancel{e^{-2cs}}}{s} + \frac{e^{-2sc}}{s^2} - \frac{c\cancel{e^{-sc}}}{s} - \frac{e^{-sc}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2cs}}\,\frac{1}{s^2}\left[1 - 2e^{-sc} + (e^{-sc})^2\right)$$

$$= \frac{1}{1-e^{-2cs}}\,\frac{1}{s^2}\left(1-e^{-sc}\right)^2$$

$$= \frac{1}{\cancel{(1-e^{-cs})}(1+e^{-cs})}\,\frac{1}{s^2}\left(1-e^{-sc}\right)^{\cancel{2}}$$

$$= \left(\frac{1-e^{-cs}}{1+e^{-cs}}\right)\frac{1}{s^2}$$

$$= \frac{1}{s^2} \times \frac{e^{+cs/2}}{e^{+cs/2}}\left(\frac{1-e^{-cs}}{1+e^{-cs}}\right)$$

$$= \frac{1}{s^2}\left(\frac{e^{+cs/2} - e^{-cs/2}}{e^{cs/2} + e^{-cs/2}}\right)$$

$$= \frac{1}{s^2}\tanh\left(\frac{cs}{2}\right).$$

Find the Laplace transform of the function $f(t) = \frac{Kt}{T}$ $\quad 0 < t < T$

and $f(t+T) = f(t)$.

Sol:- It is periodic funⁿ with period T and is defined as

$$f(t) = \frac{Kt}{T} \qquad 0 < t < T$$

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By def $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \frac{kt}{T}\, dt$$

$$= \frac{1}{1-e^{-sT}} \frac{k}{T} \int_0^T e^{-st} t\, dt$$

$$= \frac{1}{1-e^{-sT}} \frac{k}{T} \left(\frac{t e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right)_0^T$$

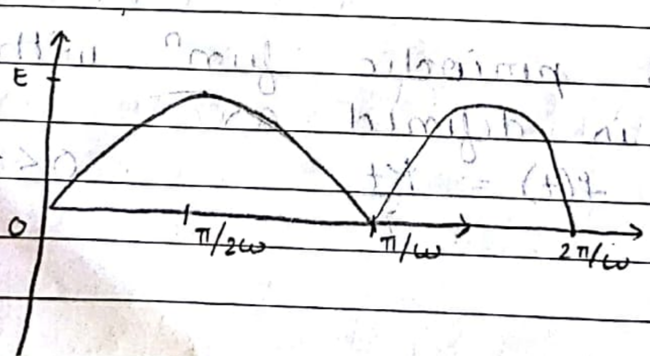
$$= \frac{1}{1-e^{-sT}} \frac{k}{T} \left(\frac{T e^{-sT}}{-s} - \frac{e^{-sT}}{s^2} + \frac{1}{s^2} \right)$$

$$= \frac{1}{1-e^{-sT}} \frac{k}{T} \left[\frac{T e^{-sT}}{-s} + \frac{1}{s^2}(1-e^{-sT}) \right]$$

$$\boxed{L\{f(t)\} = \frac{k e^{-sT}}{1-e^{-sT}} + \frac{k}{Ts^2}}$$

Find the Laplace transform of full wave rectifier $f(t) = E\sin\omega t$, $0 < t < \frac{\pi}{\omega}$ having period of π/ω and also sketch the graph of the function.

Sol:-



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$$L\{f(t)\} = \frac{1}{1-e^{-st}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-st}}\int_0^T e^{-st} E\sin\omega t\,dt$$

$$= \frac{1}{1-e^{-st}} E \int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt$$

$$= \frac{1}{1-e^{-st}}(E)\left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t-\omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{E}{1-e^{-st}}\left[\frac{e^{-sT}}{s^2+\omega^2}(-s\sin\omega T-\omega\cos\omega T)+\frac{\omega}{s^2+\omega^2}\right]$$

$$= \frac{E}{1-e^{-s\pi/\omega}}\left[\frac{e^{-s\pi/\omega}}{s^2+\omega^2}(0+\omega)+\frac{\omega}{s^2+\omega^2}\right]$$

$$= \frac{E\omega}{s^2+\omega^2}\left(\frac{1+e^{-s\pi/\omega}}{1-e^{-s\pi/\omega}}\right)$$ coth

Laplace transform of unit step funs.

Unit step fun^n (Heaviside's step) - denoted by

$u(t-a)$ or $H(t-a)$

and is defined as

$$u(t-a) = 1 \quad t \geq a$$
$$= 0 \quad t < a$$

If $a=0$, then $u(t) = 1 \quad t \geq 0$
$= 0 \quad t < 0$

From the def^n of Laplace transform, we can prove that

$$L\{u(t-a)\} = \frac{e^{-as}}{s}$$

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$$L\{u(t)\} = \frac{1}{s}$$

NOTE:- If $g(t) = f(t).\,u(t-a)$ then using the defn of Laplace transform, we can prove that $L\{g(t)\} = L\{f(t).\,u(t-a)\} = e^{-as} L\{f(t+a)\}$

Example:-

i) Find the Laplace transform of folln funn

i) $f(t) = (t+1)^2\, u(t-1)$

sol $L\{f(t)\} = L\{(t+1)^2\, u(t-1)\}$

$a = +1$

WKT

$$L\{f(t).\,u(t-a)\} = e^{-as}\, L\{f(t+a)\}$$

$$L\{(t+1)^2.\,u(t-1)\} = e^{-s}\, L\{f(t+1)\}$$

$$= e^{-s}\, L\{(t+2)^2\}$$

$$= e^{-s}\, L\{t^2 + 4t + 4\}$$

$$= e^{-s}\left(\frac{\boxed{2+1}}{s^{2+1}} + \frac{4}{s^2} + \frac{4}{s}\right)$$

$$= e^{-2}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right]$$

ii) $f(t) = e^{-t}\cos t \cdot H(t - \pi/2)$

sol $a = \pi/2$

$$L\{e^{-t}\cos t\, H(t-\pi/2)\} = e^{-\pi/2 \times s}\, L\{f(t+\pi/2)\}$$

$$= e^{-\pi s/2}\, L\{e^{-(t+\pi/2)}\cos(t+\pi/2)\}$$

$$= e^{-\pi s/2}\, L\{-e^{-t}.\,e^{-\pi/2}.\sin t\}$$

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$= e^{-\pi/2(s+1)} \; L\{e^{-t}.\sin t\}$

$= e^{-\pi/2(s+1)} \; L\{\sin t\}\Big|_{s\to s+1}$ (frequency)

$= e^{-\pi/2(s+1)} \; \frac{\cancel{s}1}{s^2+1}\Big|_{s+1}$

$= e^{-\pi/2(s+1)} \; \frac{\cancel{s+}1}{(s+1)^2+1}.$

iii) $f(t) = (t+\sin t).\,u(t-1)$

Sol:- $L\{f(t)\} = L\{(t+\sin t).\,u(t-1)\}$

$a = 1$

$L\{(t+\sin t).\,u(t-1)\} = e^{-s}\,L\{(t+1)+\sin(t+1)\}$

$= e^{-s}\left[L\{(t+1) + \sin(t+1)\}\right]$

$= e^{-s}\left[\frac{1}{s^2} + \frac{1}{s} + L\{\sin(t+1)\}\right]$

$= e^{-s}\left[\frac{1}{s^2} + \frac{1}{s} + L\{\sin t.\cos 1 + \sin 1\cos t\}\right]$

$= e^{-s}\left(\frac{1}{s^2} + \frac{1}{s} + \frac{\cos 1}{s^2+1} + \frac{s(\sin 1)}{s^2+1}\right).$

Result on unit step fun^n :-

If $f(t) = \phi(t) \quad a<t<b$

$= 0 \quad$ otherwise.

then

using def^n of unit step fun^n, we can define

$f(t) = \phi(t)\left[u(t-a) - u(t-b)\right] \quad \forall t.$

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Example:-

L.P !! i) Find the Laplace transform of foll^n fun^n by expressing given fun^n interms of unit step fun^n:

i) $f(t) = \sin t \quad 0<t<\pi/2$

$= \cos t \quad t>\pi/2.$

Sol:- Given:-

In unit step fun^n:-

$f(t) = \sin t\,(u(t-0) - u(t-\pi/2))$
$+ \cos t\,(u(t-\pi/2))$

$= \sin t\,(u(t-0)) + u(t-\pi/2)(\cos t - \sin t)$

$L\{f(t)\} = L\{\sin t\,(u(t-0)) + (\cos t - \sin t)\,u(t-\pi/2)\}$

$= e^{-0s} L\{\sin(t+0)\} + e^{-\pi/2 s} L\{\cos(t+\pi/2) - \sin(t+\pi/2)\}$

$= L\{\sin(t+0)\} + e^{-\pi/2 s} L\{-\sin t - \cos t\}$

$= L\{\sin t.\cos 0 - \cos t.\sin 0\} + e^{-\pi/2 s} L\{\cos t + \sin t\}$

$= \frac{1}{s^2+1} - e^{-\pi/2 s}\left[\frac{s}{s^2+1} + \frac{1}{s^2+1}\right]$

$L\{f(t)\} = \frac{1}{s^2+1}\left[1 - e^{-\pi/2 s} s - e^{-\pi/2 s}\right]$

iii) $f(t) = \cos t \quad 0<t<\pi$

$= \cos 2t \quad \pi<t<2\pi$

$= \cos 3t \quad t>2\pi$

Sol:- $f(t) = \cos t\,(u(t-0) - u(t-\pi)) + \cos 2t\,(u(t-\pi) - u(t-2\pi))$
$+ \cos 3t\,(u(t-2\pi))$

$= \cos t\,(u(t-0)) + u(t-\pi)(\cos 2t - \cos t)$

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$+ u(t-2\pi)(\cos 3t - \cos 2t)$

$L\{f(t)\} = L\{\cos t\,(u(t-0)) + u(t-\pi)(\cos 2t - \cos t)$
$\quad + u(t-2\pi)(\cos 3t - \cos 2t)\}$

$= e^{-0s} L\{\cos(t+0)\} + e^{-\pi s} L\{\cos 2(t+\pi) - \cos(t+\pi)\}$
$\quad + e^{-2\pi s} L\{\cos 3(t+2\pi) - \cos 2(t+\pi)\}$

$= L\{\cos t\} + e^{-\pi s} L\{\cos 2t + \cos t\}$
$\quad + e^{-2\pi s} L\{\cos 3t - \cos 2t\}$

$$= \frac{s}{s^2+1} + e^{-\pi s}\left[\frac{s}{s^2+4} + \frac{s}{s^2+1}\right] + e^{-2\pi s}\left[\frac{s}{s^2+9} - \frac{s}{s^2+4}\right]$$

$$L\{f(t)\} = \frac{s}{s^2+1}\left(1 + e^{-\pi s}\right) + \frac{s}{s^2+4}\left(e^{-\pi s} - e^{-2\pi s}\right) + e^{-2\pi s}\frac{s}{s^2+9}$$

iv) $f(t) = e^t \quad 0 < t < 1$
$\quad\;\; = 1 \quad t > 1$

sol:-

$f(t) = e^t[u(t-0) - u(t-1)] + 1[u(t-1)]$
$\quad\; = e^t u(t-0) + u(t-1)[1 - e^t]$

$L\{f(t)\} = L\{e^t u(t-0)\} + L\{u(t-1)[1-e^t]\}$
$\quad = e^{-0s} L\{e^{t+0}\} + e^{-s} L\{1 - e^{t+1}\}.$
$\quad = L\{e^t\} + e^{-s} L\{1 - e^t.e\}$

$$= \frac{1}{s-1} + e^{-s}\left[\frac{1}{s} - \frac{e}{s-1}\right]$$

$$L\{f(t)\} = \frac{1}{s-1}\left(1 - e^{-s+1}\right) + \frac{e^{-s}}{s}$$

v) $f(t) = e^t \sin t \qquad 0 < t < \pi$
$\quad = e^t \cos t \qquad t > \pi$

Sol:- $f(t) = e^t \sin t \,[u(t-0) - u(t-\pi)] + e^t \cos t \,[u(t-\pi)]$
$= e^t \sin t\,[u(t-0)] + u(t-\pi)\,[e^t \cos t - e^t \sin t]$

$L\{f(t)\} = L\{e^t \sin t\,[u(t-0)] + u(t-\pi)\,[e^t \cos t - e^t \sin t]\}$
$= e^{0s}\,[L\{e^t \sin t\}] + e^{-\pi s}\, L\{e^{t+\pi} \cos(t+\pi) - e^{t+\pi} \sin(t+\pi)\}$

$= L\{\sin t\}\Big|_{s \to s-1} + e^{-\pi s}\, e^{\pi}\, L\{e^t(-\cos t) + e^t \sin t\}$

$= \dfrac{1}{(s-1)^2+1} + e^{\pi - \pi s} \left[\dfrac{-s}{s^2+1}\Big|_{s \to s-1} + \dfrac{1}{s^2+1}\Big|_{s \to s-1} \right]$

$= \dfrac{1}{(s-1)^2+1} + e^{\pi - \pi s} \left[\dfrac{-(s-1)}{(s-1)^2+1} + \dfrac{1}{(s-1)^2+1} \right]$

$= \dfrac{1}{(s-1)^2+1} \left[1 + e^{\pi - \pi s}\,(-(s-1)) + e^{\pi - \pi s} \right]$

$= \dfrac{1}{(s-1)^2+1} \left[1 - e^{\pi - \pi s}\, s + 2e^{\pi - \pi s} \right]$

Dirac delta funn or unit impulse funn

Unit impulse funn is defined as

$\delta(t-a) = \infty \qquad t = a$
$\qquad = 0 \qquad \text{otherwise}$

such that $\int_0^\infty \delta(t-a)\, dt = 1.$

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Laplace transform of such funⁿ is given by

$$L\{\delta(t-a)\} = e^{-as}$$

If $a = 0$

$$L\{\delta(t)\} = e^{-0s} = 1$$

Inverse Laplace transform

i) If $L\{f(t)\} = F(s)$, then $f(t)$ is called inverse laplace transform of $F(s)$ and is written as

$$f(t) = L^{-1}\{F(s)\}$$

Formulas for inverse laplace transform

i) $L\{\delta(t)\} = 1$

$\delta(t) = L^{-1}\{1\}$.

ii) $L\{1\} = 1/s$

$L^{-1}\{1/s\} = 1$

iii) $L\{e^{at}\} = 1/s-a$

$L^{-1}\{1/s-a\} = e^{at}$

iv) $L\{\sin\omega t\} = \dfrac{\omega}{s^2+\omega^2}$

$L^{-1}\left\{\dfrac{1}{s^2+\omega^2}\right\} = \dfrac{\sin\omega t}{\omega}$

v) $L\{\cos\omega t\} = \dfrac{s}{s^2+\omega^2}$ $\quad L^{-1}\left\{\dfrac{s}{s^2+\omega^2}\right\} = \cos\omega t$

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The coffiaint of s or s^2 in denominator should always be 1.

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vi) $L\{\sinh\omega t\} = \frac{\omega}{s^2-\omega^2}$

$L^{-1}\left\{\frac{1}{s^2-\omega^2}\right\} = \frac{1}{\omega}\sinh\omega t$

vii) $L\{\cosh\omega t\} = \frac{s}{s^2-\omega^2}$

$L^{-1}\left\{\frac{s}{s^2-\omega^2}\right\} = \cosh\omega t$

viii) $L\{t^n\} = \frac{\lceil n+1}{s^{n+1}}$

$L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{1}{\lceil n+1} t^n$

$L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{\lceil n} t^{n-1}$

Properties

1) If $L\{f(t)\} = F(s)$, then $L\{e^{at} f(t)\} = F(s-a)$

If $f(t) = L^{-1}\{F(s)\}$ then $L^{-1}\{F(s-a)\} = e^{at} L^{-1}\{F(s)\}$

2) If $L\{f(t)\} = F(s)$, then $L\{f(at)\} = F(s/a) \times \frac{1}{a}$

If $f(t) = L^{-1}\{F(s)\}$, then $L^{-1}\{F(s/a)\} = a f(at)$

3). If $L\{f(t)\} = F(s)$ then $L^{-1}\{x'(t)\} = sX(s) - x(0)$

then

$L^{-1}\left\{s^n F(s)\right\} = \frac{d^n}{dt^n} f(t)$ provided

$f(0) = f'(0) = \_\_\_ = 0.$

4). If $L\left\{\int_0^t f(t)\,.dt\right\} = \frac{F(s)}{s}$, then

$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\,.dt$

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5) If $L\{t^n f(t)\} = \frac{d^n}{ds^n} F(s)$.

$$L^{-1}\left\{\frac{d^n}{ds^n} F(s)\right\} = (-1)^n t^n f(t).$$

6) If $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds$

$$L^{-1}\left\{\int_s^\infty F(s)\, ds\right\} = \frac{f(t)}{t}.$$

Partial fractions

i) $\frac{Px+q}{(x-a)^2} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2}$

ii) $\frac{Px+q}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$

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L.P 12) Find the inverse Laplace transform of given funⁿ.

i) $\frac{(s^2-1)^2}{2s^5}$

Sol. $\frac{s^4-2s^2+1}{2s^5}$

$\frac{1}{2s} - \frac{1}{s^3} + \frac{1}{2s^5}$

$L^{-1}\left\{\frac{1}{2s} - \frac{1}{s^3} + \frac{1}{2s^5}\right\} = \frac{1}{2} - \frac{1}{\lfloor 3}t^2 + \frac{1}{2}\frac{1}{\lfloor 5}t^4$

$= \frac{1}{2} - \frac{t^2}{2} + \frac{1}{2\times 24}t^4$

$= \frac{1}{2}\left(1 - t^2 + \frac{t^4}{24}\right)$

ii) $\frac{3s+5\sqrt{2}}{s^2+8}$

Sol $F(s) = \frac{3s}{s^2+8} + \frac{5\sqrt{2}}{s^2+8}$

$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{3s}{s^2+8} + \frac{5\sqrt{2}}{s^2+8}\right\}$

$L^{-1}\{F(s)\} = 3\cos\sqrt{8}t + \frac{5\sqrt{2}}{\sqrt{8}}\sin\sqrt{8}t$

$L^{-1}\{F(s)\} = 3\cos\sqrt{8}t + \frac{5}{2}\sin\sqrt{8}t$

iii) $\frac{4s+15}{16s^2-25}$

Sol:- $\frac{1}{16}\left(\frac{4s+15}{s^2-25/16}\right)$.

$F(s) =$

$L^{-1}(F(s$

$L^{-1}(F($

iv) $\frac{s+1}{s^2+s}$

Sol:- $F(s$

L^{-1}

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$$F(s) = \frac{1}{16}\left[\frac{4s}{s^2-25/16} + \frac{15}{s^2-25/16}\right]$$

$$L^{-1}(F(s)) = \frac{1}{16} L\left\{\frac{4s}{s^2-25/16} + \frac{15}{s^2-25/16}\right\}$$

$$= \frac{1}{16}\left[4\left(\cosh\left(\frac{5}{4}\right)t + \frac{15}{5/4}\sinh\left(\frac{5}{4}\right)t\right]$$

$$L^{-1}(F(s)) = \frac{1}{4}\left[\cosh\left(\frac{5}{4}\right)t + 3\sinh\left(\frac{5}{4}\right)t\right]$$

iv) $\frac{s+1}{s^2+s+1.}$

Sol:- $F(s) = \frac{s+1}{s^2+s+1-1/4+1/4}$

$$= \frac{s+1+1/2-1/2}{(s+1/2)^2+3/4}$$

$$= \frac{s+1/2}{(s+1/2)^2+3/4} + \frac{1/2}{(s+1/2)^2+3/4}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s+1/2}{(s+1/2)^2+3/4}\right\} + L^{-1}\left\{\frac{1/2}{(s+1/2)^2+3/4}\right\}$$

$$= e^{-1/2s} L^{-1}\left\{\frac{s}{s^2+3/4}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{(s+1/2)^2+3/4}\right\}$$

$$= e^{-1/2s}\cos\frac{3}{4}t + \frac{1}{2}e^{-1/2s} L^{-1}\left\{\frac{1}{s^2+3/4}\right\}$$

$$= e^{-1/2s}\cos\frac{\sqrt{3}}{\sqrt{4}}t + \frac{e^{-1/2s}}{2}\cdot\frac{1}{\sqrt{3}/2}\sin\frac{\sqrt{3}}{\sqrt{4}}t$$

$$= e^{-1/2s}\cos\frac{\sqrt{3}}{2}t + \frac{e^{-1/2s}}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t$$

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i) $L^{-1}\left\{\dfrac{2s-3}{s^2+4s+3}\right\}$

Sol:- $F(s) = \dfrac{2s-3}{s^2+4s+3} = \dfrac{2s-3}{s^2+s+3s+3} = \dfrac{2s-3}{(s+1)(s+3)}$

$$\frac{2s-3}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$2s-3 = A(s+3) + B(s+1)$ —— ①

Put $s = -3$

$-6-3 = B(-2)$

$\dfrac{9}{2} = A$

Put $s = -1$ in eq^n ①

$-2-3 = A(2)$

$\dfrac{-5}{2} = A$

$\therefore \dfrac{2s-3}{(s+1)(s+3)} = \dfrac{-5/2}{s+1} + \dfrac{9/2}{s+3}$

$L^{-1}F(s) = L^{-1}\left\{\dfrac{-5/2}{s+1} + \dfrac{9/2}{(s+3)}\right\}$

$L^{-1}F(s) = \dfrac{-5}{2}e^{-t} + \dfrac{9}{2}e^{-3t}$

ii) $L^{-1}\left\{\dfrac{s}{(s+1)^2(s^2+1)}\right\}$

Sol:- $F(s) = \dfrac{s}{(s+1)^2(s^2+1)} = \dfrac{A}{(s+1)} + \dfrac{B}{(s+1)^2} + \dfrac{Cs+D}{(s^2+1)}$

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$$\frac{s}{(s+1)^2(s^2+1)} = \frac{A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2}{(s+1)^2(s^2+1)}$$

Put $s = -1$

$$-1 = B(2)$$

$$B = \frac{-1}{2}$$

$$s = As^3 + As + As^2 + A + Bs^2 + B + (Cs+D)(s^2+2s+1)$$

$$= As^3 + As + As^2 + A + Bs^2 + B + Cs^3 + 2Cs^2 + Cs + Ds^2 + 2Ds + D$$

$$s = s^3(A+C) + s^2(A+B+2C+D) + s(A+C+2D) + (A+B+D)$$

Equating the coefficient of like terms on both sides

$$A + C = 0 \qquad A = -C$$

$$A + B + 2C + D = 0$$

$$A + B + D = 0$$

$$A + C + 2D = 1 \quad \Rightarrow \quad -C + C + 2D = 1$$

$$\boxed{D = 1/2}$$

$$A + -1/2 + 1/2 = 0$$

$$\boxed{\begin{array}{l} A = 0 \\ C = 0 \end{array}}$$

$$F(s) = \frac{-1/2}{(s+1)^2} + \frac{1/2}{(s^2+1)}$$

$$L^{-1}\left\{\frac{-1/2}{(s+1)^2} + \frac{1/2}{s^2+1}\right\} = \frac{-1}{2}e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{2}\sin\omega t.$$

$$= -\frac{1}{2}e^{-t}t + \frac{1}{2}\sin\omega t.$$

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iii) $\dfrac{s+3}{(s-1)(s^2+2s+10)}$

Sol:- $F(s) = \dfrac{s+3}{(s-1)(s^2+2s+1/4-1/4+10)}$

$= \dfrac{s+3}{(s-1)((s+1)^2+9)}$

$\dfrac{s+3}{(s-1)(s^2+2s+10)} = \dfrac{A}{s-1} + \dfrac{Bs+D}{s^2+2s+10}$

$s+3 = A(s^2+2s+10) + (Bs+D)(s-1)$

$= A(s^2+2s+10) + Bs^2 - Bs + Ds - D$

$= (A+B)s^2 + (2A-B+D)s + 10A - D$

$A = 4/13$

$A+B = 0 \Rightarrow A = -B \Rightarrow B = -4/13$

$2A - B + D = 1 \Rightarrow$

$10A - D = 3$ $\quad 10A = 3+D$

$A = \dfrac{3+D}{10}$

$\dfrac{+40}{13} = 3 + D$

$\dfrac{40-39}{13} = D$

$\boxed{D = \dfrac{1}{13}}$

$F(s) = \dfrac{4/13}{s-1} + \dfrac{-4/13\, s + 1/13}{((s+1)^2+9)}$

$L^{-1}\{F(s)\} = \dfrac{4}{13}e^t + \dfrac{-4}{13}\left(\dfrac{s+1}{(s+1)^2+9}\right) + \dfrac{1/13 + 4/13}{(s+1)^2+9}$

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$= \frac{4}{13}e^{t} - \frac{4}{13}e^{-t}L^{-1}\left\{\frac{s}{s^2+9}\right\} - \frac{5}{13}e^{-t}L^{-1}\left\{\frac{1}{(s+1)^2+9}\right\}$

$= \frac{4}{13}e^{t} - \frac{4}{13}e^{-t}\cos 3t - \frac{15}{13}\left(\frac{1}{3}\right)\sin 3t \cdot e^{-t}$

$= \frac{4}{13}e^{t} - \frac{4}{13}e^{-t}\cos 3t - \frac{5}{39}\sin 3t \cdot e^{-t}$

L.P 12)

5) $\frac{2s^2-1}{(s^2+1)(s^2+4)}$

Sol. $F(s) = \frac{2s^2-1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{(s^2+4)}$ X

$\therefore$ Put $s^2 = t$.

$F(s) = \frac{2t-1}{(t+1)(t+4)} = \frac{A}{(t+1)} + \frac{B}{(t+4)}$

$2t-1 = A(t+4) + B(t+1)$

Put $t = -4$

$\frac{-9}{-3} = B$

$\boxed{3 = B}$

Put $t = -1$

$\frac{-3}{3} = A$

$\boxed{A = -1}$

$\therefore \frac{2s^2-1}{(s^2+1)(s^2+4)} = \frac{-1}{s^2+1} + \frac{3}{s^2+4}$

$L^{-1}\left\{\frac{-1}{s^2+1} + \frac{3}{s^2+4}\right\} = -1\sin t + \frac{3}{2}\sin 2t.$

If $\frac{s^3}{(s^2+a^2)(s^2+b^2)} = s\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] =$ Put $s^2 = t$.

NOTE :-

1) If F(s) involves logarithmic function or inverse trigonometric function, in such case :-

a) We differentiate the given funⁿ w.r.t x.

b) We find the inverse laplace transform

c) We solve the equation for f(t)

Ex :-

L.P 12

10) $F(s) = \cot^{-1}(s+1)$

Sol :- $F'(s) = \frac{-1}{1+(s+1)^2}$

$$L^{-1}\{F'(s)\} = -L^{-1}\left\{\frac{-1}{1+(s+1)^2}\right\}$$

$$= -t f(t)$$

$$-t f(t) = -(e^{-t}\sin t)$$

$$f(t) = \frac{e^{-t}\sin t}{t}.$$

9) $F(s) = \log\left(\frac{s(s+1)}{s^2+4}\right)$

Sol $F(s) = \log s + \log s+1 - \log s^2+4$

$F'(s) = \frac{1}{s} + \frac{1}{s+1} - \frac{2s}{s^2+4}$

$$L^{-1}\{F's\} = L^{-1}\left\{\frac{1}{s} + \frac{1}{s+1} - \frac{2s}{s^2+4}\right\}.$$

$-tf(t) = 1 + e^{-t} - 2\cos 2t.$

$f(t) = \dfrac{2\cos t - e^{-t} - 1}{t}$

L.P 12

6) $\dfrac{5s+3}{(s+1)(s^2+2s+5)}$

Sol:- $F(s) = \dfrac{5s+3}{(s+1)(s^2+2s+5)}$

$\dfrac{5s+3}{(s+1)(s^2+2s+3)} = \dfrac{A}{s+1} + \dfrac{Bs+D}{(s^2+2s+3)}$

$5s+3 = A(s^2+2s+5) + (Bs+D)(s+1)$

Put $s = -1$

$-2 = A(1-2+5)$

$-2 = 4A$

$\boxed{A = -1/2}$

Put $s = 0$

$3 = 5A + D$

$3 + \dfrac{5}{2} = D$

$\boxed{\dfrac{11}{2} = D}$

$5s+3 = As^2 + 2As + 3A + Bs^2 + Bs + Ds + D$

$5s+3 = s^2(A+B) + s(2A+B+D) + 5A + D$

$5 = 2A + B + D$

$0 = A + B$ $\quad$ $A = -B$

$3 = 5A + D$ $\quad$ $\boxed{B = 1/2}$

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$$F(s) = \frac{-1/2}{s+1} + \frac{1/2 s + 11/2}{s^2+2s+5}$$

$$=$$

$$L^{-1}F(s) = L^{-1}\left\{\frac{-1/2}{s+1} + \frac{1/2 s + 11/2}{s^2+2s+1-1+5}\right\}$$

$$= L^{-1}\left\{\frac{-1/2}{s+1} + \frac{1/2 s + 11/2}{(s+1)^2+4}\right\}$$

$$= \frac{-1}{2} e^{-t} L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1/2(s+11)}{(s+1)^2+4}\right\}$$

$$= \frac{-1}{2} e^{-t} + \cancel{\tfrac{1}{2}}\, L^{-1}\left\{\frac{1/2 s + 1/2 - 1/2 + 11/2}{(s+1)^2+4}\right\}$$

$$= \frac{-1}{2} e^{-t} + L^{-1}\left\{\frac{1}{2}\frac{(s+1)}{(s+1)^2+4} + \frac{5}{(s+1)^2+4}\right\}$$

$$= \frac{-1}{2} e^{-t} + \frac{1}{2} e^{-t} L^{-1}\left\{\frac{s}{s^2+4}\right\} + 5e^{-t} L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$L^{-1}F(s) = \frac{-1}{2} e^{-t} + \frac{1}{2} e^{-t}\cos 2t + \frac{5e^{-t}}{2}\sin 2t.$$

vii) $\dfrac{1}{s(s+1)^3}$

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = L^{-1}\left\{\frac{1/(s+1)^3}{s}\right\}$$

$$= \int_0^t L^{-1}\left\{\frac{1}{(s+1)^3}\right\} dt$$

$$= \int_0^t e^{-t} \cdot \frac{t^2}{2}\, dt$$

sol:- $F(s) = \dfrac{1}{s(s+1)^3}$

$$F(s) = \frac{1}{((s+1)-1)(s+1)^3}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{((s+1)-1)(s+1)^3}\right\}$$

$$= e^{-t} L^{-1}\left\{\frac{1}{(s-1)s^3}\right\}$$

$$= e^{-t} L^{-1}\left\{\frac{1}{(s-1)s^3}\right\}.$$

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WKT $L^{-1}\left\{\frac{1}{s-1}\right\} = e^{t}.$

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viii) $F(s) = \frac{1}{s^2(s^2+4)}$

Sol:- $\frac{1}{s^2(s^2+4)}$ Put $s^2 = t$

$$\frac{1}{t(t+4)} = \frac{A}{t} + \frac{B}{t+4}$$

$1 = A(t+4) + Bt$

Put $t = 0$

$1 = 4A$

$\boxed{A = 1/4}$

Put $t = -4$

$1 = -4B$

$B = -1/4$

$$\frac{1}{s^2(s^2+4)} = \frac{1/4}{t} - \frac{1/4}{t+4}$$

$$= \frac{1/4}{s^2} - \frac{1/4}{s^2+4}$$

$$L^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \frac{1}{4}t - \frac{1}{4} \times \frac{1}{2}\sin 2t$$

$$= \frac{1}{4}t - \frac{1}{8}\sin 2t.$$

(xi) $\tan^{-1}\left(\frac{2}{s}\right)$

Sol:- $F(s) = \tan^{-1}\left(\frac{2}{s}\right)$

$$F'(s) = \frac{1}{\left(\frac{2}{s}\right)^2 + 1} \times \frac{-2}{s^2}.$$

$$= \frac{\cancel{s^2}}{4+s^2} \times \frac{-2}{\cancel{s^2}} = \frac{-2}{s^2+4}.$$

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$F^{-1}(s) = \frac{-2}{s^2+4}$

$L^{-1}\{F'(s)\} = L^{-1}\left\{\frac{-2}{s^2+4}\right\}$.

$-tf(t) = \frac{-2}{2}\sin 2t$

$f(t) = \frac{\sin 2t}{t}$

Finding inverse laplace transform when it involves e^{-as}, then

$L^{-1}\{e^{-as}F(s)\} = u(t-a).\, L^{-1}\{F(s)\}\Big|_{t\to t-a}$

$= u(t-a).\, f(t-a)$

Example:-

1) $L^{-1}\left\{\frac{e^{-3s}}{(s-1)^4}\right\}$

Sol:- $L^{-1}\left\{e^{-as}F(s)\right\} = u(t-a).\, L^{-1}\{F(s)\}\Big|_{t\to t-a}$.

$L^{-1}\left\{\frac{e^{-3s}}{(s-1)^4}\right\} = u(t-3).\left(L^{-1}\left\{\frac{1}{(s-1)^4}\right\}\right|_{t\to t-3}$

$= u(t-3)\, e^t\, L^{-1}\left\{\frac{1}{s^4}\right\}\Big|_{t\to t-3}$

$= u(t-3)\, e^t \frac{t^3}{\lceil 4} \Big|_{t\to t-3}$

$= u(t-3)\, e^{t-3}\frac{(t-3)^3}{6}$

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2) $L^{-1}\left\{\frac{e^{-\pi s}\cdot s}{s^2+4}\right\}$

Sol:- $L^{-1}\left\{e^{-as}\,F(s)\right\}$

$$L^{-1}\left\{\frac{e^{-\pi s}\,s}{s^2+4}\right\} = u(t-\pi).\,L^{-1}\left\{\frac{s}{s^2+4}\right\}\Big|_{t\to t-\pi}$$

$$= u(t-\pi)\,(\cos 2t)_{t\to t-\pi}$$

$$= u(t-\pi)\,(\cos 2t-2\pi)$$

$$= u(t-\pi)\,\cos 2t.$$

L.P 12

13) $\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$

Sol:- $L^{-1}\left\{\frac{se^{-s/2}}{s^2+\pi^2}+\frac{\pi e^{-s}}{s^2+\pi^2}\right\}$

$\Rightarrow u(t-1/2)\,L^{-1}\left\{\frac{s}{s^2+\pi^2}\right\}\Big|_{t\to t-1/2} + \pi\left(u(t-1)\,L^{-1}\left\{\frac{1}{s^2+\pi^2}\right\}\right)\Big|_{t\to t-1}$

$\Rightarrow u(t-1/2)\,\cos\pi t\Big|_{t\to t-1/2} + \pi.\frac{u(t-1)}{\pi}\sin\pi t\Big|_{t\to t-1.}$

$= u(t-1/2)\,\cos\pi(t-t/2) \mp u(t-1)\sin\pi t-\pi.$

$= u(t-1/2)\,\sin\pi t - u(t-1)\sin\pi t.$

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Convolution

Convolution of 2 functions $f(t)$ and $g(t)$ denoted by $f(t) * g(t)$ or $(f*g)(t)$ is a new function and is given by

$$f(t) * g(t) = \int_0^t f(u).\, g(t-u).\, du$$

NOTE :-

1) Convolution is commutative

$$f(t) * g(t) = g(t) * f(t)$$
$$= \int_0^t g(u).\, f(t-u).\, du.$$

Laplace transform of convolution

Statement :- Let $f(t)$ and $g(t)$ be 2 functions with laplace transforms $F(s)$ and $G(s)$ then

$$L\{f*g\} = F(s).\, G(s)$$

or $$L^{-1}\{F(s).\, G(s)\} = f*g$$

$$L^{-1}\{F(s).\, G(s)\} = L^{-1}\{F(s)\}.\, L^{-1}\{G(s)\}$$

Examples :-

L.P 13

1). $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$

Sol:- $H(s) = \frac{s}{(s^2+a^2)^2}$

$= \frac{1}{(s^2+a^2)} \cdot \frac{s}{(s^2+a^2)}$

$= F(s) \cdot G(s)$

By convolution theorem

$L^{-1}\{F(s).G(s)\} = L^{-1}\{F(s)\} * L^{-1}\{G(s)\}$

$L^{-1}\left\{\frac{1}{(s^2+a^2)} \cdot \frac{s}{(s^2+a^2)}\right\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} * L^{-1}\left\{\frac{s}{s^2+a^2}\right\}$

$= \frac{1}{a} \sin at * \cos at.$

$= \frac{1}{a}\left(\sin at * \cos at\right)$

$= \frac{1}{a}\int_0^t \sin au \cdot \cos a(t-u)\, du$

$= \frac{1}{2a}\int_0^t 2\sin au \cdot \cos(at-au)\, du$

$= \frac{1}{2a}\int_0^t \left[\sin(at) - \sin(2au-at)\right] du$

$= \frac{1}{2a}\int_0^t \sin at - \sin(2au-at)] du$

$= \frac{1}{2a}\left[\sin at \cdot u + \frac{\cos(2au-at)}{2a}\right]_0^t$

$= \frac{1}{2a}\left[\sin at \cdot t + \frac{\cos at}{2a} - 0 - \frac{\cos(-at)}{2a}\right]$

$= \frac{1}{2a}\left[t \sin at + \cancel{\frac{1}{2a}\cos at} - \cancel{\frac{\cos at}{2a}}\right]$

$= \frac{1}{2a} t \sin at.$

$\frac{d}{dt}\sinh t = \cosh t$ $\frac{d}{dt}\cosh t = \sinh t$ $\frac{d}{dt}\operatorname{sech} t = -\operatorname{sech} t \cdot \tanh t$

If both are starting with same letter then we have to take − sign.

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iii) $L^{-1}\left\{\frac{1}{s(s^2-a^2)}\right\}$

Sol. $H(s) = \frac{1}{s} \cdot \frac{1}{s^2-a^2}$

$= F(s) \cdot g(s)$

By convolution theorem

$L^{-1}\{F(s) \cdot g(s)\} = L^{-1}\{F(s)\} * L^{-1}\{g(s)\}$

$L^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s^2-a^2}\right\} = L^{-1}\left\{\frac{1}{s}\right\} * L^{-1}\left\{\frac{1}{s^2-a^2}\right\}$

$= 1 * \frac{1}{a}\sinh at$

$= \frac{1}{a}(\sinh at * 1)$

$= \frac{1}{a}\int_0^t \sinh at\,(1)\,dt$

$= \frac{1}{a}\left[\frac{\cosh at}{a}\right]_0^t$

$= \frac{1}{a^2}[\cosh at - 1]$

Extra

i) $L^{-1}\left\{\frac{s}{(s-1)(s^2+4)}\right\}$

Sol $H(s) = \frac{1}{s-1} \cdot \frac{s}{s^2+4}$

$= F(s) \cdot g(s)$

By convolution theorem

$L^{-1}\left\{\frac{1}{s-1} \cdot \frac{s}{s^2+4}\right\} = e^t * \cos 2t$

Date: / /

$$= \int_0^t \cos 2u . e^{t-u} du$$

$$= e^t \int_0^t \cos 2u . e^{-u} du$$

$$= e^t \left[\frac{e^{-u}}{1+4} \left[-\cos 2u + 2\sin 2u \right] \right]_0^t$$

$$= e^t \left(\left[\frac{e^{-t}}{5} \right] \left[-\cos 2t + 2\sin 2t \right] + \frac{e^0}{5} \right)$$

$$= \frac{1}{5} \left[-\cos 2t + 2\sin 2t + e^t \right]$$

ii) $L^{-1} \left\{ \frac{s}{(s^2+a^2)(s^2+b^2)} \right\}$

Sol:- $H(s) = \frac{s}{s^2+a^2} \cdot \frac{1}{s^2+b^2}$

By convolution theorem

$$L^{-1}\left\{ H(s) \right\} = \cos at * \frac{\sin bt}{b}$$

$$= \frac{1}{b} \left[\cos at * \sin bt \right]$$

$$= \frac{1}{b} \int_0^t \sin bu . \cos a(t-u) \, du$$

$$= \frac{1}{b} \int_0^t \sin bu . \cos at - au \, du$$

$$= \frac{1}{2b} \int_0^t (\sin bu + at - au) + \sin(bu - at + au) du$$

$$= \frac{1}{2b} \int_0^t \sin(at + u(b-a)) + \sin((a+b)u - at) du$$

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$$= \frac{1}{2b}\left[\frac{-\cos(at+u(b-a))}{b-a} + \frac{\cos((a+b)u-at)}{a+b}\right]_0^t$$

$$= \frac{1}{2b}\left[\frac{-\cos(at+bt-at)}{b-a} + \frac{\cos(at+bt-at)}{a+b} + \frac{\cos at}{b-a} - \frac{\cos at}{a+b}\right]$$

$$= \frac{1}{2b}\left[\frac{-\cos bt}{b-a} + \frac{\cos bt}{a+b} + \frac{\cos at}{b-a} + \frac{\cos at}{b+a}\right]$$

$$\times\left[= \frac{1}{2b}\left[\frac{2\cos bt}{a+b} + \frac{2\cos at}{b-a}\right]\right.$$

$$= \frac{1}{b}\left[\frac{\cos bt - \cos at}{a+b}\right]\Big]\times$$

$$= \frac{1}{2b}\left[\cos bt\left[\frac{1}{a+b} - \frac{1}{b-a}\right] + \cos at\left[\frac{1}{b-a} + \frac{1}{b+a}\right]\right]$$

$$= \frac{1}{2b}\left[\frac{\cos bt(-2a)}{b^2-a^2} + \frac{\cos at(2a)}{b^2-a^2}\right)$$

$$= \frac{2a}{2b}\left[\frac{\cos at - \cos bt}{b^2-a^2}\right]$$

$$= \quad \text{Ans} \quad \boxed{\frac{\cos bt - \cos at}{a^2-b^2}}$$

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Solution of differential eqⁿ using Laplace transform.

i) $y'' - 2y' + 8y = 0$ given:- $y(0) = 3$, $y'(0) = 6$
(Homogenous)

Sol:- Take laplace transform

$= L\{y'' - 2y' + 8y\} = 0$

$\Rightarrow L\{y''\} - 2L\{y'\} + 8L\{y\} = 0$

$= s^2L\{y\} - sy(0) - y'(0) - 2[sL\{y\} - y(0)] + 8L\{y\} = 0$

$= s^2L\{y\} - s(3) - 6 - 2[sL\{y\} - 3] + 8L\{y\} = 0$

$= L\{y\}[s^2 - 2s + 8] - s(3) - 6 + 6 = 0$

$= L\{y\}[s^2 - 2s + 8] = 6 + 3s - 6$

$= L\{y\}[s^2 - 2s + 8] = 3s$

$L\{y\} = \dfrac{3s}{s^2 - 2s + 8}$

Take inverse both sides.

$= y = L^{-1}\left\{\dfrac{3s}{s^2 - 2s + 8}\right\}$

$= 3L^{-1}\left\{\dfrac{s}{s^2 - 2s + 1 - 1 + 8}\right\}$

$= 3L^{-1}\left\{\dfrac{s}{(s-1)^2 + 7}\right\}$

$= 3L^{-1}\left\{\dfrac{s-1+1}{(s-1)^2 + 7}\right\}$

$= 3L^{-1}\left\{\dfrac{s-1}{(s-1)^2 + 7} + \dfrac{1}{(s-1)^2 + 7}\right\}$

$= 3\left(e^t L^{-1}\left\{\dfrac{s}{s^2 + 7}\right\} + e^t L^{-1}\left\{\dfrac{1}{s^2 + 7}\right\}\right)$

$y = 3\left(e^t \cos\sqrt{7}t + \dfrac{e^t}{\sqrt{7}} \sin\sqrt{7}t\right)$

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Date: / /

t.p.15

ii) $y'' - 3y' + 2y = 4t + e^{3t}$ given $y(0) = 1$
$y'(0) = -1$

Sol: Take laplace transformation

$L\{y'' - 3y' + 2y\} = L\{4t + e^{3t}\}$

$L\{y''\} - 3L\{y'\} + 2L\{y\} = \frac{4}{s^2} + \frac{1}{s-3}$

$s^2L\{y\} - sy(0) - y'(0) - 3[sL\{y\} - y(0)] + 2L\{y\} = \frac{4}{s^2} + \frac{1}{s-3}$

$L\{y\}[s^2 - 3s + 2] - s(1) - (-1) + 3(1) = \frac{4}{s^2} + \frac{1}{s-3}$

$L\{y\}[s^2 - 3s + 2] - s + 1 + 3 = \frac{4+1}{s^2(s-3)}$

$L\{y\}[s^2 - 3s + 2] = \frac{4}{s^2} + \frac{1}{s-3} - 4 + s$

$L\{y\}[s^2 - 3s + 2] = \frac{4s - 12 + s^2 - 4 + s}{s^2(s-3)}$

$= \frac{4s - 12 + s^2 - (4-s)(s^2)(s-3)}{s^2(s-3)}$

$= \frac{4s - 12 + s^2 - [4s^3 - 12s^2 - s^4 + 3s^3]}{s^2(s-3)}$

$= \frac{4s - 12 + s^2 - 7s^3 + 12s^2 + s^4}{s^2(s-3)}$

$L\{y\} = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-3)(s^2 - 3s + 2)}$

$L\{y\} = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-3)(s-1)(s-2)}$

$\frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-3)(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{s-1} + \frac{E}{s-2}$

$= As(s-3)(s-1)(s-2) + B(s-3)(s-1)(s-2) + C(s^2)(s-1)(s-2)$
$+ D(s^2)(s-3)(s-2) + E(s^2)(s-3)(s-1)$

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Put $s=0$

$-12 = B(-3)(-1)(-2)$

$B = 12/6 = 2$

Put $s=1$

$1-7+13+4-12 = D(1)(-2)(-1)$

$-1 = D(2)$

$\boxed{D = \frac{-1}{2}}$

$s=2$

$16-7(8)+13(4)+8-12 = E(4)(-1)(1)$

$16-56+52-4 = -4E$

$\frac{8}{-4} = E$

$\boxed{E = -2}$

Put $s=3$

$81-7(27)+13(9)+4(3)-12 = C(9)(2)(1)$

$\cancel{81}\ \cancel{189}\ \frac{9}{18} = C$

$\boxed{C = 1/2}$

Equating coefficient of s^4

$1 = A + C + D + E$

$\boxed{1 = A + 1/2 - 1/2 - 2} \Rightarrow \boxed{A = 3}$

$$L\{y\} = \frac{3}{s} + \frac{2}{s^2} + \frac{1}{2(s-3)} + \frac{-1/2}{s-1} + \frac{(-2)}{s-2}$$

$$L\{y\} = \frac{3}{s} + \frac{2}{s^2} + \frac{1/2}{s-3} - \frac{1/2}{s-1} - \frac{2}{s-2}$$

Taking inverse

$$y = 3 + 2t + \frac{1}{2}e^{3t} - \frac{1}{2}e^{t} - 2e^{2t}$$

(iii) $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = e^{-t}\sin t$ $\quad x(0)=0$, $x'(0)=1$

Sol:- $x'' + 2x' + 5x = e^{-t}\sin t$

$L\{x''+2x'+5x\} = L\{e^{-t}\sin t\}$

$s^2L\{x\} - sx(0) - x'(0) + 2[sL\{x\} - x(0)] + 5L\{x\}$

$= \frac{1}{s^2+1}\Big|_{s\to s+1}$

$L\{x\}[s^2+2s+5] - s(0) - 1 = \frac{1}{(s+1)^2+1}$

$L\{x\}[s^2+2s+5] = \frac{1}{(s+1)^2+1} + 1$

$= \frac{1+(s+1)^2+1}{(s+1)^2+1}$

$= \frac{(s+1)^2+2}{(s+1)^2+1}$

$L\{x\} = \frac{(s+1)^2+2}{((s+1)^2+1)(s^2+2s+5)}$

$= \frac{(s+1)^2+2}{((s+1)^2+1)((s+1)^2+4)}$

$x = L^{-1}\left\{\frac{(s+1)^2+2}{((s+1)^2+1)((s+1)^2+4)}\right\}$

$x = e^{-t}L^{-1}\left\{\frac{s^2+2}{(s^2+1)(s^2+4)}\right\}$

Put $s^2 = t$

$\frac{s^2+2}{(s^2+1)(s^2+4)} = \frac{t+2}{(t+1)(t+4)}$

$\frac{t+2}{(t+1)(t+4)} = \frac{A}{t+1} + \frac{B}{t+4}$

Put $t=-1$

$t+2 = A(-1+4) \Rightarrow 1 = 3A \Rightarrow \boxed{A = 1/3}$

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Put $t = -4$

$-4+2 = B(-3)$

$\frac{2}{3} = B$

$$x = e^{-t} L^{-1}\left\{\frac{1/3}{s^2+1} + \frac{2/3}{s^2+4}\right\}$$

$$= e^{-t}\left(\frac{1}{3} \times \frac{1}{2} \sin t + \frac{2}{3} \times \frac{1}{2} \sin 2t\right)$$

$$= \frac{e^{-t}}{3} \sin t + \frac{e^{-t}}{3} \sin 2t$$

L.P 15

v) $y'' + y = \sin 2t . \sin t \qquad y(0) = 1$
$\qquad y'(0) = 0$

Sol. $L\{y'' + y\} = \frac{1}{2} L\{\cos t - \cos 3t\}$

$s^2 L\{y\} - s y(0) - y'(0) + L\{y\} =$

$= s^2 L\{y\} - s(1) - 0 + L\{y\}$

$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+9}\right] = s^2 L\{y\} - s + L\{y\}$

$= L\{y\}[s^2+1] - s$

$= L\{y\}(s^2+1) - s = \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+9}\right]$

$= L\{y\}(s^2+1) = \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+9} + 2s\right]$

$= L\{y\} = \frac{1}{2}\left[\frac{s}{(s^2+1)^2} - \frac{s}{(s^2+9)(s^2+1)} + \frac{2s}{s^2+1}\right]$

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Taking inverse

$$y = \frac{1}{2} L^{-1}\left[\frac{s}{(s^2+1)^2} - \frac{s}{(s^2+9)(s^2+1)} + \frac{2s}{s^2+1} \right] \quad \text{--- (1)}$$

I^{st} term

$$L^{-1}\left\{ \frac{s}{(s+1)^2} \right\} = L^{-1}\left[\frac{s}{(s^2+1)} \times \frac{1}{s^2+1} \right]$$ (By convoluti theorem

$$= L^{-1}\left\{ \frac{s}{s^2+1} \right\} * L^{-1}\left\{ \frac{1}{s^2+1} \right\}$$

$$= \cos t * \sin t$$

$$= \cancel{\int_0^t \cos(t-u)\sin(u)\, du}$$

$$= \int_0^t \sin u . \cos(t-u)\, du$$

$$= \int_0^t \frac{1}{2}\left(\sin t + \sin 2u - t\right) du$$

$$= \frac{1}{2}\left[\sin t . u - \frac{\cos(2u-t)}{2} \right]_0^t$$

$$= \frac{1}{2}\left[t.\sin t - \cancel{\frac{\cos t}{2}} - [0] + \cancel{\frac{\cos t}{2}} \right]$$

$$\boxed{L^{-1}\left\{ \frac{s}{(s+1)^2} \right\} = \frac{t \sin t}{2}}$$

II term

$$L^{-1}\left\{ \frac{s}{(s^2+9)(s^2+1)} \right\} = L^{-1}\left\{ \frac{s}{s^2+9} \times \frac{1}{s^2+1} \right\}.$$

$$= \cos 3t * \sin t.$$

$$= \sin t * \cos 3t.$$

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$$= \int_0^t \sin u \cdot \cos 3(t-u)\, du$$

$$= \int_0^t \sin u \cdot \cos 3t - 3u \;\, du$$

$$= \frac{1}{2}\int_0^t \sin 3t - 2u + \sin(4u-3t)\, du$$

$$= \frac{1}{2}\left[\frac{-\cos(3t-2u)}{-2} - \frac{\cos(4u-3t)}{4}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\cos t}{2} - \frac{\cos t}{4} - \frac{\cos 3t}{2} + \frac{\cos 3t}{4}\right]$$

$$= \frac{1}{2}\left[\frac{\cos t}{4} - \frac{\cos 3t}{4}\right]$$

$$= \frac{1}{8}\left[\cos t - \cos 3t\right] \quad \text{—— (2)}$$

$$y = \frac{1}{2}\left[\frac{t \sin t}{2} - \frac{1}{8}(\cos t - \cos 3t) + 2\cos t\right]$$

$$y = \frac{t \sin t}{4} - \frac{\cos t + \cos 3t}{16} + \cos t$$

$$y = \frac{t \sin t}{4} + \frac{15}{16}\cos t + \frac{\cos 3t}{16}$$

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Date: / /

LP15

4) $y'' + 9y = 18t$ $\quad y(0) = 0$

$y\left(\frac{\pi}{2}\right) = 0$

Sol: $L\{y'' + 9y\} = L\{18t\}$.

$s^2 L\{y\} - s\,y(0) - y'(0) + 9L\{y\} = \frac{18}{s^2}$

Assume $y'(0) = A$.

$s^2 L\{y\} + 9\{y\} - A = \frac{18}{s^2}$

$L\{y\}[s^2 + 9] = \frac{18}{s^2} + A$

$L\{y\}[s^2+9] = \frac{18}{s^2} + A$

$L\{y\} = \frac{18}{s^2(s^2+9)} + \frac{A}{s^2+9}$

$y = L^{-1}\left\{\frac{18}{s^2(s^2+9)} + \frac{A}{s^2+9}\right\}$.

first term

$\frac{18}{s^2(s^2+9)} =$

$L^{-1}\left\{\frac{18}{s^2} \times \frac{1}{s^2+9}\right\} = L^{-1}\left\{\frac{18}{s^2}\right\} \times L^{-1}\left\{\frac{1}{s^2+9}\right\}$

$= 18 \frac{1}{\sqrt{2}} t \times \frac{\sin 3t}{3}$

$= 18t * \frac{\sin 3t}{3}$

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Date: / /

$$= \int_0^t \frac{\sin 3u}{3} \cdot 18(t-u)\, du.$$

$$= 6\int_0^t \sin 3u \cdot t - u\sin 3u\, du.$$

$$= 6\left[-\frac{\cos 3u \cdot t}{3} - \left(\frac{u(-\cos 3u)}{3} + \frac{\sin 3u}{9}\right)\right]_0^t$$

$$= 6\left[-\frac{\cos 3u}{3}\right.$$

—— x ——

$$y = L^{-1}\left[\frac{18 + As^2}{s^2(s^2+9)}\right]$$

$$\frac{18 + As^2}{s^2(s^2+9)} \qquad \text{Put } s^2 = t.$$

$$\frac{18 + At}{t(t+9)} = \frac{B}{t} + \frac{C}{t+9}$$

$$18 + At = Bt + 9B + Ct.$$

$$\boxed{B = 2}$$

$$A = 2 + C$$

$$C = A - 2$$

$$y = L^{-1}\left[\frac{2}{t} + \frac{A-2}{t+9}\right]$$

$$= L^{-1}\left[\frac{2}{s^2} + \frac{A-2}{s^2+9}\right]$$

$$y = 2t + \frac{A-2}{3}\sin 3t$$

$$y\left(\frac{\pi}{2}\right) = 2\left(\frac{\pi}{2}\right) + \frac{A-2}{3}\sin\frac{3\pi}{2}$$

$$9 = \pi - \frac{A-2}{3}$$

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$$0 = 3\pi - (A - 2)$$

$$\boxed{A = 3\pi + 2}$$

$$y = 2t + \frac{3\pi}{3} \sin 3t$$

$$= 2t + \pi \sin 3t$$

L.P

17) $V(t) = 10 \sin 5t$

$$V_{in} = IR + L\frac{dI}{dt}$$

$$10 \sin 5t = 4I + 2\frac{dI}{dt}$$

$$\int 10 \sin 5t - 4I \, dt = \int 2\, dI$$

$$2I' + 4I = 10 \sin 5t$$

$$L\{2I' + 4I\} = L\{10 \sin 5t\}$$

$$2[s^2 L\{I\} - sI\{0\} - I'(0)] + 4L\{I\} = 10\left(\frac{5}{s^2+25}\right)$$

$$L\{I\}[2s^2 + 4] = \frac{50}{s^2+25}$$

$$L\{I\} = \frac{50}{(s^2+25)(2s^2+4)}$$

$$I = L^{-1}\left\{\frac{25}{(s^2+25)(s^2+2)}\right\}$$

$$L^{-1}\left[\frac{25}{s^2+25} \times \frac{1}{s^2+2}\right] = \cancel{5 \sin 5t} * \cancel{\cos}$$

$$= \cancel{\cos\sqrt{2}t} * \cancel{\sin 5t}.$$

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$$= \cancel{\int_0^t \cos\sqrt{2}u \ \sin 5(t-u)\, du}$$

$$= \cancel{\int_0^t \cos\sqrt{2}u \cdot \sin 5t - 5u \, du}$$

$$= \cancel{\frac{1}{2}\int_0^t \sin(}$$

$$L^{-1}\left\{\frac{25}{s^2+25} \times \frac{1}{s^2+2}\right\} = \frac{25}{5}\sin 5t * \frac{1}{\sqrt{2}}\sin\sqrt{2}t.$$

$$= \int_0^t \frac{1}{\sqrt{2}}\sin\sqrt{2}t \times 5\sin 5t-5u \ du$$

$$I = \frac{5}{\sqrt{2}}\int_0^t \sin\sqrt{2}t \cdot \sin(5t-5u)\, du$$

$$= \frac{5}{\sqrt{2}} \cdot \frac{1}{2}\int_0^t \cos(\sqrt{2}t-5t+5u) - \cos(\sqrt{2}t+5t-5u)\, du$$

$$= \frac{5}{2\sqrt{2}}\int_0^t \cos(\sqrt{2}t-5t+5u) - \cos(\sqrt{2}t+5t-5u)$$

$$= \frac{5}{2\sqrt{2}}\left[\frac{\sin(\sqrt{2}t-5t+5u)}{5} + \frac{\sin(\sqrt{2}t+5t-5u)}{5}\right]$$

$$= \frac{5}{2\sqrt{2}}\left[\frac{\sin\sqrt{2}t}{5} + \frac{\sin\sqrt{2}t}{5}\right.$$

$$I = L^{-1}\left\{\frac{25}{(s^2+25)(s^2+2)}\right\}$$

$$= 25\ L^{-1}\left\{\frac{1}{(s^2+25)(s^2+2)}\right\}.$$

$$\frac{1}{(s^2+25)(s^2+2)} = \frac{As+B}{s^2+25} + \frac{Cs+D}{s^2+2}$$

$$1 = As^3 + 2As + Bs^2 + 2B + Cs^3 + 25Cs + Ds^2 + 25D.$$

$$1 = s^3(A+C) + s^2(B+D) + s(2A+25C) + 2B + 25D$$

comparing the coefficients

$A + C = 0 \Rightarrow A = -C$

$B + D = 0 \qquad B = -D$

$2A + 25C = 0 \qquad C = 0 \qquad A = 0$

$2B + 25D = 1$

$-2D + 25D = 1$

$D = \frac{1}{23} \qquad B = \frac{-1}{23}$

$$I = 25\ L^{-1}\left[\frac{-1}{23}\left(\frac{1}{s^2+25}\right) + \frac{1}{23}\left(\frac{1}{s^2+2}\right)\right]$$

$$= \frac{25}{23}\left[\frac{-\sin 5t}{5} + \frac{1}{\sqrt{2}}\sin\sqrt{2}t\right]$$

$=$

Final value theorem :- If $f(t)$ and its first derivative are laplace transformable then $F(\infty) = \lim_{s \to 0} F(s)$

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Initial and ~~Final~~ value theorem.

Initial value theorem :-

If $f(t)$ and its derivatives are laplace transformable then the initial value is $f(0^+) = \lim_{s \to \infty} s\,F(s)$

14)

i) $X(s) = \dfrac{808}{s(s^2+2s+101)}$

sol:- By initial value theorem

$x(0) = f(0^+) = \lim_{s \to \infty} s \cdot X(s)$

$= \lim_{s \to \infty} \cancel{s} \times \dfrac{808}{\cancel{s}(s^2+2s+101)}$

$= \lim_{s \to \infty} \dfrac{808}{s^2+2s+101}$

$= \dfrac{808}{\infty} = 0.$

By final value theorem

$f(\infty) = \lim_{s \to 0} s\,F(s)$

$= \lim_{s \to 0} \dfrac{808}{\cancel{s}(s^2+2s+101)} \times \cancel{s}$

$= \dfrac{808}{101}$

$= 8.$

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iii) $F(s) = \dfrac{s^2+2s+4}{s^3+3s^2+2s}$

Sol: By initial value theorem

$$f(0^+) = \lim_{s\to\infty} s\,X(s)$$

$$= \lim_{s\to\infty} \cancel{s} \times \frac{s^2+2s+4}{\cancel{s}(s^2+3s+2)}$$

$$= \lim_{s\to\infty} \frac{s^2+2s+4}{s^2+3s+2}$$

$$= \lim_{s\to\infty} \frac{\cancel{s^2}}{\cancel{s^2}}\left(\frac{1+2/s+4/s^2}{1+3/s+2/s^2}\right)$$

$$= \frac{1+0+0}{1+0+0}$$

$$= 1.$$

By final value theorem

$$f(\infty) = \lim_{s\to 0} s\,X(s)$$

$$= \lim_{s\to 0} \cancel{s} \times \frac{s^2+2s+4}{\cancel{s}\,(s^2+3s+2)}$$

$$= \frac{4}{2}$$

$$= 2.$$

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Probability

Basic def^n's :-

1) Experiment :-

An experiment is some process or procedure that we do. A process for which outcome cannot be predicted in advance is called random expt or stochastic expt.

2) Sample space :-

It is the set of all possible outcomes of an expt and is also called universal set. It is generally denoted by S.

Ex: a) Rolling a die has sample space

$S = \{1, 2, 3, 4, 5, 6\}$

b) Throwing 2 coins

$S = \{HH, HT, TH, TT\}$

3) Event :-

The outcome of random expt is called an event. Thus every subset of a sample space is called an event. Generally denoted by E.

An event with only one element is called simple event or element ary event. Otherwise it is called compound event. Empty set is a subset of S and it is called null event. The probability of happening of such event is 0 also called as impossible event.

S is a subset of S $(S \subseteq S)$ It is called certain or sure event. Probability of such event is always 1.

→ The total number of outcomes in sample space is called exhaustive no. of cases.

→ Total number of elements of an event E is called favourable no. of cases to the happening of event E.

Ex:- Throwing a die is a random expt. Then sample space = {1, 2, 3, 4, 5, 6}

i) Getting the number 6, then set $E_1 = \{6\}$ (It is a simple ~~elemen~~ event)

ii) Getting an even number, $E_2 = \{2, 4, 6\}$ [It is a compound event.

iii) Getting a number 7, $E_3 = \{0\}$ [null event]

iv) Getting a number less than 7 $E_4 = \{1, 2, 3, 4, 5, 6\}$ It is a sure event.

v) Exhaustive numbers of cases = 6

vi) a) Favourable no of cases $E_1 = 1$, $E_2 = 3$, $E_3 = 0$ $E_4 = 6$

4) Equally likely events - The outcomes of an experiment are equally likely, if they have same chance of occuring.

5) Mutually ~~exc~~ exclusive events - Two events A & B are said to be mutually exclusive if $A \cap B = \{\phi\}$

6) Compliment of an event - Complement of an even A, contains elements of set S, that are not in A and denoted by $\bar{A}$ or A^C.

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7) Probability - Probability of an event E is a number or measure used to measure the chance of occurence of that event and is denoted by P(E). This value lies bet^n 0 & 1.

There are 3 distinct ways of assigning probability

i) Statistical or impirical approach:- If trails be repeated for large number of times N. under same conditions and certain event E happens on f occasions. then probability of event E is given by

$$P(E) = \lim_{N \to \infty} \frac{f}{N}$$

ii) Axiomatic def^n of probability:- Let S be sample space, consisting of N simple elements, the probability of simple event E_i ($E_i \in S$) denoted by $P(E_i)$ is a definite number and satisfies fol^n condition

a) $0 \leq P(E_i) \leq 1$.

b) $P(E_1) + P(E_2) + P(E_3) + \dots + P(E_n) = P(S) = 1$

(iii) Mathematical or classical def^n:- If a random expt can produce N exhausitive mutually exclusive equally likely cases and m of these are favourable to the happening of event E, the probability of happening of event E is given by P(E).

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$$P(E) = \frac{\text{no of F.C}}{\text{no E.N.C}} = \frac{n(E)}{n(S)}$$

NOTE:-

i) If an event E, can happen in m ways out of n exhausitive events, then the favourable number of events for happening of event

$\bar{E} = n-m$

$$P(\bar{E}) = \frac{n-m}{n} = 1-\frac{m}{n} = 1-P(E)$$

$$\boxed{P(E)+P(\bar{E}) = 1}$$

ii) If m+n are exhausitive number of cases, to the happening of ~~event~~, for random expt m are favourable to the happening of event E, then n are favourable to the happening of $\bar{E}$.

Then we say that odds in favour of E is given by m:n

the odds against E is given by n:m

L.P 1)

i) S = {OOO, OOF, OFO, FOO, OFF, FOF, FFO, FFF}

E.N.C = 8

ii) A = {OFF, FOF, FFO, FOO, OFO, OOF, OOO}

B = {OFF, OOO, OFO, OFF}

C = {FFF}

D = ~~Vacant~~ ϕ

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iii) $A \cap B = \{OOO, OOF, OFO, OFF\} \neq \phi$

$\therefore$ A and B are not mutually exclusive

$A \cap C = \phi$; A and C are mutually exclusive

$A \cap D = \phi$ A and D " " "

iv) Impossible event

v) $P(\text{no switch is on}) = \frac{1}{8}$

2) Total number of balls in a bag = 11

a) One ball is randomly drawn :-

E.N.C = 11 (selecting one ball at a time)

$^{11}C_1 = 11$

i) $E_1 \to$ ball is white

F.N.C for $E_1 = {}^5C_1 = 5$

$P(E_1) = \frac{^5C_1}{^{11}C_1} = \frac{5}{11}$

ii) $E_2 \to$ ball is black

F.N.C for $E_2 = {}^6C_1 = 6$

$P(E_2) = \frac{FNC}{ENC} = \frac{6}{11}$

b) Two balls are drawn

$ENC = {}^{11}C_2$

i) $E_3 \to$ both balls are white

F.N.C for $E_3 = {}^5C_2 =$

$P(E_3) = \frac{^5C_2}{^{11}C_2} = \frac{10}{55} = 0.1818.$

ii) $E_4 \to$ both are black

F.N.C for $E_4 = {}^6C_2 =$

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$$P(E_4) = \frac{{}^6C_2}{{}^{11}C_2} = 0.2727$$

iii) $ENC = {}^{11}C_2$

$FNC = {}^5C_2 \cdot {}^6C_1$

$$P(E_5) = \frac{{}^5C_1 \cdot {}^6C_1}{{}^{11}C_2}$$

$= 0.5454$

iv) $ENC = {}^{11}C_2$

$FNC = {}^5C_2 + {}^6C_2$

$$P(E_6) = \frac{{}^5C_2 + {}^6C_2}{{}^{11}C_2}$$

$= 0.4545$

c) i) Two balls are drawn one after other with replacement

$$P(E_1) = \frac{{}^5C_1}{{}^{11}C_1} \cdot \frac{{}^5C_1}{{}^{11}C_1} =$$

ii) $$P(E_2) = \frac{{}^6C_1}{{}^{11}C_1} \cdot \frac{{}^6C_1}{{}^{11}C_1} =$$

iii) $$P(E_3) = \frac{{}^5C_1}{{}^{11}C_1} \cdot \frac{{}^6C_1}{{}^{11}C_1} + \frac{{}^6C_1}{{}^{11}C_1} \cdot \frac{{}^5C_1}{{}^{11}C_1}$$

$=$

iv) $$P(E_4) = \frac{{}^5C_1}{{}^{11}C_1} \cdot \frac{{}^5C_1}{{}^{11}C_1} + \frac{{}^6C_1}{{}^{11}C_1} \cdot \frac{{}^6C_1}{{}^{11}C_1}$$

$= 0.5041$

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d) Two balls are drawn one after other without replacement

i) $P(E_1) = \frac{5}{11} \times \frac{4}{10} = 0.1818$

ii) $P(E_2) = \frac{6}{11} \times \frac{5}{10} = 0.2727$

iii) $P(E_3) = \frac{5}{11} \times \frac{6}{10} + \frac{6}{11} \times \frac{5}{10} = 0.5454$

iv) $P(E_4) = \frac{5}{11} \times \frac{4}{10} + \frac{6}{11} \times \frac{5}{10} =$

3) $E.N.C = {}^{15}C_8$

Total numbers of balls in a bag = 15
Selecting 8 balls at a time = ${}^{15}C_8$

$ENC = {}^{15}C_8$

∴ Now selecting 2 red balls & 6 black balls can be done in ${}^5C_2 \cdot {}^{10}C_6$ ways

$FNC = {}^5C_2 \cdot {}^{10}C_6$

$P(E_1) = \frac{{}^5C_2 \cdot {}^{10}C_6}{{}^{15}C_8}$

$= 0.3263$

4) A = odd numbers
$= \{1, 3, 5, 7, 9\}$

B = even numbers
$= \{2, 4, 6, 8, 10\}$

To get odd sum, we select one card from A & one from B

$E.N.C = {}^{10}C_2$ $F.N.C = {}^5C_1 \cdot {}^5C_1$

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i) $P(E_1) = \dfrac{{}^5C_1 \cdot {}^5C_1}{{}^{10}C_2} = 0.555$

ii) $P(E_2) = \left(\dfrac{5}{10} \cdot \dfrac{5}{9}\right) + \left(\dfrac{5}{10} \cdot \dfrac{5}{9}\right)$

$= \dfrac{25}{90} + \dfrac{25}{90} = \dfrac{50}{90} = 0.555$

iii) $P(E_3) = \left(\dfrac{5}{10} \cdot \dfrac{5}{10}\right) + \left(\dfrac{5}{10} \cdot \dfrac{5}{10}\right)$

$= 0.5$

6) Total number of students = 9

1^{st} year → 2

2^{nd} year → 3

3^{rd} year → 4

$ENC = {}^9C_3$

i) $P(E_1) = \dfrac{{}^2C_1 \cdot {}^3C_1 \cdot {}^4C_1}{{}^9C_3} = \dfrac{2 \times 3 \times 4}{{}^9C_3} = 0.285$

ii) $P(E_2) = \dfrac{{}^2C_2 \cdot {}^3C_1 \cdot {}^4C_0}{{}^9C_3} + \dfrac{{}^2C_2 \cdot {}^3C_0 \cdot {}^4C_1}{{}^9C_3}$

$+ \dfrac{{}^2C_1 \cdot {}^3C_2 \cdot {}^4C_0}{{}^9C_3} + \dfrac{{}^2C_0 \cdot {}^3C_2 \cdot {}^4C_1}{{}^9C_3}$

$+ \dfrac{{}^2C_1 \cdot {}^3C_0 \cdot {}^4C_2}{{}^9C_3} + \dfrac{{}^2C_0 \cdot {}^3C_1 \cdot {}^4C_2}{{}^9C_3}$

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ii) $P(E_2) = \frac{{}^2C_2\,{}^7C_1}{{}^9C_3} + \frac{{}^3C_2\,{}^6C_1}{{}^9C_3} + \frac{{}^4C_2\,{}^5C_1}{{}^9C_3}$

$= 0.6547$

iii) $P(E_3) = \frac{{}^3C_3}{{}^9C_3} + \frac{{}^4C_3}{{}^9C_3}$

$= 0.0595$

7) X : X settles the dispute

Y : Y settles the dispute

$P(X) = \frac{6}{6+8} = \frac{6}{14} = 0.428$

$P(Y) = \frac{14}{16+14} = \frac{14}{30} = 0.4667$

i) P(neither settles dispute) $= P(\bar{X} \cap \bar{Y})$

$= P(\bar{X}) \cdot P(\bar{Y})$

$= \left(1 - P(X)\right) \cdot \left(1 - P(Y)\right)$

$= \frac{8}{14} \cdot \frac{16}{30}$

$= 0.3047$

ii) P(settles the dispute) $= P(X \cup Y)$

$= P(X) + P(Y) - P(X \cap Y)$

$= \frac{6}{14} + \frac{14}{30} - \frac{6}{14} \times \frac{14}{30}$

$= \frac{73}{105} = 0.695$

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4) $P(X \cup Y) = 1 - (P(\overline{X \cup Y}))$

$= 1 - P(\overline{X} \cap \overline{Y})$

$= 1 - P(\overline{X}) . P(\overline{Y})$

5) E :- selecting engineering subject.

A :- selecting first group

B :- selecting second

$P(E) = P(A) . P(E/A) + P(B) . P(E/B)$

$= \frac{2}{6} \times \frac{3}{8} + \frac{4}{6} \frac{5}{8}$

$= \frac{6}{48} + \frac{20}{48}$

$= \frac{26}{48} = \frac{13}{24}.$

8) E :- selecting two engineer's

A :- selecting plant I

B :- selecting plant II

$P(E) = P(A) . P(E/A) + P(B) . P(E/B)$

$= \frac{1}{2}\left(\frac{^5C_1 . ^3C_1}{^8C_2}\right) + \frac{1}{2} . \left(\frac{^4C_1 . ^5C_1}{\cancel{^9C_2} 36}\right)$

$= \frac{1}{2}\left[\frac{15}{28} + \frac{20}{3\cancel{8}6}\right)$

$= \cancel{\frac{1}{2} \times \frac{35}{28}}$

$= \frac{35}{56} = \cancel{0.625}$ 0.545

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Baye's theorem

Let $A = \{A_1, A_2, \ldots A_n\}$ be parition of sample space Ω, with $P(A_i) \geq 0 \quad \forall i$

B is any event

Then $P(A_i/B) = \dfrac{P(A_i).P(B/A_i)}{P(B)}$

where $P(B) = \sum_{i=1}^{n} P(A_i).P(B/A_i)$.

NOTE: Base theorem gives the probability of cause A_i given that effect B has happened.

q). E - Selected output is defective

A - Even output selected from Machine 1

B - " " " " " 2

C - " " " " " 3

i) $P(A/E) = \dfrac{P(A).P(E/A)}{P(E)}$

$P(E) = P(A).P(E/A) + P(B).P(E/B) + P(C).P(E/C)$

$= \dfrac{3000}{10000} \times \dfrac{30}{3000} + \dfrac{2500}{10{,}000} \times 0.012 + \dfrac{4500}{10{,}000} \times 0.02$

≈ 0.015

i) $P(A/E) = \dfrac{P(A).P(E/A)}{P(E)} = \dfrac{3/10 \times 0.01}{0.015} = \dfrac{1}{5} = 0.2$

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$$P(B/E_1) = \frac{P(B).\ P(E/B)}{P(E)}$$

$$= \frac{1/4 \cdot 0.012}{0.015} = \frac{1}{5} = 0.2$$

$$P(C/E_1) = \frac{45/100 \times 0.02}{0.015} = 0.6.$$

M.Q

1) $\rightarrow$ n be the transistor produced by y.
2n be " " " by x &
and n " " " by z
Total number of transistors $= 4n$.

$$P(X) = \frac{1}{2}$$

D → selecting faulty transistor
X → selection of X factory
Y → " " " Y "
Z → " " " Z "

$$P(Y) = \frac{1}{4}$$

$$P(Z) = \frac{1}{4}$$

$$P(D) = P(X).\ P(D/X) + P(Y).\ P(D/Y) + P(Z).\ P(D/Z)$$

$$= \frac{1}{2} \cdot \frac{0.2}{100} + \frac{1}{4} \cdot \frac{0.2}{100} + \frac{1}{4} \cdot \frac{0.4}{100}$$

$$= 0.0019$$

$$P(Y/D) = \frac{1/4 \times 0.002}{0.0019}$$

$$= 0.263$$

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TATANAGAR 2→8

TA AT TA AN NA

1.P 10) E → only 2 consecutive letters TA are visible.

A : Letter comes from Calcutta

B : " " " " Jatanagar

$P(A/E) = \frac{P(A) \cdot P(E/A)}{P(E)}$ CALCUTTA

$= \frac{1/2 \cdot 1/7}{\left(\frac{1}{2}\right)\left(\frac{1}{7}\right) + \left(\frac{1}{2}\right)\left(\frac{2}{8}\right)}$ $P(E) = P(A) \cdot P(E/A)$

$= \frac{4}{11}$

$P(A/E) = 4/11$

1.P 9) $P(M_1) = 3/10$ $P(M_2) = 25/100 = 1/4$

$P(M_3) = \frac{45}{100} = \frac{9}{20}$

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RANDOM VARIABLE:-

A random variable is a rule or funⁿ. that assigns a real number to every sample point in sample space of random experiment.

Generally random variables are denoted by x, y, z .... Further if random variables take discrete values, then it is called discrete random variable. If random variable takes every value on continuous scale then it is called continuous random variable (CRV).

According to the category of random variable we have 2 types of probability distribution.

1) Discrete probability distribution
2) Continuous "

1) Discrete probability distribution :-

If for each value of x_i of a discrete random variable X, we assign a real number $P(x_i)$ such that

i) $P(x_i) \geq 0$.

ii) $\sum P(x_i) = 1$

then the function $P(x_i)$ is called probability mass function (pmf) and $P(x_i) = P(x = x_i) = P_i$ gives the probability of random variable at $x = x_i$ the set of values x_i, $P(x_i)$ is called discrete probability distribution.

counting $\xrightarrow{wrt}$ discrete

measuring $\xrightarrow{wrt}$ measuring

The distribution $F(x_i)$ defined as
$$f(X) = \sum p(x)$$ such that $X \leq x$
is called cumulative distribution funⁿ

The mean of probability distribution of discrete random variable X is given by
$$\mu = \sum_{x} x_i P(x)$$ It is also called

expected value of x denoted by $E(x)$

Variance of distribution is given by
$$\sigma^2 = \sum_{x} x^2 p(x) - \left[\sum x p(x)\right]^2$$

$$= \sum_{x} x^2 p[x] - \mu^2$$ and square root of variance is called standard deviation

Ex- A coin is tossed twice a random variable X denotes number of heads turning up.
i) Find the discrete probability distribution.
ii) " " cumulative " funⁿ of x
iii) " mean, variance, sd.

Sol:- Random experiment:
Tossing coins two times
$$S = \{HH, TT, HT, TH\}$$

Random variable X = no. of heads = 2, 0, 1, 1

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x	2	1	0
$p(x)$	$1/4$	$1/2$	$1/4$
x^2	4	1	0
$x^2 p(x)$	1	0.5	0
$x\,(p(x))$	$1/2$	$1/2$	0

i) cumulative distribution probability

x	0	1	2
$F(x)$	$1/2$	$3/4$	1

$$F(0) = \sum_{x \le 0} p(x) = p(0) = 0$$

$$F(1) = \sum_{x \le 1} p(x) = p(0) + p(1) = 3/4$$

$$F(2) = \sum_{x \le 2} p(x) = p(0) + p(1) + p(2) = 1$$

ii) Mean of distribution is given by

$$\mu = E(x) = \sum_x x\,p(x)$$

$$= 1/2 + 1/2 + 0$$

$$\mu = 1$$

$$\sigma^2 = \sum x^2 p(x) - \mu^2$$

$$= 1 + 0.5 - 1$$

$$\sigma^2 = 0.5 \rightarrow \text{variance}$$

$$\sigma = \sqrt{0.5}$$

$$\sigma = 0.707$$

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Q 12) Random experiment : selecting 3 times from a box of 12

Exhaustive no of cases $^{12}C_3$

X : no of defectives in selection

$X = 0, 1, 2, 3$

X	0	1	2	3
$p(x)$	$^8C_3 / {}^{12}C_3 = 14/55$	$^8C_2 \times {}^4C_1 / {}^{12}C_3 = 28/55$	$12/55$	$1/55$
$x\,p(x)$	0	$28/55$	$24/55$	$3/55$
$x^2 p(x)$	0	$28/55$	$48/55$	$9/55$

Mean $\mu = E(x) = \frac{28}{55} + \frac{24}{55} + \frac{3}{55} = 1$

$S.D = \sum x^2 p(x)$

$= 1\left(\frac{28}{55}\right) + \left(\frac{48}{55}\right) + \frac{9}{55} - 1$

$= 0.545$

$S.D = \sqrt{0.545} = 0.738$

Q.13

x	0	1	2	3	4	5	6
$P(x)$	K	$3K$	$5K$	$7K$	$9K$	$11K$	$13K$

Since given distribution is probability distribution,

we have $K \geq 0$

& $\sum P(x) = 1$.

$$49K = 1$$
$$K = \frac{1}{49}$$

i) $P(x<4) = P(0) + P(1) + P(2) + P(3)$
$$= \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{7}{49}$$
$$P(x<4) = \frac{16}{49}$$

$$P(x \geq 5) = 1 - P(x \leq 4)$$
$$= 1 - \left(\frac{16}{49} + \frac{9}{49}\right)$$
$$= 1 - \frac{25}{49}$$
$$= \frac{24}{49}$$

$$P(3 < x \leq 6) = P(4) + P(5) + P(6)$$
$$= \frac{9}{49} + \frac{24}{49}$$
$$= \frac{33}{49}$$

ii) $P(x \leq 2) > 0.3$
$$P(0) + P(1) + P(2) > 0.3$$
$$K + 3K + 5K > 0.3$$
$$9K > 0.3$$
$$K > \frac{0.1}{3}$$
$$K > \frac{1}{30}$$

Minimum value of K is $\frac{1}{30}$

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4) Given probability distribution is

X	0	1	2
p(x)	$3c^3$	$4c-10c^2$	$5c-1$

$3c^3 + 4c - 10c^2 + 5c - 1 = 1.$

$3c^3 - 10c^2 + 9c - 2 = 0$

$C_1 = 2$, $C_2 = 1$ } $\times$ are not possible. probability becomes >1 which is contradiction

$C_3 = 1/3$ $\therefore$ $C = 1/3$

X	0	1	2
p(x)	1/9	2/9	2/3

ii) $p(X<2) = p(0) + p(1)$
$= \frac{1}{9} + \frac{2}{9} = 3/9 = 1/3$

iii) $p(1 < x \leq 2) = 2/3$

iii) cummulative distribution funⁿ.

X	0	1	2
F(x)	1/9	3/9	9/9 = 1.

$F(0) = \sum_{x \leq 0} p(x) = \frac{1}{9}$

$F(1) = \sum_{x \leq 1} p(x) = 3/9$

$F(2) = \sum_{x \leq 2} p(x) = \cancel{8/9}\ 3/9 + 6/9 = 9/9 = 1.$

* Random variable X has the following probability function, x values 0, 1, 2, 3, 4, 5, 6, 7

x	0	1	2	3	4	5	6	7
$P(x)$	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

Find the K and distribution of x.

Sol $K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K) = 1$

$K(1 + 2 + 3 + K + 2K + 7K + 1) = 1$

$9K + 10K^2 = 1$

$10K^2 + 9K - 1 = 0$

$\boxed{\begin{matrix} K = 0.1 \\ K = -1 \end{matrix}}$ neglecting negative value.

$K = 0.1$

x	0	1	2	3	4	5	6	7
$P(x)$	0	0.1	0.2	0.2	0.3	0.01	0.02	$0.07 + 0.1 = 0.17$
$F(x)$	0	0.1	0.3	0.5	0.8	0.81	0.83	1

* Continuous probability distribution:-

If X is a continuous random variable if each $x \in X$, we assign a real number $f(x)$ satisfying the conditions

$f(x) \geq 0 \quad \forall x$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

then $f(x)$ is called continuous probability function or probability density fun^n

- Cummulative distribution fun^n - If X is CRV with probability density fun^n f(x) is ~~given~~ by then CDF is given by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\, dx.$$

- Mean and variance of CPD are given by

$$\mu = E(x) = \int_{-\infty}^{\infty} x f(x)\, dx.$$

$$\text{Variance}(x) = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

$$S.D = \sqrt{Var(x)}$$

NOTE :-

1) If X is a continuous random variable $P(a \le x \le b)$ is given by $\int_a^b f(x)\, dx$.

$P(a \le x)$ is given by $\int_a^{\infty} f(x)\, dx$.

$P(x \le b)$ is given by $\int_{-\infty}^{b} f(x)\, dx$

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Find the constant k such that

$$f(x) = kx^2 \qquad 0 < x < 3$$
$$= 0 \qquad \text{otherwise}$$

is a probability density funⁿ. also compute probability of

i) $P(1 < x < 2)$

ii) $P(x \leq 1)$

iii) $P(x \leq 2)$ iv) mean, variance

v) cummulative distribution funⁿ.

Sol. The given $f(x) = kx^2 \quad ; 0 < x < 3$
$= 0 \quad$ otherwise

is a continuous probability funⁿ

$\therefore \quad k > 1$

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_{-\infty}^{0} f(x)\, dx + \int_{0}^{3} f(x) dx + \int_{3}^{\infty} f(x) dx = 1$$

$$0 + \left.\frac{kx^3}{3}\right|_0^3 + 0 = 1$$

$$9k = 1$$

$$\boxed{k = 1/9}$$

i) $P(1 < x < 2) = \int_1^2 f(x)\, dx$

$$= \int_1^2 kx^2\, dx$$

$$= \left.\frac{kx^3}{3}\right|_1^2$$

$$= \frac{1}{9}\left(\frac{8}{3} - \frac{1}{3}\right)$$

$$= \frac{7}{27}.$$

ii) $P(x \leq 1) = \int_{-\infty}^{1} f(x)\,dx$

$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$

$= 0 + \frac{kx^3}{3}\Big|_0^1$

$= \frac{1}{27}.$

iii) $P(x \geq 2) = \int_{-\infty}^{\infty} f(x)\,dx.$

$= \int_{+3}^{\infty} f(x)\,dx + \int_{2}^{3} f(x)\,dx$

$= 0 + \frac{kx^3}{3}\Big|_2^3$

$= -\frac{1}{9} \times \left(\frac{8}{3}\right) + \frac{27}{9 \times 3}$

$= \frac{8}{27} \quad \frac{27}{27} - \frac{8}{27} = \frac{19}{27}.$

iv) $\mu = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$= \int_{-\infty}^{0} x f(x)\,dx + \int_{0}^{3} x f(x)\,dx + \int_{3}^{\infty} x f(x)\,dx$

$= 0 + \int_{0}^{3} kx^3\,dx + 0$

$= \frac{kx^4}{4}\Big|_0^3$

$= \frac{81}{4} \times \frac{1}{9} = \frac{9}{4}.$

v) Variance $= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

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$$= \int_0^3 kx^4\,dx - \frac{81}{16}$$

$$= \frac{kx^5}{5}\Big|_0^3 - \frac{81}{16}$$

$$= \frac{1}{\cancel{9}_1}\left(\frac{\cancel{81}^{9}\times 3}{5}\right) - \frac{81}{16}$$

$$= \frac{9\times 3}{5} - \frac{81}{16}$$

v) C.D.F

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

case 1) $x \in (-\infty, 0)$ $f(x) = 0$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx$$

$$= 0$$

case 2) $x \in (0, 3)$, $f(x) = kx^2$

$$F(x) = \int_{-\infty}^{0} \cancel{\ldots} \int_0^3 \cancel{kx^3\,dx}$$

$$= 0 + \frac{1}{9}\left[\frac{\cancel{x^3}}{3}\right]_0^3$$

$$= \frac{\cancel{27}}{\cancel{27}} = 1$$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_0^x kx^2\,dx$$

$$= \frac{1}{9} \times \frac{x^3}{3}$$

$$= \frac{x^3}{27}$$

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case 3 $x \in (3, \infty)$

$$F(x) = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{0} \cancel{f(x)}\,dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{x} \cancel{f(x)}\,dx$$

$$= \int_{0}^{3} Kx^2\,dx$$

$$= \frac{1}{9}\left(\frac{27}{3}\right)$$

$$= 1$$

Thus C.D.F

$$F(x) \begin{cases} = 0 & -\infty < x < 0 \\ = x^3/27 & 0 < x < 3 \\ = 1 & x > 3 \end{cases}$$

Q If probability density funⁿ for cotⁿ random funⁿ is given by

$$f(x) = |x| \qquad -1 < x < 3$$

$$= 0 \qquad \text{otherwise}$$

Find probability of i) $P(0 < x < 3)$

ii) $P(2x+1 > 2)$

Sol: $P(0 < x < 3) = \int_0^3 x\,dx$

$$= 9/2$$

$P(2x+1>2) = P(2x > 1)$

$$= P(x > 1/2) = \int_{1/2}^{3} f(x)\,dx$$

$$= \left.\frac{x^2}{2}\right|_{1/2}^{3} = \frac{9}{2} - \frac{1}{8} = \frac{35}{8}$$

$$\lim_{x\to\infty} x^2 \cdot e^{-ax} = 0.$$

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The lifetime in hours of electronic tubes is a random variable with probability of density funⁿ given by

$$f(x) = a^2 x e^{-ax} \qquad x \geq 0$$

1) compute expected life time of such tube

2) find probability of it $\left(\frac{1}{a} < x < \frac{2}{a}\right)$

Sol:- Given :- $f(x) = a^2 x e^{-ax}$

Expected life time of tube $= \mu$

$$\mu = \int_{-\infty}^{\infty} f(x) \cdot x \, dx.$$

$$= \int_0^{\infty} x^2 a^2 \cdot e^{-ax} \, dx$$

$$= a^2 \left[\frac{x^2 e^{-ax}}{-a} - \frac{2x e^{-ax}}{a^2} + \frac{2e^{-ax}}{-a^3} \right.$$

$$= \cancel{a^2} \left[\frac{\cancel{a^2 e^{-ax}}}{\cancel{-a}} - \frac{\cancel{2ae^{-ax}}}{\cancel{a^2}} + \frac{\cancel{2e^{-ax}}}{\cancel{-a^3}} \right.$$

$$= a^2 \left[\frac{x^2 e^{-ax}}{-a} - \frac{2x e^{-ax}}{a^2} + \frac{2e^{-ax}}{-a^3} \right]_0^{\infty}$$

$$= a^2 \left[0 - 0 + \frac{1}{\infty} + \frac{2}{a^3} \right]$$

$$= \frac{2}{a^2}.$$

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ii) $P\left(\frac{1}{a} < x < \frac{2}{a}\right) = \int_{1/a}^{2/a} f(x)\,dx$

$$= \int_{1/a}^{2/a} a^2 x e^{-ax}\,dx$$

$$= a^2 \left[\frac{xe^{-ax}}{-a} - \frac{e^{-ax}}{a^2} \right]_{1/a}^{2/a}$$

$$= a^2 \left[\frac{2e^{-2}}{-a^2} - \frac{e^{-2}}{a^2} + \frac{e^{-1}}{a^2} + \frac{e^{-1}}{a^2} \right]$$

$$= 2e^{-1} - 3e^{-2}$$

L.P.

15) Given lifetime of one tube is given by probability density funⁿ

$$f(x) = \frac{100}{x^2} \quad x \geq 100$$

$$= 0 \quad \& \text{ otherwise}$$

Probability for a tube S.T it is replaced before 150 hours is given by

$$P(x < 150) = \int_{-\infty}^{150} f(x)\,dx$$

$$= \int_{100}^{150} f(x)\,dx$$

$$= \int_{100}^{150} \frac{100}{x^2}\,dx$$

$$= 100 \left[\frac{-1}{x} \right]_{100}^{150} = 100 \left(\frac{-1}{150} + \frac{1}{100} \right)$$

$$= 100 \left(\frac{-10 + 15}{1500} \right) = \frac{1}{3}$$

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Now, if we use 3 tubes B_1, B_2, B_3 then the probability that all 3 tubes replaced before 150 is given

$$P(B_1 \cap B_2 \cap B_3) = P(B_1) \cdot P(B_2) \cdot P(B_3)$$
$$= \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}.$$

The probability ~~for~~ of a tube that is not replaced before 150 hours is

$$P = 1 - P(\text{tube is replace before 150 h})$$
$$= 1 - \frac{1}{3}$$
$$= \frac{2}{3}$$

∴ If we use 3 independent tubes then probability that all tubes are not replaced

$$P = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$$
$$= \frac{8}{27}.$$

Theoretical Distributions

1) Binomial distribution :- A random experiment with only two possible outcomes categorised as success and failure is called binomial experiment.

Probability distribution of binomial random variable is given by.

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$p(x) = {}^nC_x p^x q^{n-x}$

$n \rightarrow$ size of sample

$x \rightarrow 0, 1, 2, \ldots\ldots n.$

$p \rightarrow$ probability of success in single trial

$q \rightarrow 1-p$ = probability of failure " " ".

and p^x gives Probability of x success in one sample

NOTE :- 1) In binomial distribution parameters are n and p. If x follows the binomial variable

$$X \sim B(n, p)$$

2) In binomial distribution mean = np

variance = npq

3) Generally this distribution is applied if $n < 30$.

4) mean = np

5) variance = npq

16) $X \rightarrow$ number of defective items

By binomial distribution,

Probability of x success in one sample $\Big\}$ $p(x) = {}^nC_x p^x . q^{n-x}$

$n = 12$

$p = \frac{1}{10}$ $\quad q = \frac{9}{10}.$

$\therefore p(x) = {}^{12}C_x \left(\frac{1}{2}\right)^x \left(\frac{9}{10}\right)^{12-x}.$

i) $P(2) = {}^{12}C_2 \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^{10}.$

$= 66 \left(\frac{1}{100}\right) \left(\frac{9}{10}\right)^{10}$

$= 0.2301$

ii) $P(2 \le x) = P(2) + P(3) + \dots P(12)$

$= 1 - [P(1) + P(0)]$

$= 1 - \left[{}^{12}C_1 \left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^{11} + {}^{12}C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{12}\right]$

$= \cancel{0.1822} \quad 0.3409.$

iii) $P(x=0) = {}^{12}C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{12}$

$= 0.2824.$

20) By binomial distribution

$P(x) = {}^nC_x \, p^x q^{n-x}$

$n = 4.$

$p = 0.6$

$q = 0.4.$

$P(x) = {}^4C_x (0.6)^x (0.4)^{4-x}.$

$P(x \ge 2) = 1 - [P(0) + P(1)]$

$= 1 - [(0.6)^0 (0.4)^4 + 4(0.6)(0.4)^3]$

$= 0.8208.$

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17) $X \rightarrow$ number of defective parts

$p \rightarrow \frac{2}{20} = \frac{1}{10}$

$q = \frac{9}{10}$

$n = 20$

$\therefore$ Probability of x defectives in one sample using the binomial distribution

$$P(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$$
$$= {}^{20}C_{x}\left(\frac{1}{10}\right)^{x}\left(\frac{9}{10}\right)^{20-x}$$

i) P(atleast 3 defectives) $= P(x \geq 3)$

$$= 1 - P(x < 3)$$
$$= 1 - \left[P(x=0) + P(x=1) + P(x=2)\right]$$
$$= 1 - \left[{}^{20}C_{0}\left(\frac{9}{10}\right)^{20} + {}^{20}C_{1}\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^{19} + {}^{20}C_{2}\left(\frac{1}{10}\right)^{2}\left(\frac{9}{10}\right)^{18}\right]$$
$$= 1 - \left[\left(\frac{9}{10}\right)^{20} + \frac{20\,(9)^{19}}{(10)^{20}} + \frac{{}^{20}C_{2}\,(9)^{18}}{(10)^{20}}\right]$$

$$= 0.32307$$

$\therefore$ This implies propability of atleast 3 defective in one sample is 0.32307. Hence if we use 1000 such ~~econmy~~ samples $\therefore$ Expected no. of samples with atleast 3 defectives is given by $= 1000 \times 0.32307$

$$= 323.07$$
$$\approx 323.$$

19). X – It denotes number of machines need adjustment in a day

total number of machines = 10.
∴ 3 machines are old and 7 are new

$P_1(x) \Rightarrow$ probability of x success with respect to old machine

$P_2(x) \Rightarrow$ probability of x success w.r.t to new machine

By Binomial distribution, the probability distribution w.r.t to old and new machines are given by :-

$$P_1(x) = {}^nC_x (p)^x q^{n-x}$$
$$= {}^3C_x \left(\frac{1}{11}\right)^x \left(\frac{10}{11}\right)^{3-x}.$$

$$P_2(x) = {}^7C_x \left(\frac{1}{21}\right)^x \left(\frac{20}{21}\right)^{7-x}.$$

i) Probability of just 2 old and no new machines need adjustment

P(~~A~~ ~~pro~~) = P(2 old machines need adjustment) . P(no new machine need adjustment)

$$= {}^3C_2 \left(\frac{1}{11}\right)^2 \left(\frac{10}{11}\right) \cdot {}^3C_0 \left(\frac{20}{21}\right)^7$$
$$= (0.022) \cdot (0.710)$$
$$= 0.015.$$

while calculating n

If p is nearer 0 poisson

$p > 0.1$ Binomial.

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ii) P(just 2 machines need adjustment which are of same type)

$= P\left(\begin{matrix}\text{old machine} \\ \text{need adjustment}\end{matrix}\right)^{2} + P\left(\begin{matrix}\text{new machine} \\ \text{need adjustment}\end{matrix}\right)^{2}$

$$= \left[\left({}^{3}C_2\left(\frac{1}{11}\right)^2\left(\frac{10}{11}\right)\right)\left(\frac{20}{21}\right)^7 + \left({}^{7}C_2\left(\frac{1}{21}\right)^2\left(\frac{20}{21}\right)^5 \times \left(\frac{10}{11}\right)^3\right)\right]$$

$= 0.0160 + 0.028$

$= 0.044$

18) X = denotes the number of bombs hitting the target

Given $P = 0.5$

we apply binomial distribution

suppose we hit n bombs on target

$$P(x) = {}^{n}C_x\, p^x\, q^{n-x}$$

$$= {}^{n}C_x (0.5)^x (0.5)^{n-x}$$

$$= {}^{n}C_x (0.5)^n$$

Here the target destroys completely if atleast 2 successive bomb hits the target.

Here we need to find value of n.

$P(x \geq 2) \leq$ ~~...~~ 0.99

$0.99 \leq 1 - P(x < 2)$

$\leq 1 - (P(0) + P(1))$

$$P(x=0) + P(x=1) \le 1 - 0.99$$
$$^nC_0 (0.5)^n + {}^nC_1 (0.5)^n \le 0.01$$

$$(1+n)(0.5)^n \le 0.01$$
$$(1+n)\left(\frac{1}{2}\right)^n \le 0.01$$

$$(1+n) \le 2^n \cdot 0.01$$
$$2^n \ge \frac{1+n}{0.01}$$

Here we need to find min value of n for which inequality holds good

$$\therefore n = 11.$$

Hence we require min 11 hits to destroy the target 99% or above.

Poisson distribution :-

The discrete random variable X is said to have poisson distribution with parameter λ, if probability distribution of X is given by

$$P(x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

where $x = 0, 1, 2, 3 \ldots$

λ = mean number of success in given interval.

NOTE :- 1) The mean and variance of poisson distribution are equal.

(mean = variance = λ)

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2) poisson distribution is limiting case of binomial distribution as $n \to \infty$ & $p \to 0$

3) λ = parameter = constant = np.

21) $X \to$ number of individuals will get bad reaction

$p = 0.001$ (nearer to zero)

By Poisson distribution

$$P(x \text{ success}) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

Here $n = 2000$ $\quad p = 0.001$

$\therefore \lambda = np$

$= 2$

$$P(x) = \frac{e^{-2} \cdot (2)^x}{x!}$$

$$P(x \geq 2) = 1 - P(x \leq 2)$$

$$= 1 - [P(x=0) + P(x=1) + P(x=2)]$$

$$= 1 - \left[e^{-2} + e^{-2}(2)^1 + \frac{e^{-2}(2)^2}{2}\right]$$

$$= 0.3233$$

22) $X \to$ number of blades defective

$p = 0.002$

$n = 10$

$\lambda = np$

$= 0.02$

mean number of defectives out of 10 $\lambda = 0.02$

$$P(x) = \frac{e^{-0.02}(\lambda)^x}{x!}$$

$$P(x) = \frac{e^{-0.02}(0.02)^x}{x!}$$

$P(x=0) = e^{-0.02}$
$= 0.981$ in 1 packet
$\therefore$ in 10,000 $= 9801.98$

$P(x=1) = e^{-0.02}(0.02)^1$
$= 0.0196$

25) In a quality control process 0.2 rejects are done per minute on the average. Find the probability of rejecting
i) one in 3 min
ii) atleast 2 in 5 min
iii) atmost 1 in 15 min.

Sol: X: no. of rejects/min
Since 0.2 rejects in one min.
Expected No. of rejections in 3 min.
$\lambda = np$
$= 3 \times 0.2$
$\lambda = 0.6$

:) $P(x) = \frac{e^{-0.6}(0.6)^x}{x!}$

i) $\therefore$ P(one rejection in 3 min) $= \frac{e^{-0.6}(0.6)^1}{1!}$
$\cdot P(1)$ $=$
$= 0.329$

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ii) Expected number of rejections in 5 min = 5 × 0.2 = 1

$$p(x) = \frac{e^{-1}(1)^x}{x!}$$

$$p(x \geq 2) = 1 - p(x < 2)$$

$$= 1 - [p(x=0) + p(x=1)]$$

$$= 1 - (e^{-1} + e^{-1})$$

$$= 1 - \frac{2}{e}$$

$$= 0.2642$$

iii) Expected number of rejections in 15 min

$$= 15 \times 0.2 = 3$$

$$p(x) = \frac{e^{-3}(1)^x}{x!}$$

$$p(x = 1) = \frac{e^{-3}(3)^1}{}$$

$$= 0.149$$

23) X :- no. of calls wrongly connected

Here we have to find the value of n.

$P(x \geq 1) = 0.9$

Since p is nearer to zero, we apply poisson distribution.

By poisson distribution

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

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since $P(x \geq 1) = 0.9$

$1 - P(x=0) = 0.9$

$1 - 0.9 = P(x=0$

$0.1 = P(x=0)$

$= \frac{e^{-\lambda}(\lambda)^0}{0!}$

$0.1 = e^{-\lambda}$

$e^{\lambda} = \frac{1}{0.1}$

$\boxed{\lambda = 2.302}$

$\lambda = np$

$\frac{\lambda}{p} = n$

$\frac{2.302}{0.001} = n$

$n = 2302$

$\therefore$ 2302 calls are wrongly connected.

24)

x	0	1	2	3	4	
f	122	60	15	2	1	$\Sigma f = 200$

For poisson fit $P(x) = \frac{e^{-\lambda}(\lambda)^x}{x!}$

λ is mean/average value

$\therefore$ This average value, we calculate from the given table

$$\text{Mean} = \frac{\Sigma f x}{\Sigma f}$$

$$= \frac{60+30+6+4}{200} = 0.5$$

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$$p(x) = \frac{e^{-0.5}(0.5)^x}{x!}$$

$$p(x=0) = e^{-0.5}$$
$$= 0.606$$

$$p(x=1) = e^{-0.5} \times 0.5$$
$$= 0.303$$

$$p(x=2) = \frac{e^{-0.5} \times 0.25}{2}$$
$$= 0.075$$

$$p(x=3) = \frac{e^{-0.5} \times (0.5)^3}{6}$$
$$= 0.0126$$

$$p(x=4) = \frac{e^{-0.5} \times (0.5)^4}{24}$$
$$= 1.57 \times 10^{-3}$$
$$= 0.00157$$

x	f	$p(x)$	Expected value $\Sigma f x \cdot P(x)$	
0	122	0.606	121.2	$\approx$ 121
1	60	0.303	60.6	61
2	15	0.075	15	15
3	2	0.0126	2.52	3
4.	1.	0.00157	0.314	0
	200			Σ 200

Since observed frequency is approximately equal to theoretical frequencies

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Date: / /

$\therefore$ Poisson fit is good fit for the exp

$\Rightarrow$ Fit Binomial Distribution

$x = 0 \ldots n.$

$n = 4.$

$\text{mean} = 0.5$

$np = 0.5$

$p = \frac{0.5}{4}$

$p = 0.125 = 1/8$

$q = 7/8$

$$P(x) = {}^4C_x \left(\frac{1}{8}\right)^x \left(\frac{7}{8}\right)^{4-x}.$$

i) $P(x=0) = {}^4C_0 \left(\frac{1}{8}\right)^0 \left(\frac{7}{8}\right)^4$

$= \left(\frac{7}{8}\right)^4$

$= 0.5861.$

ii) $P(x=1) = {}^4C_1 \left(\frac{1}{8}\right)^1 \left(\frac{7}{8}\right)^3$

$= \frac{343}{1024} = 0.3349$

iii) $P(x=2) = {}^4C_2 \left(\frac{1}{8}\right)^2 \left(\frac{7}{8}\right)^2.$

$= \frac{147}{2048}$

$= 0.07177$

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Date: / /

iv) $p(x=3) = {}^4C_3 \left(\frac{1}{8}\right)^3 \left(\frac{7}{8}\right)$

$= \frac{7}{1024}$

$= 0.00683$

v) $p(x=4) = (1/8)^4$

$= 0.000244$

x	f	$p(x)$	Expected value. $\Sigma f(x) \cdot p(x)$
0	122	0.5861	71.504 $\approx$ 72
1	60	0.3349	20
2	15	0.07177	1.07 $\approx$ 1
3	2	0.00683	0.01366
4.	1	0.000244	0.000244.
	200		

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Date: / /

Continuous probability distribution

i) Uniform distribution – Uniform distribution is a constant probability distribution for which the probability density funⁿ over the interval (a, b) is given by

$$f(x) = \frac{1}{b-a} \quad \text{if } a < x < b$$

$$= 0. \quad \text{otherwise}$$

Mean of distribution is given by

$$\mu = \frac{a+b}{2}$$

Variance of distribution is

$$V = \frac{(b-a)^2}{12}$$

Ex:- suppose x is a continuous random variable with uniform distribution over the interval $(-\alpha, \alpha)$. Find α such that i) $P(x > 1) = 1/3$

ii) $P(|x| < 1) = P(|x| > 1)$

Sol. Since $f(x)$ is uniformly distributed over $(-\alpha, \alpha)$ density funⁿ $f(x)$ is given

$$f(x) = \frac{1}{\alpha+\alpha} \quad -\alpha < x < \alpha$$

$$= 1/2\alpha$$

$$= 0 \quad \text{otherwise}$$

i) To find α for $P(x > 1) = 1/3$

$$1 - P(x \leq 1) = 1/3$$

$$1 - 1/3 = P(x < 1)$$

$$\frac{2}{3} = \int_{-\alpha}^{1} f(x)\, dx$$

Date 24 / 9 / 18

Probability

1) A resistor R, is a random variable with uniform distribution between (900, 1100 Ω) determine the probability that R is betⁿ 950 Ω and 1050 Ω. What are mean and variance of distribution.

Sol:- R follow uniform distribution over (900, 1100) its pdf is

$$f(x) = \frac{1}{1100-900}$$

$$= \frac{1}{200} \quad \text{if} \quad 900 < x < 1100$$

$$= 0 \quad \text{otherwise}$$

i) $$P(950 < x < 1050) = \int_{950}^{1050} \frac{1}{200}\, dx$$

$$= \frac{1}{200} \times 100$$

$$= 0.5$$

ii) $$\text{Mean} = \frac{a+b}{2}$$

$$= \frac{900+1100}{2}$$

$$= 1000$$

iii) $$\text{Variance} = \frac{(b-a)^2}{12}$$

$$= 3333.33$$

2) If X is uniformly distributed with mean = 1 and variance = 4/3 Find

Date ___/___/___

probability of $x<0$.

Sol: suppose X is uniformly distributed over the interval (a,b) if $a<x<b$

its pdf $f(x) = \frac{1}{b-a}$

$= 0$ otherwise

Given mean $= 1$

$\frac{a+b}{2} = 1$

$a+b = 2$ — ①

Variance $= \frac{4}{3}$

$\frac{(b-a)^2}{12} = \frac{4}{3}$

$(b-a)^2 = 16$

$b-a = 4$ — ②.

on solving ①, ②

$4+2a = 2$

$a = -1$

$b = 3$

∴ density function is given by

$f(x) = \frac{1}{4}$, $-1 < x < 3$

$= 0$ otherwise

$P(x<0) = \int_{-\infty}^{0} f(x)\,dx$

$= \int_{-1}^{0} f(x)\,dx$

$= \frac{1}{4}[0+1]$

$= 1/4$

Page No. ___

Saathi

Date ___ / ___ / ___

3) Find the cummulative dist^n fun^n for uniform distribution

Ans:- For uniform distr^n of random variable X

$f(x) = \frac{1}{b-a}$ $a < x < b$

$= 0$ otherwise

Let $F(x)$ denotes cummulative dist^n probability distr^n.

$F(x) = P(X \leq x)$

$= \int_{-\infty}^{x} f(x)\,dx$

case 1:- $x \in (-\infty, a)$ then $f(x) = 0$.

$F(x) = \int_{-\infty}^{x} f(x)\,dx$

$= 0.$

case 2:- $x \in (a, b)$ then $f(x) = 1/b-a$

$F(x) = \int_{a}^{x} \frac{1}{b-a}\,dx$

$= \frac{x-a}{b-a}$

$= x-a / b-a$

case 3 :- $x \in (b, \infty)$ then $f(x) = 0$

$F(x) = \int_{-\infty}^{x} f(x)\,dx$

$= \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx + \int_{b}^{x} f(x)\,dx$

$= 0 + \frac{b-a}{b-a} + 0$

$= 1$

$\therefore$ $F(x) = 0$ $-\infty < x < a$

$= \frac{x-a}{b-a}$ $a < x < b$

$= 1$ $b < x < \infty$

Date ___ / ___ / ___

Exponential distribution

A continuous random variable X is said to have exponential distribution with α (known as rate parameter). If X has the probability density funⁿ &

$$f(x) = \alpha e^{-\alpha x} \quad x \geq 0$$
$$= 0 \quad \text{otherwise}$$

$X \sim exp(\alpha)$

Mean and variance of exponential distrⁿ are $\frac{1}{\alpha}$ and $\frac{1}{\alpha^2}$ resp.

Ex:- 1) Length of time a person speaks over a phone follows exponential distribution with mean = 6 min. What is the probability that a person will talk for

i) more than 8 min

ii) betⁿ 4 & 8 min

iii) 8 min or less

Sol X denotes length of time of telephone conversation of a person.

X follows exponential distⁿ

Mean = 6

P. D. f = $f(x) = \alpha e^{-\alpha x} \quad x \geq 0$
$= 0 \quad$ otherwise

Mean = 6

$$\frac{1}{\alpha} = 6$$

Page No.

Date ___/___/___

$\alpha = 1/6$

$$f(x) = 1/6\, e^{(-1/6)x} \quad x \geq 0$$
$$= 0 \quad \text{otherwise}$$

i) $P(x>8) = \int_8^{\infty} f(x)\,dx$

$$= \left(\frac{1}{6} \frac{e^{-1/6\,x}}{-1/6} \right)_8^{\infty}$$

$$= +e^{-8/6}$$

$$P(x>8) = \frac{+1}{e^{4/3}}$$

$$= 0.2636$$

ii) $P(4 < x < 8) = \int_4^8 f(x)\,dx$

$$= \left(\frac{1}{6} \frac{e^{-1/6\,x}}{-1/6} \right)_4^8$$

$$= (-1)\left[e^{-4/3} - e^{-2/3} \right]$$

$$= 0.2498$$

iii) $P(x \leq 8) = \int_0^8 f(x)\,dx$

$$= 1 - \int_8^{\infty} f(x)\,dx$$

$$= 1 - 0.2636$$

$$= 0.7364$$

2) In a construction site, 3 lorry's unload material per hour on an average, What is the probability that the time

Page No.

Date ___ / ___ / ___

bet^n arrival of successive lorry's will be
i) atleast 30 min
ii) less than 10 min

Sol X denotes time bet^n number of lorrys arrive per hour.

Since lorries unload per hour

$$\alpha = \lambda = 3$$

Hence $f(x) = \alpha e^{-\alpha x} = 3e^{-3x}$ $\quad x \geq 0$

$= 0. \quad = 0$ $\quad$ otherwise

i) 30 min = 0.5 hrs

$\therefore$ P(time bet^n arrival of successive lorries will be atleast 30 min)

$= P(x \geq 0.5)$

$= P(x \geq 0.5)$

$$P(x \geq 0.5) = \int_{0.5}^{\infty} 3e^{-3x}\, dx$$

$$= \left(-e^{-3x}\right)_{0.5}^{\infty}$$

$= 0 + 0.223$

$= 0.2231$

ii) 10 min = 1/6 hrs

$$P(x < 1/6) = \int_0^{1/6} 3e^{-3x}\, dx$$

$$= \left(-e^{-3x}\right)_0^{1/6}$$

$= -e^{-1/2} + 1$

$= 0.3934$

Saathi

Date __ / __ / __

3) Jobs are sent to a printer at an average of 3 jobs per hour.

i) What is the expected time betⁿ jobs?

ii) What is the probability that next job is sent within 5 min?

Sol. X = number of jobs are sent to a printer

$\alpha = 3$

p.d.f $\quad f(x) = 3e^{-3x} \qquad x \geq 0$

$\qquad = 0 \qquad$ otherwise

i) expected time = mean = $1/\alpha = 1/3$

ii) $P(x < 1/12) = \cancel{\int_{1/12}^{\infty} f(x)\,dx} = \int_0^{1/12} f(x)\,dx$

$= \cancel{(-1)\, e^{-3x} \Big|_{1/12}^{\infty}} = \left(-e^{-3x}\right)_0^{1/12}$

$= \cancel{0 + e^{-1/4}} = 1 - e^{-1/4}$

$= 0.221$

4) The time in hours required to repair is an exponential distribution with $\alpha = 1$

i) What is the probability that time exceeds 2 hours?

ii) What is the conditional probability that repair takes atleast 3 hours, that its duration exceeds two hours?

Sol X - time in hours required to repair

$\alpha = 1$

$f(x) = e^{-x} \qquad x \geq 0$

$\quad = 0 \qquad$ otherwise

i) $P(x > 2) = \int_2^{\infty} f(x)\,dx = \left[\frac{e^{-x}}{-1}\right]_2^{\infty} = e^{-2}$

$= 0.135$

Page No.

Date

ii) $P(x>3/x>2) = \frac{P(x>3) \cap P(x>2)}{P(x>2)}$

$= \frac{P(x>3)}{P(x>2)}$

$= \frac{\int_3^\infty e^{-x}\,dx}{\int_2^\infty e^{-x}\,dx}$

$= \frac{e^{-3}}{e^{-2}}$

$= 0.3678$

3)

$\lambda = 4$ per minute)

$P(x \le 0.5) = \int_0^{0.5} 4e^{-4x}\,dx$

$= \frac{4e^{-4x}}{-4}\Big]_0^{0.5}$

$= [1 - e^{-4(0.5)}]$

$= 0.864$

$P(x \le k) = 0.95$

$0.95 = \int_0^k 4e^{-4x}\,dx$

$= (-1)[e^{-4k} - 1]$

$0.95 = 1 - e^{-4k}$

$e^{-4k} = 1 - 0.95$

$e^{-4k} = 0.05$

$-4k = -2.995$

$k = 0.7489$

$\therefore$ The maximum waiting time between 2 jobs is 0.75 mins $\approx$ 45 seconds

Page No.

Saathi

Date ___ / ___ / ______

L.P 30)

Given :- mean = 10 hour

$\lambda = \frac{1}{10}$

$\lambda = 0.1$ hours

i) $P(x \geq 12) = \int_{12}^{\infty} 0.1 e^{-0.1x} dx$

$= \frac{0.1}{-0.1} \left[e^{-0.1x} \right]_{12}^{\infty}$

$= (-1)(0 - e^{-0.1 \times 12})$

$= 0.30119$

ii). $P(x > 12 / x > 8) = \frac{P(x > 12) \cap P(x > 8)}{P(x > 8)}$

$= \frac{P(x > 12)}{P(x > 8)}$

$= \frac{\int_{12}^{\infty} 0.1 e^{-0.1x} dx}{\int_{8}^{\infty} 0.1 e^{-0.1x} dx}$

$= \frac{0.30119}{(-1)(-e^{-0.1 \times 8})}$

$= \frac{0.30119}{0.4493}$

$= 0.6703.$

L.P 29) $P(x \geq 4) = \int_{4}^{\infty} 3 e^{-3x} dx$

$= (-1)(-e^{-3 \times 4})$

$= 6.144 \times 10^{-6}$

Page No.

Saat

Date __ / __ / ____

$\alpha = 3$

Mean $= 1/\alpha$

$= 1/3$

$= 0.333$

Variance $= \dfrac{1}{\alpha^2}$

$= 0.1111$

S.D $= \sqrt{\text{variance}}$

$= 0.333$

Normal Distribution

It is a continuous probability distributi with probability density funⁿ given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where x varies from $-\infty < x < \infty$

$-\infty < \mu < \infty$

$\sigma > 0$

in this distribution μ and σ are paramet

$\therefore$ x follows normal distribution

NOTE 1) mean of distribution $= \mu$

2) S.D $= \sigma$

3) The graph of probability funⁿ is bell shaped. symmetrical with respect to the line $x = \mu$ are the total area under curve $= 1$. Hence the area to left and right of $x = \mu$ is 0.5.

4) To solve problems on normal distributio we convert given normal variate in.

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Date __/__/__

standard variate with substitution

$$z = \frac{x-\mu}{\sigma}$$

Then probability density funⁿ becomes

$$F(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

and mean of the normal variate = 0
and S.D = 1
Hence, if x follows std normal variate, $X \sim N(\mu, \sigma) = X \sim N(0, 1)$
then we write

5) To solve the problems, we use normal dist^n table. This table is only w.r.t +ve values of z. Using symmetric conditions to -ve values of z we evaluate the table values also

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2} \, dz$$

gives area under curve

Ex:- i) x is a normal variable mean = 30, S.D = 5.
find a) $P(20 \leq x \leq 40)$
b) Find $P(x \geq 45)$
c) Find $P(x \leq 35)$

Sol: $\mu = 30$
$\sigma = 5$

Let $z = \frac{x-\mu}{\sigma}$ corresponding normal variate

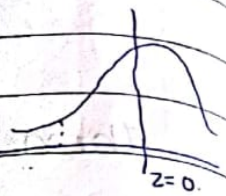
$$z = \frac{x-30}{5}$$

when $x = 20$, $z = -2$
$x = 40$, $z = 2$

i) $P(20 \leq x \leq 40) = P(-2 \leq z \leq 2)$

Date ___/___/___

$$P(-2 \le z \le 2) = \frac{1}{\sqrt{2\pi}} \int_{-2} \phi(|-2|) + \phi(2)$$



$= 2\phi(2)$
$= 2(0.4772)$
$= 0.9544$

ii) $P(x \ge 45)$
when $x = 45$, $z = \frac{45-30}{5} = 3$

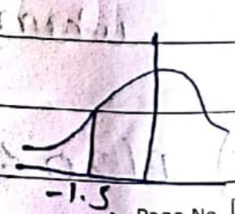
$P(z \ge 3) = 0.5 - \phi(3)$
$= 0.5 - 0.4987$
$= 0.0013$

iii) $P(x \le 35)$
when $x = 35$, $z = \frac{35-30}{5} = 1$

$P(z \le 1) = 0.5 + \phi(1)$
$= 0.5 + 0.3413$
$= 0.8413$

28 X- lifetime of electric lamp
$\mu = 1000$
$\sigma = 200$
$z = \frac{x-1000}{200}$ be corresponding std normal variate

i) $P(x \le 700)$
when $x = 700$ $z = -1.5$
$P(z \le -1.5) = 0.5 - \phi(|-1.5|)$
$= 0.5 - \phi(1.5)$



Page No.

probability is zero

Date __/__/__

$= 0.5 - 0.4332$

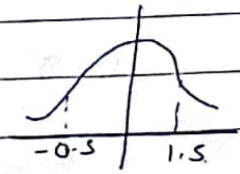
$= 0.0668$

ii) $P(900 \le x \le 1300) = P(-0.5 \le z \le 1.5)$

$= \phi(0.5) + \phi(1.5)$

$= 0.1915 + 0.4332$

$= 0.6247$



iii) Expected number of lamps out 2000 fail betⁿ 900 and 1300 $= 2000 \times P(900 \le x \le 1300)$

$= 2000 \times 0.6247$

$= 1249.4$

≈ 1249

iv) $P(x = 900) = P(z = -0.5)$

$= 0$

v) $P(899 \le x \le 901) = P(-0.505 \le z \le -0.495)$

$= \phi(0.505) - \phi(0.495)$

$= 0.1915 - 0.1879$

$= 0.0036$

vi)

a) Let x_1 be the required hour such that only 10% of the lamps will fail

$P(x \le x_1) = 0.1$

when $x = x_1$ $\quad z_1 = \dfrac{x_1 - 1000}{200}$

$P(x \le x_1) = P(z \le z_1)$

$P(z \le z_1) =$

Date ___/___/____

since P is < 0.5 ∴ z_1 is negative

$$P(z \le z_1) = 0.5 - \phi(|z_1|)$$
$$0.1 = 0.5 - \phi(|z_1|)$$
$$\phi(|z_1|) = 0.4$$
$$z_1 = \phi^{-1}(0.4)$$ z_1 is negative
$$z_1 = -1.28$$

$$z_1 = \frac{x_1 - 1000}{200}$$
$$(z \times 200) + 1000 = x_1$$
$$x_1 = 744 \text{ hrs}$$

∴ This implies 10% of lamp will fail before 744 hrs

b). Let x_2 be number of hrs required such that 10% of lamps survive.

$$P(x \le x_2) = 0.9$$
$$z_2 = \frac{x_2 - 1000}{200}$$
z_2 is +ve.

$$P(z \le z_2) = 0.5 + \phi(|z_2|)$$
$$0.9 = 0.5 + \phi(z_2)$$
$$0.4 = \phi(z_2)$$
$$z_2 = 1.28$$
$$z_2 = \frac{x_2 - 1000}{200}$$
$$x_2 = 1256$$

∴ This implies 90% of lamp will fail before 1256 hrs

$p(z \le z_1) < 0.5$ z_1 is $-ve$
$p(z \le z_1) > 0.5$ z is $+ve$

Saathi

Date __/__/__

LP 26) X - time in hours needed to locate and correct a problem.

$\mu = 10$
$\sigma^2 = 9$
$\sigma = 3$

Let z be its std normal variate.

$$z = \frac{x-\mu}{\sigma}$$

$$= \frac{x-10}{3}$$

i) $P(x \le 15)$

when $x = 15$, $z = \frac{15-10}{3} = \frac{5}{3}$

$$P(z \le \tfrac{5}{3}) = 0.5 + \phi\left(\tfrac{5}{3}\right)$$

$$= 0.5 + \phi(1.67)$$

$$= 0.5 + 0.4525$$

$$= 0.9525$$

ii) $P(x \le x_1)$

when $x = x_1$, $z_1 = \frac{x_1 - 10}{3}$

$P(z \le z_1) = 0.05$

z_1 is a negative value

$P(z \le z_1) = 0.05$

$0.5 - \phi(|-z_1|) = 0.05$

$0.5 - 0.05 = \phi(|z_1|)$

$0.45 = \phi(|z_1|)$

$z_1 = -1.65$

$3(-1.65) + 10 = x_1$

$x_1 = 5.05$

≈ 5 hrs

Page No

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Date ___/___/___

L.P 27)

~~$P(x \le 45) = 0.31$~~

Let X be normal variate with mean $= \mu$ and S.D $= \sigma$ and $z = \frac{x-\mu}{\sigma}$

be the std normal variate.

Given $P(x < 45) = 0.31$

$P(x > 64) = 0.08$

when $x = 45$, $z = \frac{45-\mu}{\sigma} = z_1$

$x = 64$, $z = \frac{64-\mu}{\sigma} = z_2$

$P(z < z_1) = 0.31$

$0.5 - \phi(z_1) = 0.31$ $\quad z_1$ is negative

$0.5 - 0.31 = \phi(z_1)$

$0.19 = \phi(z_1)$

$z_1 = -0.5$

$P(z < z_2) = 0.08$

$0.5 - \phi(z_2) = 0.08$ $\quad z_2$ is negative

$0.5 - 0.08 = \phi(z_2)$

$0.42 = \phi(z_2)$

$z_2 = -1.41$

$P(z \le z_2) = 0.92$

$0.5 + \phi(z_2) = 0.92$

$\phi(z_2) = 0.42$

$z_2 = 1.41$

$$-0.5 = \frac{45-\mu}{\sigma}$$

$$\mu - 0.5\sigma = 45 \quad —① $$

$$1.41 = \frac{64-\mu}{\sigma}$$

$$\mu + 1.41\sigma = 64 \quad —②$$

$$\mu = 49.9$$
$$\approx 50$$

$$\sigma = 9.9$$
$$\approx 10$$

→ In a class of students the height of student is normally distributed 6% has height below 60 inches, 39% are betⁿ 60 and 70 inches. Find mean & std of height.

Sol: Let X be normal variate

$$P(x<60) = 0.06$$
$$P(60<x<70) = 0.39$$

when $x = 60$, $Z = \frac{60-\mu}{\sigma} = Z_1$

when $x = 70$, $Z = \frac{70-\mu}{\sigma} = Z_2$

$$P(x<60) = 0.06$$

$P(Z<Z_1) = 0.06$ (Z_1 is negative)

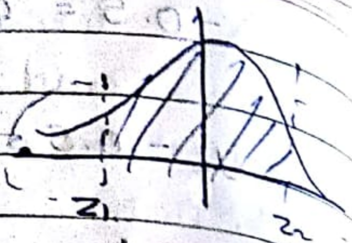
$$P(Z<Z_1) = 0.06$$
$$0.5 - \phi(Z_1) = 0.06$$
$$\phi(Z_1) = 0.5 - 0.06$$
$$= 0.44$$
$$Z_1 = -1.56.$$

Date ___/___/___

$P(z_1 < z < z_2) = 0.39$

$P(z < z_1) + P(z_2 < z) = 1 - 0.39$

$= 0.61$



Since $P(z < z_1) = 0.06$

$0.06 + P(z_2 < z) = 0.61$

$P(z_2 < z) = 0.55$

$P(z < z_2) = 0.45$

$0.5 - \phi(z_2) = 0.45$ z_2 is negat

$0.05 = \phi(z_2)$

$z_2 = -0.13$

$= -0.13$

$\mu - 1.56\sigma = 60$ — (1)

$\mu - 0.13\sigma = 70$ — (2)

$\mu = 70.19$

≈ 71

$\sigma = 6.99$

≈ 7

→ The life of a component is normally distributed with mean value 250 hrs and variance of σ^2 hrs. Find the maximum value of σ so that probability of a component to have a life between 200 and 300 hrs is 0.7

Sol: $P(200 < x < 300) = 0.7$

Date ___/___/___

$\mu = 250.$

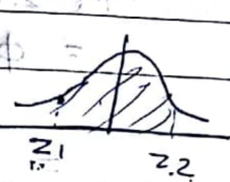
when $x = 200$, $z_1 = \frac{200-250}{\sigma} = \frac{-50}{\sigma} = A$

$x = 300$ $\quad z_2 = \frac{300-250}{\sigma} = \frac{50}{\sigma} = B$

$z_1 = -z_2$

$P(200 < x < 300) = 0.7$

$P(z_1 < x < z_2) = 0.70$



$\phi(z_1) + \phi(z_2) = 0.7$

$\phi(z_1) + \phi(1-z_1) = 0.7$

$2\phi(z_1) = 0.7$

$\phi(z_1) = 0.35$

$z_1 = -1.04.$

$z_2 = +1.04.$

$\sigma = \frac{-50}{-1.04}$

$\sigma = 48.07$

L.P

25) $\mu = 1000h$

$\sigma = 24h.$

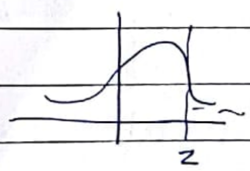
$P(x \geq 1050)$

when $x = 1050$ $\quad z = \frac{50}{24} = \frac{25}{12} = 2.083$

$P(z \geq 2.083) = 0.5 - \phi(2.08)$

$= 0.5 - 0.4812$

$= 0.0188$



Date __/__/__

ii) $P(950 \le x \le 1025)$

when $x = 950$

$z = \frac{950-1000}{24}$

$= -2.083$

when $x = 1025$

$z = \frac{1025-1000}{24}$

$= 1.041$

$P(-2.083 \le z \le 1.041)$

$= \phi(2.083) + \phi(1.041)$

$= 0.4812 + 0.3508$

$= 0.832$

-2.03

Page No.

Date 28/9/18 Fourier Series

→ Fourier series is used to represent a periodic signal in terms of an infinite sum of sine and cosine wave components.

→ Periodic signals :- A signal is said to be periodic, if it repeats itself after regular intervals of time. This interval is called time period of the signal.

Mathematically, a signal $x(t)$ is periodic if

$$x(t) = x(t+T)$$

where $T = nT_0$; n is integer, T_0 is fundamental period. and fundamental frequency $= 1/T_0$.

and fundamental angular frequency

$$\omega = \frac{2\pi}{T_0} = 2\pi f_0.$$

Ex:- Find the fundamental period for the given signals, if they are periodic

i) $x(t) = e^{j\pi t}$

Sol: Given signal is $x(t) = e^{j\pi t}$

$x(t)$ is periodic, if $x(t) = x(t+T)$

$$x(t+T) = e^{j\pi(t+T)}$$

$$= e^{j\pi t} \cdot e^{j\pi T}$$

$$x(t) = x(t+T)$$

$$\cancel{e^{j\pi t}} = e^{j\pi T} \cdot \cancel{e^{j\pi t}}$$

$$1 = e^{j\pi T} \qquad \boxed{e^{j(2k\pi)} = 1}$$

$$e^{j(2k\pi)} = e^{j\pi T}$$

$$T = 2k.$$

take K = smaller +ve integer

$$\boxed{T = 2}$$

(or) Std form $e^{j\omega t}$. $\omega = \pi$

$$\frac{2\pi}{T_0} = \pi \qquad \boxed{T_0 = 2}$$

Page No.

Date ___/___/___

2). $x(t) = 2\cos(3t + \pi/4)$

Sol. comparing with std form

$x(t) = A\cos(\omega t + \phi)$

Since given signal is trigonometric

$\therefore$ It is periodic

$\omega = 3$

$\frac{2\pi}{T_0} = 3$

$$\boxed{\frac{2\pi}{3} = T_0}$$

L.P 1

a) $x(t) = \cos 2t + \sin 3t$

Sol:- Let $f(t) = \cos 2t$

It is periodic

$\frac{2\pi}{T_{01}} = 2$

$\pi = T_{0(1)}$ — ①

$g(t) = \sin 3t$

It is periodic

$\frac{2\pi}{T_2} = 3$

$\frac{2\pi}{3} = T_{20}$ — ②

$\frac{T_1}{T_2} = \frac{\pi}{2\pi} \times 3 = \frac{3}{2}$

$\therefore$ The ratio is rational $\therefore x(t)$ is periodic

$\therefore$ The period of $x(t) = \pi \times 2$

$= 2\pi$

Page No.

Saathi

Date ___ / ___ / ___

3) Sol: $x(t) = \cos 2t + \sin \pi t$

Sol:- Let $x_1(t) = \cos 2t$ & it is periodic

$$\frac{2\pi}{T_1} = 2$$

$$T_1 = \pi \quad - (1)$$

$$x_2(t) = \sin \pi t$$

$$\frac{2\pi}{T_2} = \pi$$

$$2 = T_2 \quad - (2)$$

$\frac{T_1}{T_2} = \frac{\pi}{2}$ is irrational

$\therefore$ It is aperiodic

4) $x(t) = \sin 6\pi t + \cos 5\pi t$

Sol:-

$$\frac{2\pi}{T_1} = 6\pi$$

$$T_1 = \frac{2}{6} = \frac{1}{3} \quad - (1)$$

$$\frac{2\pi}{T_2} = 5\pi$$

$$T_2 = \frac{2}{5} \quad - (2)$$

$\frac{T_1}{T_2} = \frac{1}{3} \times \frac{5}{2} = \frac{5}{6}$ It is rational

$\therefore$ It is periodic.

The period of $x(t) = \frac{1}{3} \times 6$

$= 2$ seconds

5) $x(t) = \sin^2(t - \pi/6)$

Sol:- $x(t) = \frac{1 - \cos(2t - \pi/3)}{2}$

Page No.

Date __/__/__

$$x(t) = \frac{1}{2} - \frac{\cos(2t - \pi/3)}{2}$$

$\therefore \quad \omega = 2$

$$\frac{2\pi}{T_0} = 2$$

$$\boxed{T_0 = \pi}$$

Here the given signal is periodic signal with period π

I.P 1

6) $x(t) = e^{j(\pi t - 1)}$

Sol:- $x(t) = e^{j\pi t} \cdot e^{-j}$

$$= \frac{e^{j\pi t}}{e^{j}}$$

$$= \frac{e^{j\pi t}}{e^{j}}$$

$\omega = \pi$

$$\frac{2\pi}{T_0} = \pi$$

$$\boxed{T_0 = 2 \text{ seconds}}$$

NOTE 1) Time period is independent of time shifting, time reversal, phase shift, amplitude reversal, amplitude scaling

2) But it depends on time scaling i.e $x(t)$ is periodic signal with period T then $x(at)$ is also periodic signal with period $\frac{T}{|a|}$ for $a \neq 0$.

Date ___/___/___

Different forms of contⁿ time fourier series :-

i) Trigonometric form :- A periodic funⁿ $x(t)$ with fundamental period T_0 can be expressed as

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad \text{①}$$

where $\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0$

is the fundamental frequency of $x(t)$ and coefficients a_0, a_n, b_n are referred as trigonometric fourier coefficients

where $a_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt$

$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos n\omega_0 t \, dt$$

$$b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin n\omega_0 t \, dt$$

here the coefficient a_0 represents average value known as dc component of $x(t)$ and the equation ① is called synthesis equation.

Page No.

Saa

Date ___/___/___

2) Harmonic form of fourier series (or compac
A real periodic signal $x(t)$ wi
fundamental period T_0 is said to be i
compact form, if $x(t)$ is expressed as

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

where $\omega_0 = \frac{2\pi}{T_0}$

this equation is derived from trigonomet
form of fourier series.
The term C_0 is known as dc componen
and $C_n \cos(n\omega_0 t + \theta_n)$ is known as
nth harmonic component. The coefficient
C_n is harmonic amplitude and θ_n is
called phase angles. C_n and θ_n in
terms of a_n and b_n are given.

$$C_0 = a_0$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

$$\theta_n = \tan^{-1} \frac{b_n}{a_n}$$

3) Complex form of fourier series :-
An arbitary periodic funn with
period T_0 is said to be complex
form of fourier series if $x(t)$ is
expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} X(k) e^{j\omega_0 kt} \quad -①$$

or

$$x(t) = \sum_{k=-\infty}^{\infty} C(k) e^{j\omega_0 kt}$$

where

$$X(t) = \frac{1}{T_0} \int_{T_0} x(t) e^{-j\omega_0 kt} dt \quad -②$$

Page No.

Date __ / __ / __

known as complex exponential fourier coefficient.

eqⁿ ① is time domain representation of periodic complex signal as a sum of exponential signals, this eqⁿ is known as synthesis eqⁿ

The terms with $k = \pm 1$ in this series are collectively called 1st harmonic its fundamental frequency is ω_0

eqⁿ ② is called analysis equation. It gives us frequency domain repreⁿ of the signal as fourier coefficient of spectral coefficients of $x(t)$. each of these coefficients tells us how much the corresponding harmonic component of given frequency contributes to the signal.

$|X(k)|$ = magnitude of spectrum

$\arg(X(k))$ = phase spectrum of $x(t)$

$X(0)$ is called average value of dc value of the signal.

Imp Derivation of Fourier coefficient :-

Statement :- If $x(t) = \sum_{k=-\infty}^{+\infty} X(k) e^{jk\omega_0 t}$ is the fourier expansion of periodic signal $x(t)$ with fundamental period T_0 i.e $T_0 = 2\pi/\omega_0$. Then its fourier coefficient is given by

$$X(k) = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$$

⇒ Given :- $x(t)$ is a periodic signal with fourier series expansion

Date ___ / ___ / ______

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t}$$

Multiply both sides by $e^{-jn\omega_0 t}$

$$x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} X(k)\, (e)^{jk\omega_0 t - jn\omega_0 t}$$

Integrate both sides with one period.

$$\int_0^{T_0} x(t).\, e^{-jn\omega_0 t}\, dt = \int_0^{T_0} \sum_{k=-\infty}^{\infty} X(k)\, e^{(k-n)j\omega_0 t}\, dt$$

$$\int_0^{T_0} x(t).\, e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} X(k) \int_0^{T_0} e^{(k-n)j\omega_0 t}\, dt \quad -①$$

Consider $\int_0^{T_0} e^{(k-n)j\omega_0 t}\, dt = \int_0 \cos(k-n)\omega_0 t + j\sin(k-n)\omega_0 t$

Case 1 :- When $k = n$.

$$LHS = \int_0^{T_0} (1+0)\, dt$$

$$= T_0$$

Case 2 :- When $k \neq n$, $k - n = m$

$$LHS = \int_0^{T_0} \cos m\omega_0 t + j\sin m\omega_0 t\, dt$$

$$= \left. \frac{\sin m\omega_0 t}{m\omega_0} - \frac{j\cos m\omega_0 t}{m\omega_0} \right]_0^{T_0}$$

$$= \frac{\sin m\omega_0 T_0}{m\omega_0} - \frac{j\cos m\omega_0 T_0}{m\omega_0} + \frac{j}{m\omega_0}$$

$$= \frac{1}{m\omega_0}\left[\sin m\frac{2\pi}{T_0}T_0 - j\cos m\frac{2\pi}{T_0}T_0 + j\right]$$

$$= \frac{1}{m\omega_0}\left[\sin 2\pi m - j\cos 2\pi m + j\right]$$

$$= \frac{1}{m\omega_0}\left[0 - j + j\right]$$

$$LHS = 0$$

Orthogonal property: $\int_0$ ... $= 1$ $k = n$.

Saathi

Date __/__/____

using above result, eq^n ① becomes

$$\int_0^{T_0} x(t).\, e^{-jn\omega_0 t}\, dt = X(n)\,[T_0]$$

$$X(n) = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jn\omega_0 t}\, dt.$$

If $n = k$

$$\boxed{X(k) = \frac{1}{T_0}\int_0^{T} x(t).\, e^{-jk\omega_0 t}\, dt}$$

Existence of the fourier series

The given periodic signal can be expressed as fourier series if it satisfies the four conditions:-

Drichlet's conditions

i) Periodic signal should have finite number of maxima and minima over one period

ii) Fourier series should have finite number of discontinuities over the range of 1 time period

iii) Periodic signal should be absolutely integrable over the range of 1 time period.

Relation bet^n trigonometric fourier series and exponential fourier series

If $x(t)$ is a periodic function with period T, then its fourier series in trigonometric form is expressed as

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \omega_0 nt + b_n \sin \omega_0 nt) \quad -①$$

and its complex fourier signal is given by

Date __/__/__

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t}$$

$$= X(0) + \sum_{k=1}^{\infty} X(-k)\, e^{j(-k)\omega_0 t} + \sum_{k=1}^{\infty} X(k)\, e^{jk\omega_0 t}$$

$$= X(0) + \sum_{k=1}^{\infty} \Big[X(-k)(\cos k\omega_0 t + j\sin(-k)\omega_0 t) + X(k)(\cos k\omega_0 t + j\sin k\omega_0 t)$$

$$= X(0) + \sum_{k=1}^{\infty} \Big[(X(-k) + X(k))\cos k\omega_0 t + j(\sin k\omega_0 t)(X(k) - X(-k)) \Big]$$

$$= C_0 + \sum_{n=1}^{\infty} \Big[(C_{(-k)} + C_k)\cos k\omega_0 t + j(C_k - C_{-k})\sin k\omega_0 t \Big]$$

$$= C_0 + \sum_{k=1}^{\infty} \Big[A_k(\cos k\omega_0 t) + B_k \sin k\omega_0 t \Big]$$

$$= C_0 + \sum_{n=1}^{\infty} \Big[A_n \cos n\omega_0 t + B_n \sin n\omega_0 t \Big]$$

This expression is similar to trigonometric expression of fourier series.

$\therefore$ comparing the coefficients with trigonometric expression we get

$$A_n = C_{(-n)} + C_n \qquad A_0 = C_0$$

$$B_n = j(C_n - C_{(-n)})$$

These expression gives trigonometric coefficients in terms of complex fourier coefficients.

solving 2nd & 3rd expressions for A_n and B_n.

$$A_n = C_{(-n)} + C_n$$

$$jB_n = C_{(-n)} - C_n$$

$$\therefore A_n + jB_n = 2C_{(-n)}$$

Date ___ /___ /______

$$C_{(-n)} = \frac{A_n + jB_n}{2}$$

subtracting 1 eq^n from another

$$A_n - jB_n = 2C_n$$

$$C_n = \frac{A_n - jB_n}{2}$$

and $C_0 = A_0$

These gives expression for complex fourier coefficient interms of trigonometric fourier coefficient.

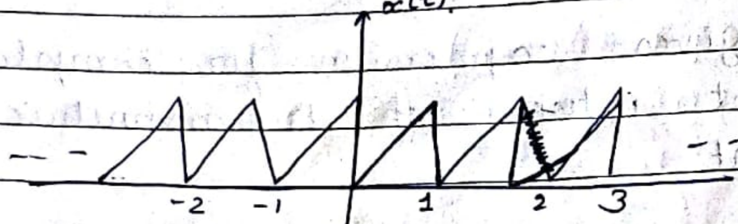
Amplitude And Phase spectra

A periodic time signal is defined by fourier coefficient and its period.
We can present graphically the description of periodic signals interms of amplitude and phase angle.
The plot of $|X(k)|$ v/s k is called amplitude / magnitude spectrum and the plot of argument of X_k i.e $\arg(X_k) \rightarrow k$ is called phase spectrum.
Further, amplitude ~~exp~~ spectra is even funct^n of k, its graph is symmetric about vertical axis.
phase spectra is odd fun^n of k, its graph is symmetric about origin / opposite quadrant.

Saat

Date ___ / ___ / ______

Ex:- Obtain the complex fourier representation for the periodic saw waveform shown in the given figure, Also
i) Find the average value of DC signal
ii) " 1st 3 non zero harmonics
iii) Sketch amplitude and phase spectra
iv) Find its trigonometric fourier series.



Sol:- The given signal is periodic with period 1 sec and $\omega_0 = 2\pi$ rad/sec

Mathematical expression.

$$x(t) = t \quad 0 < t \leq 1$$

∴ complex fourier expression is

$$x(t) = \sum_{k=-\infty}^{\infty} X(k) e^{j\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} X(k) e^{j2\pi t}$$

$$X(k) = \frac{1}{T}\int_T x(t) \cdot e^{-j2\pi kt}\, dt$$

$$= \frac{1}{T}\int_T t\, e^{-j2\pi kt}$$

$$= \frac{1}{T}\left[\frac{te^{-j2\pi kt}}{-j(2\pi)} - \frac{e^{-j2\pi kT}}{j^2(4\pi)}\right]_0$$

$$= \frac{1}{T}\left[\frac{-T e^{-j2\pi k}}{j(2\pi)k} + \frac{e^{-j2\pi k}}{4\pi^2 k^2} - \frac{1}{4\pi^2 k^2}\right]$$

$$= \frac{-e^{-2\pi jk}}{2\pi jk} + \frac{e^{-2\pi jk}}{4\pi^2 k^2} - \frac{1}{4\pi^2 k^2}$$

Date __ / __ / ____

$$= \frac{-e^{2\pi jk}}{2\pi jk} + \frac{e^{2\pi jk}}{4\pi^2k^2} - \frac{1}{4\pi^2k^2} \quad -(1)$$

$$e^{-j2\pi k} = \cos 2\pi k + j\sin 2\pi k$$

$$= 1$$

$$\therefore X(k) = \frac{-1}{2\pi jk} + \cancel{\frac{1}{4\pi^2k^2}} - \cancel{\frac{1}{4\pi^2k^2}}$$

$$X(k) = \frac{-1}{2\pi jk}$$

$$= \frac{j}{2\pi k}$$

This expression is valid for all values of k, except ~~x(t)~~ k = 0.

$\therefore$ We need to calculate X(0) separately

By definition of fourier coefficient

$$X(0) = \frac{1}{T}\int_0^T x(t) \cdot e^{-j\omega_0(0)t}\, dt.$$

$$= \frac{1}{T}\int_0^T x(t)\, dt.$$

$$X(0) = \int_0^1 x(t)\, dt$$

$$= \int_0^1 (t)\, dt$$

$$= \frac{t^2}{2}$$

$$X(0) = 1/2$$

$\therefore$ Required fourier series expression for $x(t) = t$

i.e. $$x(t) = X(0) + \sum_{\substack{k=-\infty \\ (k\neq 0)}}^{\infty} X(k) \cdot e^{jk\omega_0 t}$$

$$x(t) = \frac{1}{2} + \sum_{\substack{k=-\infty \\ (k\neq 0)}}^{\infty} \frac{j}{2\pi k} e^{j2\pi t \times k.}$$

Page No.

$\cos 2\pi t$ -
$j = \sqrt{-1}$

Date __/__/____

i) DC value of the signal
$$x(0) = 1/2$$

ii). Let H_1 denote first harmonic

$$H_1 = X(-1)e^{-j2\pi t} + X(1)e^{j2\pi t}$$

$$= \frac{-j}{2\pi}e^{-j2\pi t} + \frac{j}{2\pi}e^{j2\pi t}$$

$$= \frac{j}{2\pi}\left(e^{j2\pi t} - e^{-j2\pi t}\right)$$

$$= \frac{j}{(2\pi)}\left(\cancel{\cos 2\pi t} + j\sin 2\pi t - \cancel{\cos 2\pi t} + j\sin 2\pi t\right)$$

$$= \frac{j}{\cancel{2}\pi}\left(\cancel{2}j\sin 2\pi t\right)$$

$$H_1 = \frac{-\sin 2\pi t}{\pi}$$

Second Harmonic

$$H_2 = X(-2)e^{-j2\pi(2)t} + X(2)e^{j(4\pi)t}$$

$$= -\frac{j}{4\pi}e^{-j4\pi t} + \frac{j}{4\pi}e^{j4\pi t}$$

$$= \frac{j}{4\pi}\left[e^{j4\pi t} - e^{-j4\pi t}\right]$$

$$= \frac{j}{4\pi}\left[\cancel{\cos 4\pi t} + j\sin 4\pi t - \cancel{\cos 4\pi t} + j\sin 4\pi t\right]$$

$$= +j\frac{(2j\sin 4\pi t)}{4\pi}$$

$$= -\frac{\sin 4\pi t}{2\pi}$$

Third Harmonic

$$H_3 = X(-3)e^{-j6\pi t} + X(3)e^{j6\pi t}$$

$$= \frac{j}{6\pi}\left[e^{j6\pi t} - e^{-j6\pi t}\right]$$

$$= \frac{j}{6\pi}\left(2j\sin 6\pi t\right)$$

$$= -\frac{\sin 6\pi t}{3\pi}$$

iii) Let Amplitude spectrum

$$x(k) = \frac{j}{2\pi k} = A + jB$$

$$A = 0$$

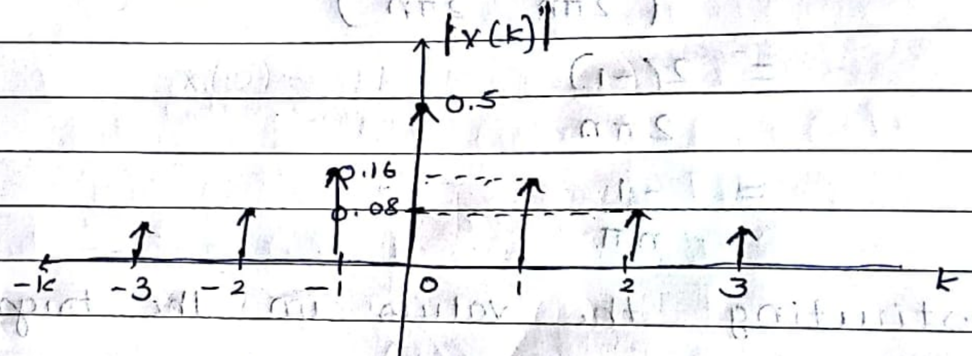
$$B = \frac{1}{2\pi k}$$

$$|X(k)| = \sqrt{0^2 + \frac{1}{4\pi^2 k^2}}$$

$$|X(k)| = 1/2\pi|k| \qquad k \neq 0.$$

$$|X(0)| = \frac{1}{2}.$$

k	0	± 1	± 2	± 3
$\lvert X(k)\rvert$	$1/2$	$1/2\pi$	$1/4\pi$	$1/6\pi$



Phase spectrum

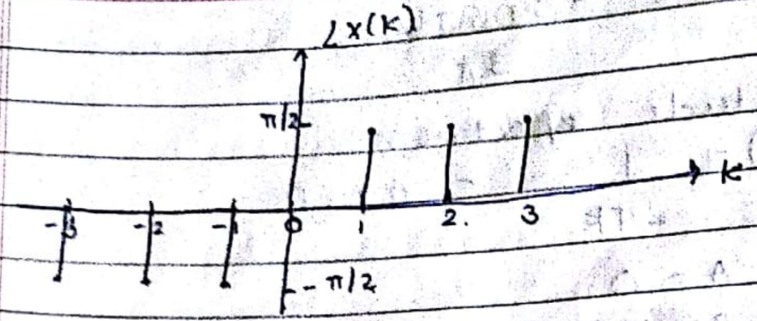
$$\angle X(k) = \tan^{-1}\frac{b}{a}$$

$$= \tan^{-1}\left(\frac{\frac{1}{2\pi k}}{0}\right)$$

$$= \tan^{-1}(\pm\infty) \qquad -\infty < k < \infty.$$

$$= \frac{\pi}{2}, \; -\frac{\pi}{2} \quad \text{for } k>0 \text{ \& } k<0 \text{ resp.}$$

Date ___ / ___ / ___



iv) Using the relation of complex fourier

$$A_n = C_n + C_{(-n)}$$

$$= \frac{j}{2\pi n} - \frac{j}{2\pi n}$$

$$A_n = 0 \qquad A_0 = C_0 = \frac{1}{2}$$

$$B_n = j\left(C_n - C_{(-n)}\right)$$

$$= j\left(\frac{j}{2\pi n} + \frac{j}{2\pi n}\right)$$

$$= \frac{2(-1)}{2\pi n}$$

$$= \frac{-1}{n\pi}$$

Substituting the values in the trignometric series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n(\cos n\omega_0 t + b_0 \sin n\omega_0 t)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-1}{n\pi} \sin n\omega_0 t$$

$$= \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\omega_0 t}{n}$$

Date __ / __ / ____

Saathi

NOTE:-

1) If $x(t)$ and $X(k)$ forms a fourier series pair, then it is denoted by

$$x(t) \xleftarrow{FS;\omega_0} X(K)$$

Properties on fourier series:-

1) Linearity property :- If $x(t)$ follows F.S with F.C $X(k)$ and $y(t)$ follows F.S with F.C $Y(k)$ and then

$$x(t) \xleftarrow{FS;\omega_0} X(k)$$

$$y(t) \xleftarrow{FS;\omega_0} Y(k)$$

$$z(t) = ax(t) + by(t) \xleftarrow{FS;\omega_0} aX(k) + bY(k) = Z(k)$$

Given :- $X(k) = \frac{1}{T}\int_T x(t) . e^{-j\omega_0 kt} \, dt$

$$Y(k) = \frac{1}{T}\int_T y(t) . e^{-j\omega_0 kt} \, dt$$

$Z(K)$ be the F.C of $z(t)$

$$\therefore Z(k) = \frac{1}{T}\int_T z(t) . e^{-j\omega_0 kt} \, dt$$

$$= \frac{1}{T}\int_T (ax(t) + by(t)) e^{-j\omega_0 kt} \, dt$$

$$= \frac{1}{T}\int_T a x(t) . e^{-j\omega_0 kt} + \frac{1}{T}\int_T by(t) . e^{-j\omega_0 kt} \, dt$$

$$= a \frac{1}{T}\int_T x(t) . e^{-j\omega_0 kt} + \frac{b}{T}\int_T y(t) . e^{-j\omega_0 kt} \, dt$$

$$z(k) = a X(K) + b Y(K)$$

Page No.

Date ___/___/___

ii) Time shift property :-

If

$$x(t) \xleftarrow{FS; \omega_0} X(k)$$

then

$$y(t) = x(t-t_0) \xleftarrow{FS; \omega_0} e^{-j\omega_0 k t_0} X(k)$$

Sol:- Given $x(t) \xleftarrow{FS; \omega_0} X(k)$

$$X(k) = \frac{1}{T}\int_T x(t) \cdot e^{-j\omega_0 kt} \, dt$$

Let $Y(k)$ be F.C of $y(t)$.

$$\therefore \; Y(k) = \frac{1}{T}\int_T y(t) \cdot e^{-j\omega_0 kt} \, dt$$

$$= \frac{1}{T}\int_T x(t-t_0) \, e^{-j\omega_0 kt} \, dt$$

Put $t - t_0 = u$ $\quad dt = du$

$t = u + t_0$

$$Y(k) = \frac{1}{T}\int_T x(u) \cdot e^{-j\omega_0 k(u+t_0)} \, du$$

$$= \frac{1}{T} \cdot e^{-j\omega_0 k t_0} \int_T x(u) \cdot e^{-j\omega_0 k u} \, du$$

$$= e^{-j\omega_0 k t_0} \cancel{X}(k)$$

$$= e^{-j\omega_0 k t_0} X(k)$$

iii) Frequency shift property :-

If

$$x(t) \xleftarrow{FS; \omega_0} X(k)$$

then

$$y(t) = e^{j k_0 \omega_0 t} x(t) \xleftarrow{FS; \omega_0} Y(k) = X(k-k_0)$$

Sol:- Let $Y(k)$ be F.C of $y(t)$.

Saathi

Date __ / __ / __

$$Y(k) = \frac{1}{T}\int_T y(t).\ e^{-j\omega_0 kt}\ dt.$$

$$= \frac{1}{T}\int_T e^{jk_0\omega_0 t}\ x(t).\ e^{-j\omega_0 kt}\ dt.$$

$$= \frac{1}{T}\int_T e^{j\omega_0 t(k_0-k)}.\ x(t)\ dt$$

$$= \frac{1}{T}\int_T e^{-j\omega_0 t(k-k_0)}.\ x(t)\ dt.$$

$$= X(k-k_0).$$

iv) Scaling property:-

If

$$x(t) \xleftarrow{FS;\omega_0} X(k).$$

then

$$z(t) = x(at) \xleftarrow{FS;\,a\omega_0} Z(k) = X(k) \quad (a>0)$$

Sol:- $Z(k)$ be F.C of $z(t)$.

If $x(t)$ is periodic, then

$z(t) = x(at)$ is also periodic

suppose $x(t)$ has fundamental time period T and angular velocity $= \omega_0$, then $x(at)$ has fundamental time period $\frac{T}{|a|}$ $(a>0)$

and angular velocity $= a\omega_0$

$$\therefore\ Z(k) = \frac{|a|}{T}\int_{T/a} e^{-ajk\omega_0 t}\ z(t)\ dt$$

$$= \frac{|a|}{T}\int_{T/a} e^{-ajk\omega_0 t}\ x(at)\ dt.$$

Put $at = u$

$dt = \frac{du}{a}$

$$Z(k) = \frac{|a|}{T}\times\frac{1}{a}\int_T x(u)\ e^{-jk\omega_0 u}\ du.$$

Date

$$= \frac{1}{T}\int_T x(u).\, e^{-j\omega_0 k u}\, du.$$

$$z(k) = X(k)$$

NOTE:- In the above case, F.C's are identical but harmonic spacing changes from to $a\omega_0$.

v) Time differentiation

If

$$x(t) \xleftrightarrow{FS;\omega_0} X(k)$$

$$\frac{d\,x(t)}{dt} \xleftrightarrow{FS;\omega_0} jk\omega_0 X(k)$$

Sol. By def^n of F.S we have

$$x(t) = \sum_{0}^{\infty} X(k).\, e^{j\omega_0 kt}$$

Differentiating both sides w.r.t t

$$\frac{dx(t)}{dt} = \sum_{k=0}^{\infty} X(k) \frac{d}{dt} e^{j\omega_0 kt}$$

$$\frac{d\,x(t)}{dt} = \sum_{k=0}^{\infty} X(k).\, (j\omega_0 k)\, e^{j\omega_0 kt}$$

$$= f$$

By comparing with def^n of F.S, we can write

$$\frac{d\,x(t)}{dt} \xleftrightarrow{FS;\omega_0} j\omega_0 k\, X(k)$$

vi) Convolution:-

If

$$x(t) \xleftrightarrow{FS;\omega_0} X(k)$$

$$y(t) \xleftrightarrow{FS;\omega_0} Y(k)$$

Date __ / __ / __

then

$$z(t) = x(t) \circledast y(t) \xleftarrow{FS;\omega_0} Z(k) = T\,X(k)\,Y(k)$$

where $T = \frac{2\pi}{\omega_0}$ and $\circledast$ denotes periodic convolution

vii) Parseval's Thm

If

$$x(t) \xleftarrow{FS;\omega_0} X(k)$$

then

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X(k)|^2$$

viii) symmetric property

If

$$x(t) \xleftarrow{FS;\omega_0} X(k)$$

then

If $x(t)$ is real and even, then imaginary part of $X(k)$ is zero

If $x(t)$ is real and odd, then real part of $X(k)$ is zero.

Ex:-

i) Determine the complex exponential fourier series for the foll^n, also sketch the magnitude and phase spectra of given signal.

i) $x(t) = \sin\omega_0 t$

Sol: Here given signal is periodic signal

angular velocity $= \omega_0$

$$T = \frac{2\pi}{\omega_0}$$

Date ___/___/___

converting exp^n interms of exponential form

$$x(t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$= \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}.$$

$$x(t) = \frac{-j}{2} e^{j\omega_0 t} + \frac{j}{2} e^{-j\omega_0 t}$$

This is required fourier series for given signal This series wrt 2 terms corresponding to to $k=1$ and $k=-1$.

Fourier coefficients for given signal are given by

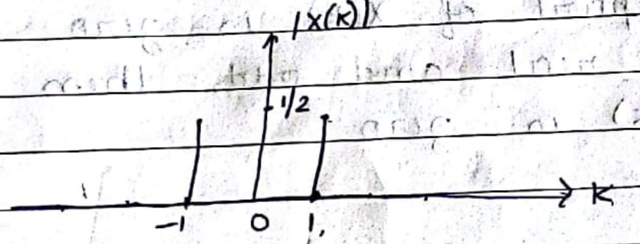
$X(1) = -j/2$

$X(-1) = j/2$

and $X(k) = 0 \;\forall\; k$ for which $k \neq 1$ & $k \neq -1$

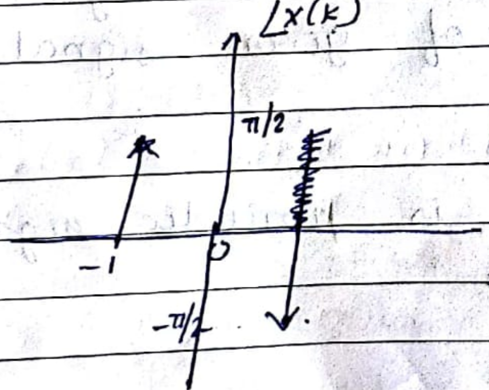
Magnitude spectrum $\quad |X(1)| = 1/2$

$|X(-1)| = 1/2$



Phase spectra $\quad \angle X(1) = \tan^{-1}\left(\frac{-1/2}{0}\right) = -\pi/2$

$\angle X(-1) = \tan^{-1}(+1/2/0) = \pi/2$



Date ___/___/___

L.P 1)

ii) $x(t) = \cos\omega_0 t$

Sol: A.V $= \omega_0$ $\quad T = \frac{2\pi}{\omega_0}$

$$x(t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

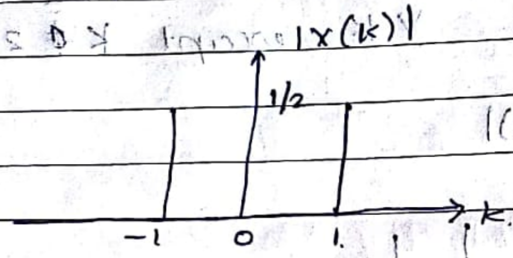
$$x(t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$$

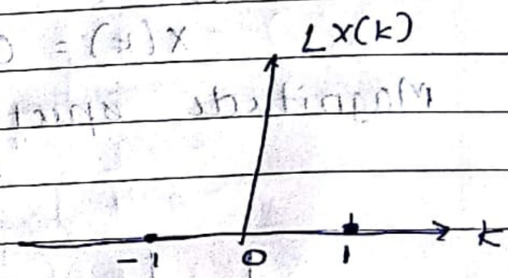
This is the required fourier series

for $k = 1$ & $k = -1$

$X(1) = 1/2$

$X(-1) = 1/2$





L.P 2)

iii) $x(t) = \cos 4t + \sin 6t$

Sol:- $T_1 = \frac{2\pi}{4} = \frac{\pi}{2}$

$T_2 = \frac{2\pi}{6} = \frac{\pi}{3}$

$\frac{T_1}{T_2} = \frac{3}{2}$ is rational.

$\therefore x(t)$ is periodic $\quad T = T_1 \times 2$

$T = \frac{\pi}{2} \times 2 = \boxed{\pi}$

$x(t) \rightarrow$ angular frequency $= 2 = \omega$ $\quad \frac{2\pi}{\pi}$

$x(t) = \cos 4t + \sin 6t$

$$= \frac{e^{j4t} + e^{-j4t}}{2} + \frac{e^{j6t} - e^{-j6t}}{2j}$$

$$x(t) = \frac{1}{2} e^{j4t} + \frac{1}{2} e^{-j4t} - \frac{j}{2} e^{j6t} + \frac{j}{2} e^{-j6t}$$

Date ___/___/___

This expression is fourier series expⁿ for given periodic signal

$$x(t) = \frac{1}{2}e^{j2(2)t} + \frac{1}{2}e^{j(2)(-2)t} + \frac{j}{2}e^{j(2)(-3)t} - \frac{j}{2}e^{j(2)(3)t}$$

comparing above expⁿ with defⁿ of fourier series weight X

$K = 2 \qquad X(2) = 1/2$

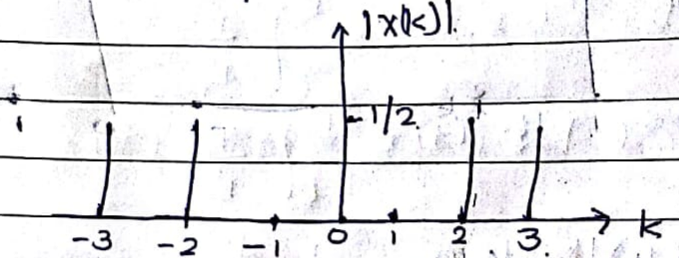
$K = -2 \qquad X(-2) = 1/2$

$K = -3 \qquad X(-3) = j/2$

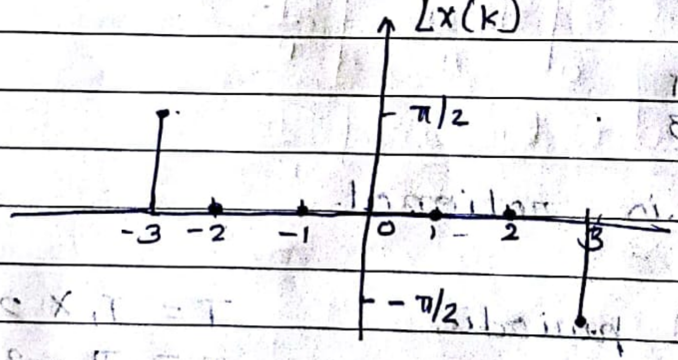
$K = 3 \qquad X(3) = -j/2$

$X(K) = 0 \qquad \forall (K)$ except $K \neq 2, -2, 3, -3$

Magnitude spectra



Phase spectra



L.P 2)

iii) $x(t) = \cos 2t + \sin \pi t$

Sol $T_1 = \frac{2\pi}{2} = \pi$ $\qquad \frac{T_1}{T_2} = \frac{\pi}{2}$ (not rational)

$T_2 = \frac{2\pi}{\pi} = 2$

Date ___/___/___

Since given funⁿ is non periodic
∴ It is not possible to express as a fourier series.

E.P 4) $x(t) = \begin{cases} \sin \pi t & 0 \le t \le 2 \\ 0 & 2 \le t \le 4 \end{cases}$

sol:- Here funⁿ is discontⁿ. ∴ we obtain its fourier series using its defⁿ.

Let

$$x(t) = \sum_{k=-\infty}^{\infty} X(k) e^{j\omega_0 t}$$ be the fourier series of given signal, where

$\omega_0 = \frac{2\pi}{T}$

$$X(k) = \frac{1}{T}\int_{T} x(t) \cdot e^{-j\omega_0 k t} dt.$$

$$= \frac{1}{4}\left[\int_0^2 \sin \pi t \cdot e^{-j\pi/2 \, kt} dt + 0\right]$$

$$= \frac{1}{4}\left[\frac{e^{-j\pi/2 \, kt}}{(j\frac{\pi}{2}k)^2 + (\pi)^2}\left[(-j\frac{\pi}{2}k)\sin \pi t - \pi \cos \pi t\right.\right.$$

$$= \frac{1}{4}\left[\frac{e^{-j\pi/2 \, kt}}{\pi^2 - \frac{\pi^2 k^2}{4}}\left[-j\frac{\pi}{2}k \sin \pi t - \pi \cos \pi t\right.\right.$$

$$= \frac{1}{4}\left[\frac{e^{-j\pi k}}{\pi^2 - \frac{\pi^2}{4}k^2}\left[0 - \pi\right] - \frac{1}{\pi^2 - \frac{\pi^2}{4}k^2}\right.$$

$$= \frac{1}{4}\left[\frac{e^{-j\pi k}}{\pi^2 - \frac{\pi^2}{4}k^2}(-\pi) + \frac{\pi}{\pi^2 - \frac{\pi^2}{4}k^2}\right]$$

$$= \frac{1}{4}\left[\frac{4}{4\pi^2 - \pi^2 k^2}\right]\left[\pi - \pi e^{-j\pi k}\right]$$

$$X(k) = \frac{1}{4\pi^2 - \pi^2 k^2}\left(\pi - \pi e^{-j\pi k}\right)$$

$$= \frac{1}{4\pi - \pi k^2}\left(1 - e^{j\pi k}\right).$$

Date ___/___/___

$$= \frac{1}{4\pi - \pi k^2}\left(1 - (\cos(-\pi)k + j\sin(-\pi)k\right]$$

$$= \frac{1}{4\pi - \pi k^2}\left(1 - \cos\pi k - j\sin\pi k\right]$$

$$= \frac{1}{4\pi - \pi k^2}\left(1 - (-1)^k - j(0)\right)$$

$$= \frac{1 - (-1)^k}{4\pi - \pi k^2}$$

$$= \frac{1 - (-1)^k}{(4 - k^2)\pi}$$

$$X(2) = \frac{1}{T}\int_T x(t)\, e^{-j(\pi/2)(2)t}$$

$\cos\pi t - j\sin\pi t$

$$= \frac{1}{4}\int_0^4 x(t).\ e^{-j\pi t}\, dt$$

$$= \frac{1}{4}\int_0^4 x(t)\, dt\, (-1)^t$$

$$= \frac{+1}{4}\int_0^2 \sin\pi t\, dt\, (-1)$$

$$= \frac{-1}{4}\left[\frac{-\cos\pi t}{\pi}\right]_0^2$$

$$= \frac{-1}{4}\left[-\cos 2\pi + \cos 0\right]$$

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Date ___/___/___

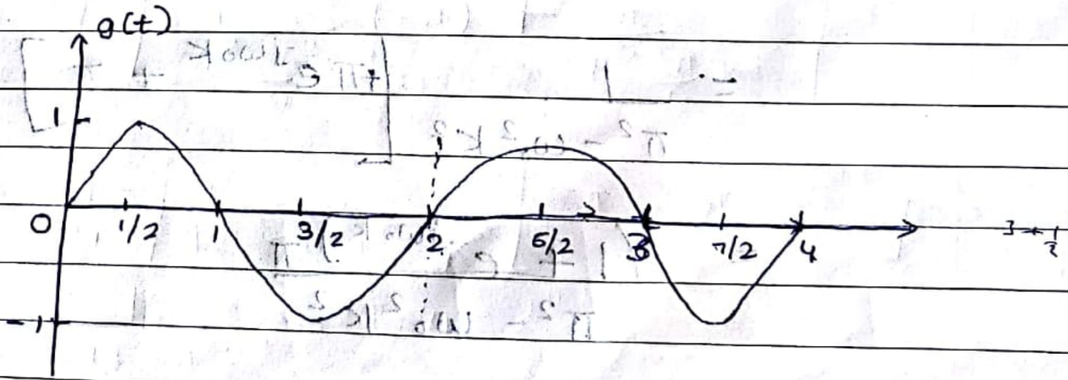
8). $x(t) = |\sin \pi t|$

sol:- $g(t) = \sin \pi t$

$\omega = \pi$

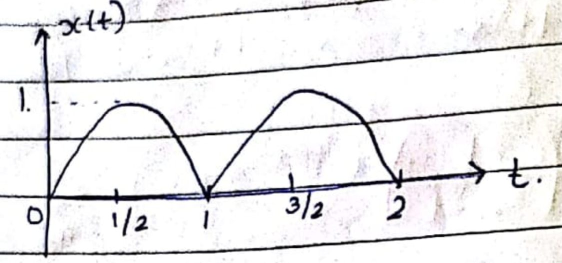
$T = 2$

t	0	1/2	1	3/2	2
$g(t)$	0	1	0	-1	0



Date ___/___/___

$x(t) = |g(t)|$



$x(t)$ is periodic function with $T = 1$.

$\therefore \omega_0 = 2\pi$.

$\therefore$ The required fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} X(k) . e^{j\omega_0 t k}$$

where $X(k) = \frac{1}{T}\int_T x(t) . e^{-j\omega_0 k t} dt$.

$$X(k) = 1 \int_0^1 \sin\pi t . e^{-j\omega_0 k t} dt$$

$$= \frac{e^{-j\omega_0 k t}}{(-j\omega_0 k)^2 + (\pi)^2}\left[\cancel{\pi\sin\pi t + j\omega_0 k\cos} \; (-j\omega_0 k)\sin\pi t - \pi\cos\pi t\right]$$

$$= \frac{e^{-j\omega_0 k}}{(-1)\omega_0^2 k^2 + \pi^2}\left[-j\omega_0 k \sin\pi - \pi\cos\pi\right]$$

$$- \frac{1}{(-1)(\omega_0)^2 k^2 + \pi^2}\left[-\pi\right]$$

$$= \frac{1}{\pi^2 - \omega_0^2 k^2}\left[+\pi e^{-j\omega_0 k} + \pi\right]$$

$$= \frac{(1 + e^{-j\omega_0 k})\pi}{\pi^2 - \omega_0^2 k^2}$$

Date ___ /___ /______

$$= \frac{\left(1+e^{-j2\pi k}\right)\pi}{\pi^2 - (2\pi)^2 k^2}$$

$$= \frac{\left(1+e^{-j2\pi k}\right)}{\pi - 4\pi k^2}$$

$$= \frac{1+\left(\cos 2\pi k - j\sin 2\pi k\right)}{\pi - 4\pi k^2}$$

$$X(k) = \frac{1+ (1)}{\pi - 4\pi k^2}$$

$$= \frac{1+(1)}{\pi - 4\pi k^2}$$

$$X(k) = \frac{2}{\pi(1-4k^2)}$$

$$\therefore \quad x(t) = \sum_{k=-\infty}^{\infty} \frac{2}{\pi(1-4k^2)} e^{j2\pi k t}$$

I.P

7) $x(t) = E_m \sin \omega_0 t \qquad 0 < t < \pi$

$\qquad \qquad \qquad \qquad \qquad \pi < t < 2\pi$

$\therefore \quad T = 2\pi$

$\omega_0 = 1$

$x(t) = E_m \sin t \qquad 0 < t < \pi$

$\quad = 0 \qquad \pi < t < 2\pi.$

$$X(k) = \frac{1}{2\pi}\int_0^{\pi} E_m \sin t \; e^{-jkt}\, dt$$

$$= \frac{E_m}{2\pi}\left(\frac{e^{-jkt}}{(-jk)^2+(1)^2}\left((-jk)\sin bx - \cos bx\right)\right]_0^{\pi}$$

Page No. ____

Date ___/___/___

$$= \frac{Em}{2\pi}\left[\frac{e^{-jk\pi}}{-k^2+1}(1) - \frac{e^0}{-k^2+1}(-1)\right]$$

$$= \frac{Em}{2\pi}\left(1+e^{-jk\pi}\right)\left(\frac{1}{1-k^2}\right)$$

$$= \frac{Em}{2\pi}\left(\cancel{T}+e^{-jk\pi}\right)\left(\frac{1}{1-k^2}\right)$$

At $k=1$

$$X(k) = \frac{1}{2\pi}\int_0^{2\pi} x(t)\, e^{-j\omega_0 kt}\, dt$$

$$X(k) = \frac{1}{2\pi}\int_0^{2\pi} x(t)\, e^{-j(1)(1)t}\, dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} Em \sin t \; e^{-jt}\, dt$$

$$= \cancel{\frac{Em}{2\pi}\left[\frac{e^{-jt}}{\ldots}\right.}$$

$$= \frac{Em}{2\pi}\left[\frac{e^{-jkt}}{j^2k^2+1^2}\left[-jk\sin t - \cos t\right]\right]_0^{\pi}$$

$$= Em$$

Page No.

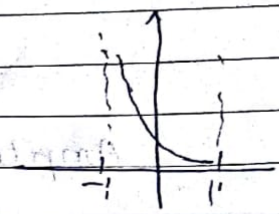
Date __/__/__

Q. 3) $x(t) = e^{-t}$ $\quad -1 < t < 1$

$T = 2$

Draw amplitude and phase spectrum.

Sol: $x(t) = \sum_{k=-\infty}^{\infty} X(k) \cdot e^{j\omega_0 k t}$



$\omega_0 = \pi$

$$X(k) = \frac{1}{T}\int_T x(t) \cdot e^{-j\omega_0 k t}\, dt$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-t} \cdot e^{-j\omega_0 k t}\, dt$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+j\omega_0 k)t}\, dt$$

$$= \frac{1}{2}\left[\frac{e^{-(1+j\omega_0 k)t}}{-(1+j\omega_0 k)}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{e^{-(1+j\omega_0 k)}}{-(1+j\omega_0 k)} - \frac{e^{1+j\omega_0 k}}{-(1+j\omega_0 k)}\right]$$

$$= \frac{1}{-(1+j\omega_0 k)\times 2}\left[e^{-(1+j\omega_0 k)} - e^{(1+j\omega_0 k)}\right]$$

$$= \frac{-1}{2(1+j\pi k)}\left[e^{-1} e^{-j\pi k} - e^{1} \cdot e^{j\pi k}\right]$$

$$= \frac{-1}{2(1+j\pi k)}\left[e^{-1}(-1)^k - e^{1}(-1)^k\right]$$

$$= \frac{(-1)^k}{2(1+j\pi k)}\left[e - \frac{1}{e}\right]$$

$$= \frac{(-1)^k}{(1+j\pi k)}\left[\frac{e - e^{-1}}{2}\right]$$

$$= \frac{(-1)^k}{(1+j\pi k)}\left(\frac{\sinh t}{2}\right)$$

Page No.

Date __/__/__

$$X(k) = (-1)^k \frac{(e - e^{-1})}{2} \times \frac{1 - j\pi k}{1 + \pi^2 k^2}$$

$$x(t) = \sum_{k=-\infty}^{\infty} (-1)^k \frac{(e - e^{-1})}{2} \left(\frac{1 - j\pi k}{1 + \pi^2 k^2}\right) e^{j\pi k t}$$

$|X(k)|$

Amplitude spectrum:- $|X(k)|$

$$|X(k)| = \left| (-1)^k \frac{(e - e^{-1})}{2} \left(\frac{1 - j\pi k}{1 + \pi^2 k^2}\right) \right|$$

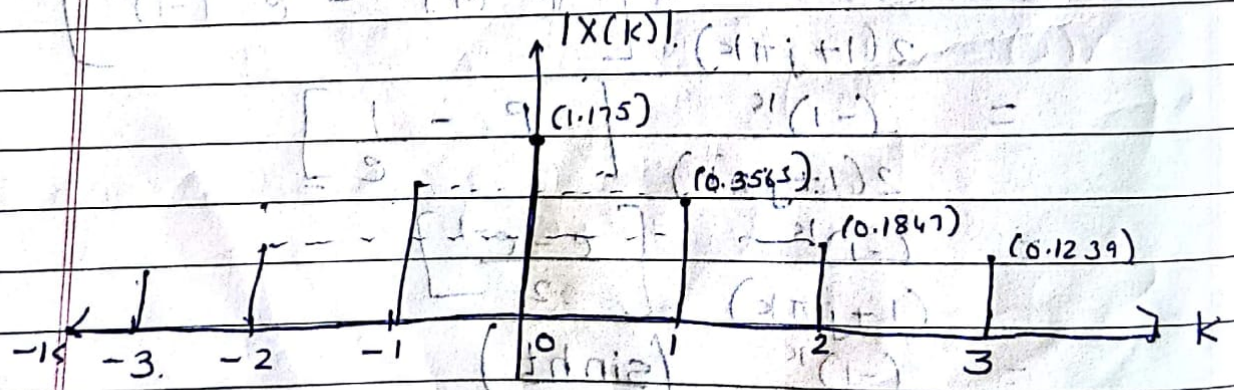
$$= \left(\frac{e - e^{-1}}{2}\right) \frac{1}{1 + \pi^2 k^2} \left| 1 - j\pi k \right|$$

$$= \left(\frac{e - e^{-1}}{2}\right) \left(\frac{1}{(1 + \pi^2 k^2)}\right) \left(\sqrt{1 + k^2 \pi^2}\right)$$

$$= \frac{e - e^{-1}}{2} \left(1 + \pi^2 k^2\right)^{-1/2}$$

$$|X(k)| = \left(\frac{e - e^{-1}}{2}\right) \frac{1}{\sqrt{1 + \pi^2 k^2}}$$

k	0	± 1	± 2	± 3
$X(k)$	$\frac{e - e^{-1}}{2}$	$\frac{e - e^{-1}}{2\sqrt{1+\pi^2}}$	$\frac{e - e^{-1}}{2\sqrt{1+\pi^2 4}}$	$\frac{e - e^{-1}}{2\sqrt{1+\pi^2 9}}$
	1.175	0.3565	0.1847	0.1239



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Date __/__/____

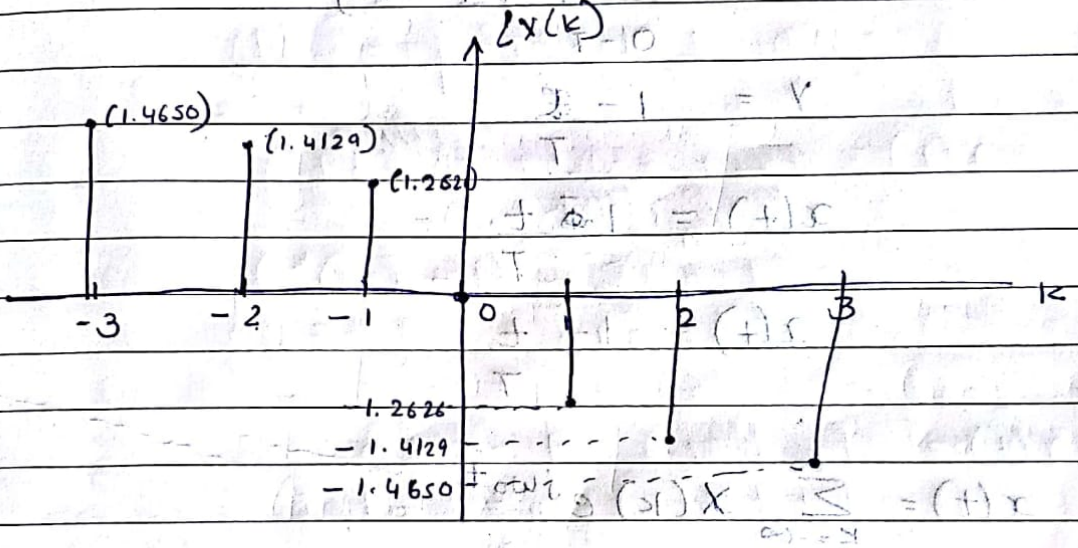
$$\angle X(k) = \left(\frac{e-e^{-1}}{2}\right)\left(\frac{1}{1+\pi^2k^2}\right)\left(1-j\pi k\right)$$

$$\theta = \tan^{-1}\frac{(-\pi k)}{1.}$$

$$\theta = \tan^{-1}(-\pi k)$$
$$= -\tan^{-1}(\pi k).$$

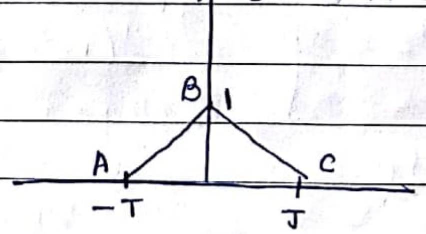
k	0	1	-1	2	-2
	0	-1.2626	1.2626	-1.4129	1.4129

k	3	-3
	-1.4650	1.4650



I.P 6) Here the signal is periodic signal for the period $2T$ and

$$\omega_0 = \frac{2\pi}{2T} = \frac{\pi}{T}$$



Page No.

Date ___/___/___

Let $A(-T, 0)$ $C(0, 1)$ $B(T, 0)$

$$y_2 - y_1 = m(x - x_1)$$

AC $$y - 0 = \frac{1-0}{0+T}(x+T)$$

$$y = \frac{1}{T}(x+T)$$

$$y = 1 + \frac{x}{T}$$

$$\boxed{x(t) = 1 + \frac{t}{T}}$$

BC = $$y - 0 = \frac{1-0}{0-T}(x-T)$$

$$y = 1 - \frac{x}{T}$$

$$x(t) = 1 - \frac{t}{T}$$

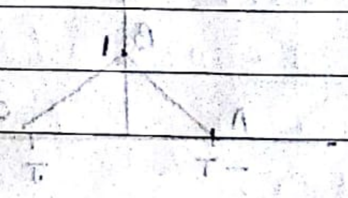
$$x(t) = 1 - \frac{t}{T}$$

$$x(t) = \sum_{k=-\infty}^{\infty} X(k) e^{j\omega_0 t}$$

$$\therefore \quad x(t) = 1 + \frac{t}{T} \qquad -T < t < 0$$

$$= 1 - \frac{t}{T} \qquad 0 < t < T$$

$$x(t) = \sum_{k=-\infty}^{\infty} X(k) . e^{j\omega_0 t}$$



Date ___/___/______

$$X(k) = \frac{1}{T}\int_0^T x(t)\, e^{-j\omega_0 t k}\, dt$$

$$= \frac{1}{2T}\int_{-T}^{T} x(t)\, e^{-j\omega_0 t k}\, dt$$

$$= \frac{1}{2T}\left[\int_{-T}^{0}\left(1+\frac{t}{T}\right)e^{-j\omega_0 t k} + \int_0^T\left(1-\frac{t}{T}\right)e^{-j\omega_0 t k}\right]$$

$$= \frac{1}{2T}\left[\left(\frac{e^{-j\omega_0 t k}}{-j\omega_0 k} + \frac{t e^{-j\omega_0 t k}}{T(-j\omega_0)k} - \frac{e^{-j\omega_0 t k}}{T(-j\omega_0)^2}\right)\right]_{-T}^{0}$$

$$+\frac{1}{2T}\left[\frac{e^{-j\omega_0 t k}}{-j\omega_0 k} - \left(\frac{t e^{-j\omega_0 t k}}{T(-j\omega_0)k} - \frac{e^{-j\omega_0 t k}}{T(-j\omega_0 k)^2}\right)\right]_{T}^{0}$$

$$= \frac{1}{2T}\left[\left(\frac{1}{-j\omega_0 k} - \cancel{\frac{1}{T(-j\omega_0)^2 k}}\right) - \left(\frac{e^{-j\omega_0 T}}{-j\omega_0 k} - \frac{e^{-j\omega_0 T}}{-j\omega_0 k} - \cancel{\frac{e^{-j\omega_0 T}}{T(-j\omega_0)^2}}\right)\right]$$

$$+\frac{1}{2T}\left[\frac{1}{-j\omega_0 k} + \cancel{\frac{1}{T(-j\omega_0)^2}} - \left(\frac{e^{-j\omega_0 T}}{-j\omega_0 T} - \frac{e^{-j\omega_0 T}}{-j\omega_0} + \cancel{\frac{e^{-j\omega_0 T}}{T(j\omega_0)^2}}\right)\right]$$

$$= \frac{1}{2T}\left(\frac{-2}{j\omega_0 k} + \cancel{\frac{e^{+j\omega_0 T}}{j\omega_0 k}} - \cancel{\frac{e^{j\omega_0 T}}{j\omega_0 k}} + \cancel{\frac{e^{-j\omega_0 T}}{j\omega_0 k}} - \cancel{\frac{e^{-j\omega_0 T}}{j\omega_0 k}}\right)$$

$$= \frac{1}{2T}\times\frac{2}{j\omega_0 k}$$

$$= -\frac{1}{T}\times\frac{j}{(-1)\omega_0 k}$$

$$= \frac{j}{T\omega_0 k} = \frac{jT}{T\pi k} =$$

Page No.

Date ___ / ___ / ______

$x(t) =$

$k = 0$

$$X(k) = \frac{1}{2T}\int_0^{2T} x(t)\cdot e^0\, dt$$

$$X(0) = \frac{1}{2T}\int_{-T}^{0} 1+\frac{t}{T} + \int_0^T 1-\frac{t}{T}\, dt$$

$$X(0) = \frac{1}{2T}\left[t+\frac{t^2}{2T}\right]_{-T}^{0} + \frac{1}{2T}\left[t-\frac{t^2}{2T}\right]_0^T$$

$$= \frac{1}{2T}\left[-T - \frac{T^2}{2T} + T - \frac{T^2}{2T}\right]$$

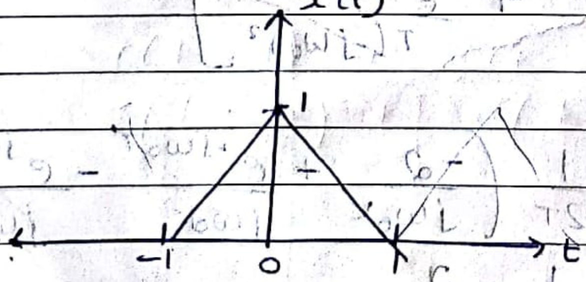
$$= \frac{1}{2T}\left[\frac{-2T^2}{2T}\right] = \frac{1}{2}$$

$$x(t) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{\pi k} e^{jk\omega_0 t}$$

$$= \frac{1}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(1-(-1)^k)}{\pi^2 k^2}$$

Q a)

i) $x(t) = \begin{cases} 1-t & -1 \le t \le 0 \\ 1+t & 0 \le t \le 1 \end{cases}$



Saathi

Date ___/___/___

$$x(t) = \sum_{k=-\infty}^{\infty} X(k).\, e^{j\omega_0 k t}.$$

$$X(k) = \frac{1}{T}\int_T x(t).\, e^{-j\omega_0 k t}\, dt.$$

$$= \frac{1}{2}\int_2 x(t).\, e^{-j\omega_0 k t}\, dt$$

$$= \frac{1}{2}\left[\int_{-1}^{0} (1-t).\, e^{-j\omega_0 k t} + \int_0^1 (1+t).\, e^{-j\omega_0 k t}\, dt\right]$$

$$= \frac{1}{2}\left[\frac{(1-t).\, e^{-j\omega_0 k t}}{-j\omega_0 k} + \frac{e^{-j\omega_0 k t}}{(-j\omega_0 k)^2}\right]_{-1}^{0}$$

$$+ \frac{1}{2}\left[\frac{(1+t)\, e^{-j\omega_0 k t}}{-j\omega_0 k} - \frac{e^{-j\omega_0 k t}}{(-j\omega_0 k)^2}\right]_0^1$$

$$= \frac{1}{2}\left[\cancel{\frac{1}{-j\omega_0 k}} + \frac{1}{(-j\omega_0 k)^2} - \frac{2e^{+j\omega_0 k}}{-j\omega_0 k} - \frac{e^{j\omega_0 k}}{(-j\omega_0 k)^2}\right.$$

$$\left. + \frac{2e^{-j\omega_0 k}}{-j\omega_0 k} + \frac{e^{-j\omega_0 k}}{(-j\omega_0 k)^2} - \cancel{\frac{1}{-j\omega_0 k}} + \frac{1}{(j\omega_0 k)^2}\right]$$

$$= \frac{1}{k^2\pi^2}\left[\cos k\pi - 1\right]$$

$$= \frac{1}{k^2\pi^2}\left[\cancel{\cos k}\, (-1)^k - 1\right)$$

$$X(0) = \frac{1}{2\pi}\int_{-T}^{T} x(t).\, e^{j\pi k t}$$

$$= \frac{1}{2\pi}\int_{-T}^{0} x(t)\, dt$$

$$= \frac{1}{2\pi}\Big[$$

Page No.

Date ___/___/___

$= 1$

10) $g(m) = \delta\left(t - \frac{1}{2} m\right) + \delta\left(t - \frac{3}{2} m\right)$

$g(-3) = \delta\left(t + \frac{3}{2}\right) + \delta\left(t + \frac{9}{2}\right)$

$g(-2) = \delta(t+1) + \delta(t+3)$

$g(-1) = \delta\left(t + \frac{1}{2}\right) + \delta\left(t + \frac{3}{2}\right)$

$g(0) = \delta(t) + \delta(t)$

$g(1) = \delta\left(t - \frac{1}{2} m\right) + \delta\left(t - \frac{3}{2}\right)$

$g(2) = \delta(t-1) + \delta(t-3)$

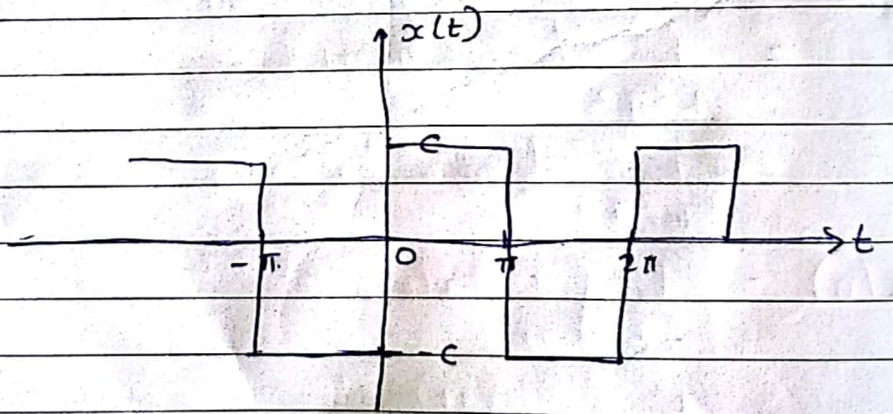
$g(3) = \delta\left(t - \frac{3}{2}\right) + \delta\left(t - \frac{9}{2}\right)$.

L.P

9)

ii) $$x(t) = \begin{cases} -C & -\pi < t < 0 \\ C & 0 < t < \pi \end{cases}$$

Sol:-



The period of periodic signal is 2π.

Date ___ / ___ / ____

$\therefore \omega_0 = 1.$

$\therefore$ From plot it is clear signal is odd signal.

$$x(t) = \sum_{k=-\infty}^{\infty} X(k).\, e^{j\omega_0 k t}$$

$$X(k) = \frac{1}{T}\int_T x(t).\, e^{-j\omega_0 k t}\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} -c.\, e^{-j\omega_0 k t} + \frac{1}{2\pi}\int_0^{\pi} c\, e^{-j\omega_0 k t}\, dt$$

$$= \frac{1}{2\pi}\left[\frac{(-c)\, e^{-jkt}}{(-jk)}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\frac{c\, e^{-jkt}}{(-jk)}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{c}{jk} - \frac{c\, e^{+jk\pi}}{j(k)}\right] + \frac{1}{2\pi}\left[\frac{c\, e^{-j\pi k}}{(-jk)} - \frac{c}{(-jk)}\right]$$

$$= \frac{1}{2\pi}\left[\frac{c}{jk} + \frac{c\,[\cos k\pi + 0]}{jk} + \frac{c\,[\cos k\pi] - c}{-jk}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2c}{jk}\right]$$

$$= \frac{c}{j\pi k}\,(1 - (-1)^{k}).$$

$$X(0) = \frac{1}{2\pi}\int_{-\pi}^{0} -c\, dt + \int_0^{\pi} c\, dt$$

$$= \frac{1}{2\pi}\left[-c\,(0+\pi] + c\,(\pi\right]$$

$$= 0$$

$\therefore$ Required fourier series is

$$x(t) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{c}{j\pi k}\, e^{jkt} + 0.$$

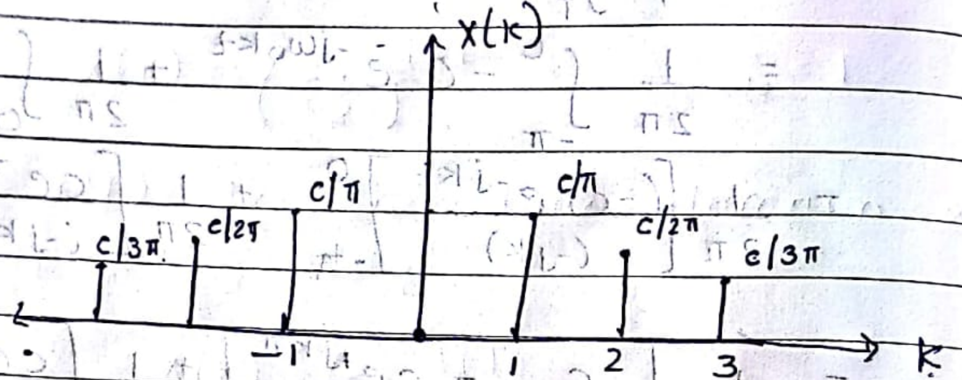
Date ___/___/___

$X(k) = -j\frac{c}{\pi k}$

$|X(k)| = \frac{c}{\pi k}$

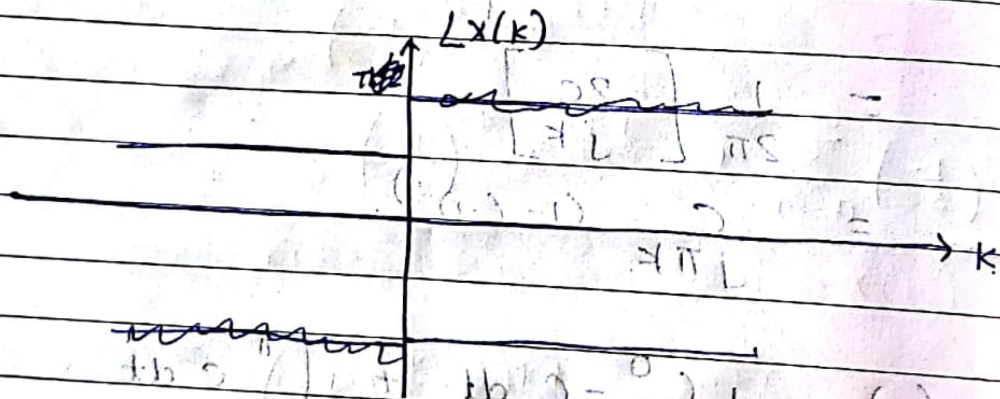
$|X(0)| = 0$

$k =$	0	1	-1	2	-2	3
$\|X(k)\|$	0	c/π	$+c/\pi$	$c/2\pi$	$+c/2\pi$	$c/3\pi$



$\angle X(k) = \tan^{-1}(-c/\pi k / 0)$

$= \pm 90°$



$$f\{\cos\omega_0 t\} = f\left(\frac{e^{+j\omega_0 t} + e^{-j\omega_0 t}}{2}\right)$$
$$= \frac{1}{2}\left[\delta(k-1) + \delta(k+1)\right]$$

Saathi

Date ___/___/___

Find the fourier series representation for the signal
$$x(t) = e^{jn\omega_0 t}$$

Sol:- The required fourier series is
$$x(t) = \sum_{k} X(k).\, e^{j\omega_0 kt}$$

$$X(k) = \frac{1}{T}\int_T x(t).\, e^{-j\omega_0 kt}\, dt$$

$$= \frac{1}{T}\int_T e^{jn\omega_0 t}\, e^{-j\omega_0 kt}\, dt$$

$$= \frac{1}{T}\int_T e^{jn\omega_0 t - j\omega_0 kt}\, dt$$

$$X(k) = 0 \quad \forall\; k \neq n.$$
$$= \frac{T}{T} \qquad k = n.$$
$$= 1.$$

$$X(k) = \delta(k-n)$$

$$\boxed{e^{jn\omega_0 t} \xleftrightarrow{FSw.} \delta(k-n)}$$

w.P

11). Given F.C $X(k)$ $\quad \omega_0 = \pi$

$$X(k) = j\,\delta(k-1) - j\,\delta(k+1) + \delta(k-3) + \delta(k+3)$$

Taking inverse fourier series both sides we get

$$x(t) = j\left(e^{j\pi t}\right) - j\,e^{-j\pi t} + e^{j3\pi t} + e^{-j3\pi t}$$

This gives complex fourier dripⁿ for the given fourier coefficient.

Date ___/___/___

$$x(t) = j\left(e^{j\pi t} - e^{-j\pi t}\right) + e^{j3\pi t} + e^{-j3\pi t}$$

$$= \cancel{2j} \frac{\cancel{e^{j\pi t}}}{j} \cancel{\sin \pi t} + \cancel{2 \sin 3\pi} t$$

$$= \cancel{25}$$

$$= -2\sin \pi t + 2\cos 3\pi t.$$

12). $X(k) = \left(-\frac{1}{2}\right)^{|k|}$ $\omega_0 = 1.$

Let $x(t)$ be the time domain signal corresponding to

$$X(k) = \left(-\frac{1}{2}\right)^{|k|}$$

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{j\omega_0 k t}$$

$$= \sum_{k=-\infty}^{\infty} \left(-\frac{1}{2}\right)^{|k|} e^{jkt}$$

$$= \sum_{k=-\infty}^{-1} \left(-\frac{1}{2}\right)^{-k} e^{jkt} + \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^{k} e^{jkt}$$

$$= \quad I \quad + \quad II$$

$$I \Rightarrow \sum_{k=-\infty}^{-1} \left(-\frac{1}{2}\right)^{-k} e^{jkt}$$

$$= \sum_{k=-\infty}^{k=-1} \left(-\frac{1}{2} e^{-jt}\right)^{-k}$$

Put $-k = m$

$$= \sum_{k=1}^{\infty} \left(-\frac{1}{2} e^{-jt}\right)^{m}$$

Now above series is G.S.

Date ___/___/___

with $\gamma = \frac{-1}{2} e^{-jt}$

$|\gamma| = \left| \frac{-1}{2} e^{-jt} \right|$

$= 1/2 \left| e^{-jt} \right|$

$= \frac{1}{2} \left(\sqrt{\cos^2 t + \sin^2 t} \right)$

$= \frac{1}{2} (1)$

$|\gamma| = 1/2 \quad \therefore \quad |\gamma| < 1.$

Series is convergent

sum of series I $= \dfrac{-1/2\, e^{-jt}}{1 + 1/2\, e^{-jt}}$

$S_1 = \dfrac{-e^{-jt}}{2 + e^{-jt}}$.

II $\rightarrow$ $a = 1$

$\gamma = \frac{-1}{2} e^{+jt}$

$|\gamma| = \left| \frac{-1}{2} e^{+jt} \right|$

$= \frac{1}{2} (1)$

$= 1/2 \rightarrow (|\gamma| < 1.$

Series is convergent

sum of series $= \dfrac{1}{1 + 1/2\, e^{+jt}}$

$= \dfrac{2}{2 + e^{+jt}}$

$\therefore \; x(t) = \dfrac{-e^{-jt}}{2 + e^{-jt}} + \dfrac{2}{2 + e^{+jt}}$.

Page No.

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Date ___/___/_____

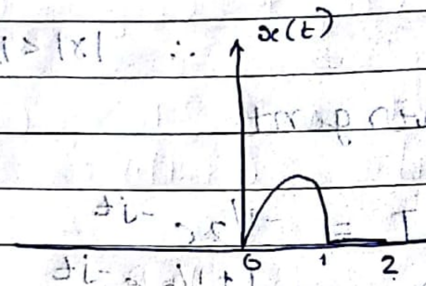
$$= \frac{-e^{-jt}(2+e^{jt}) + 2(2+e^{-jt})}{4+2e^{jt}+2e^{-jt}+1}$$

$$= \frac{-1+4}{5+2(\cos t+\cos t)}$$

$$= \frac{3}{5+4\cos t}$$

L.P

15)
i)



$x(t)$ is periodic funⁿ ; $T = 2$ $\omega = \pi$

Here figure 1 is obtained by shifting $x(t)$ to the right side by one unit

$y(t) = a(t) = x(t-1)$.

By time shifting property

$$y(t) = x(t-1) \xleftrightarrow{FS;\omega_0} e^{-j\omega_0 k t_0} X(k)$$

$$y(t) \xleftrightarrow{FS;\omega_0} e^{-j\pi k(1)} X(k)$$

$$e^{-j\pi k} X(k)$$

Given $X(0) = \frac{1}{\pi}$

$X(1) = -j0.25$

$$X(k) = \frac{1}{\pi(1-k^2)}$$

Date ___/___/___

$Y(k) = e^{-jk\pi} X(k)$

$\Rightarrow$ $Y(0) = e^{0} X(0)$

$Y(0) = \frac{1}{\pi}$

$Y(1) = e^{-j\pi} X(1)$

$= e^{-j\pi}(-j)(0.25)$

$= (-1)(-j)(0.25)$

$= j\,0.25$

$Y(k) = e^{-jk\pi} X(k)$

$= e^{-jk\pi} \frac{1}{\pi(1-k^2)}$ if k is even.

$= \frac{(-1)^k}{\pi(1-k^2)}$ k is even

$= \frac{1}{\pi(1-k^2)}$

ii) Second figure is obtained by reflecting $y(t)$ w.r.t t axis i.e $-y(t)$ then adding it to the $x(t)$, then we ~~fig~~ get $b(t)$.

$b(t) = x(t) - y(t)$

Using linearity property, the fourier coefficient of $b(t)$ is given by

$B(k) = X(k) - Y(k)$

$B(0) = \frac{1}{\pi} - \frac{1}{\pi}$

$= 0$

$B(1) = -j\,0.25 - j\,0.25$

$= -0.5j$

$B(k) = \frac{1}{\pi(1-k^2)} - \frac{1}{\pi(1-k^2)}$ if k even.

$= 0$

Page No.

Date ___/___/___

iii) fig is given by

$$c(t) = x(t) + y(t)$$

$$c(k) = X(k) + Y(k)$$

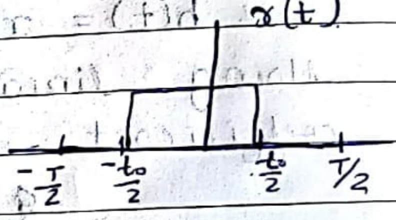
$$c(0) = \frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi}$$

$$c(1) = -j0.25 + j0.25 = 0$$

$$c(k) = \frac{1}{\pi(1-k^2)} + \frac{1}{\pi(1-k^2)}$$

$$= \frac{2}{\pi(1-k^2)} \quad \text{when } k \text{ is even.}$$

16) ~~Here~~ given signal is periodic signal with period T



$$x(t) = 0 \qquad -\frac{T}{2} < t < \frac{t_0}{2}$$

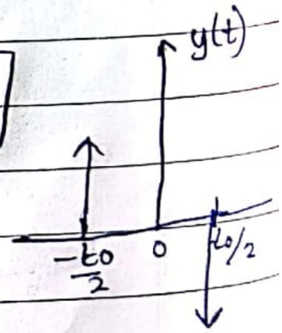
$$= 1 \qquad -\frac{t_0}{2} < t < \frac{t_0}{2}$$

$$= 0 \qquad \frac{t_0}{2} < t < \frac{T}{2}$$

$$x(t) = 1\left[u\left(t+\frac{t_0}{2}\right) - u\left(t-\frac{t_0}{2}\right)\right]$$

$$\frac{d\,x(t)}{dt} = \frac{d}{dt}\left[u\left(t+\frac{t_0}{2}\right) - u\left(t-\frac{t_0}{2}\right)\right]$$

$$y(t) = \delta\left(t+\frac{t_0}{2}\right) - \delta\left(t-\frac{t_0}{2}\right)$$



Page No.

$\int \delta(t-t_0).x(t)\,dt = x(t_0) \rightarrow$ shifting

Date __/__/__

Saathi

using differentiation property

$$\frac{d}{dt}x(t) \xleftarrow{FS;\omega_0} jk\omega_0 X(k)$$

$$\therefore \; Y(k) = jk\omega_0 X(k)$$

$$\therefore \; X(k) = \frac{1}{jk\omega_0} Y(k)$$

$$Y(k) = \frac{1}{T}\int_T y(t).\,e^{-j\omega_0 kt}\,dt$$

$$= \frac{1}{T}\int_T \delta(t+\tfrac{t_0}{2}) - \delta(t-\tfrac{t_0}{2})\; e^{-j\omega_0 kt}\,dt$$

$$= \frac{1}{T}\int_T \delta(t+\tfrac{t_0}{2}).\,e^{-j\omega_0 kt} - \frac{1}{T}\int_T \delta(t-\tfrac{t_0}{2}).\,e^{-j\omega_0 kt}$$

$$= \frac{1}{T}\left[e^{-j\omega_0 kt}\Big|_{t\to -t_0/2} - \frac{1}{T}\left[e^{-j\omega_0 kt}\right]_{t\to t_0/2}\right.$$

[By impulse shifting property]

$$= \frac{1}{T}\left[e^{j\omega_0 \frac{t_0}{2}k} - e^{-j\omega_0 \frac{t_0}{2}k}\right]$$

$$= \frac{1}{T}\left[2j\sin\omega_0 k\frac{t_0}{2}\right]$$

$$j\left[\frac{2}{T}\sin\omega_0 k\frac{t_0}{2}\right]$$

$$\therefore \; X(k) = \frac{1}{\cancel{j}k\omega_0} \times \cancel{j} \times \frac{2}{T}\sin\omega_0 k\frac{t_0}{2}$$

$$= \frac{2}{k\omega_0 T}\sin\omega_0 k\frac{t_0}{2} \qquad \forall\; k \text{ except } k\neq 0$$

$$X(0) = \frac{1}{T}\int_T x(t)\,dt$$

Page No.

Date __/__/__

$$= \frac{1}{T}\int_{-t_0/2}^{t_0/2} 1\, dt$$

$$= \frac{1}{T}\left[\frac{t_0}{2} + \frac{t_0}{2}\right]$$

$$X(0) = \frac{t_0}{T}$$

$$X(k) = \frac{2}{k\omega_0 T} \sin \omega_0 \frac{t_0}{2} k$$

$$= \frac{2}{\cancel{k}\cancel{\omega_0}T} \frac{\sin \omega_0 t_0/2\, k}{k\omega_0 \frac{t_0}{2}} \times \cancel{\omega_0} \frac{t_0}{2} \cancel{k}$$

$$= \frac{t_0}{T} \frac{\sin \omega_0 t_0/2\, k}{k\omega_0 t_0/2}$$

$$X(0) = \frac{t_0}{T} \lim_{k\to 0} \frac{\sin \omega_0 t_0/2\, k}{k\omega_0 t_0/2}$$

$$X(0) = \frac{t_0}{T}.$$

If $x(t)$ is a periodic fun^n with T.P = 1

F.C $X(k) = \left(\frac{-1}{3}\right)^k \quad k \geq 0$

$= 0 \quad$ otherwise

compute $x(0)$

Sol- Let $x(t)$ be the fourier series with F.S.C $X(k)$

$$x(t) = \sum_{k=-\infty}^{\infty} X(k).\, e^{j\omega_0 kt}$$

$$= \sum_{k=0}^{\infty} X(k).\, e^{j\omega_0 kt}$$

$$= \sum_{k=0}^{\infty} \left(\frac{-1}{3}\right)^k e^{j\omega_0 kt}$$

Page No.

Saathi

Date ___/___/___

$$x(t)\ \sum_{k=0}^{\infty}\left(\frac{-1}{3}e^{j\omega_0 t}\right)^k$$

It is convergent $\qquad a=1$

$$x(t)=\frac{-1/3\,e^{j\omega_0 t}}{\cancel{1+\frac{1}{3}e^{j\omega_0 t}}}\qquad \frac{1}{1+1/3\,e^{j\omega_0 t}}$$

$$=\frac{\cancel{-1e^{j\omega_0 t}}}{\cancel{3+e^{j\omega_0 t}}}\qquad \frac{3}{3+e^{j\omega_0 t}}$$

$$x(0)=\frac{3}{3+e^0}=\frac{3}{3+1}=\frac{3}{4}$$

14)

i) $T_1=\frac{2\pi}{3}$

$T_2=\frac{2\pi}{4}$

$\frac{T_1}{T_2}=$ rational

$\therefore T=\frac{2\pi}{3}\times 3$

$=2\pi$

ii) $P=\frac{1}{T}\int_0^T |(x^2(t))|\,dt$

$=\frac{1}{2\pi}\int_0^{2\pi}$

Using Parseval's identity

$$P_{avg}=\sum_{k=-\infty}^{\infty}(X(k))^2$$

$$P_{avg}=16+1+1+\frac{9}{4}+\frac{9}{4}$$

$$=18+\frac{18}{4}=\frac{72+18}{4}=\frac{90}{4}=\frac{18}{\cancel{8}}$$

Page No.

Date ___/___/___

19/10/18

$$= \frac{45}{2}$$

$$= 22.5$$

iii) $T = 2\pi$

$\omega = 1$

$$x(t) = 4 + 2\cos 3t + 3\sin 4t$$

$$= 4 + 2\left(\frac{e^{j3t} + e^{-j3t}}{2}\right) + 3\left(\frac{e^{j4t} - e^{-j4t}}{2j}\right)$$

$$= 4 + e^{j3t} + e^{-j3t} - \frac{3j}{2} e^{j4t} + \frac{3e^{-j4t}}{2} j$$

$$= 4e^{j0\omega_0 t} + e^{j3(1)t} + e^{j(-3)(1)t} - \frac{3}{2} j e^{j4t} + \frac{3}{2} e^{j(-4)t}$$

$X(0) = 4$

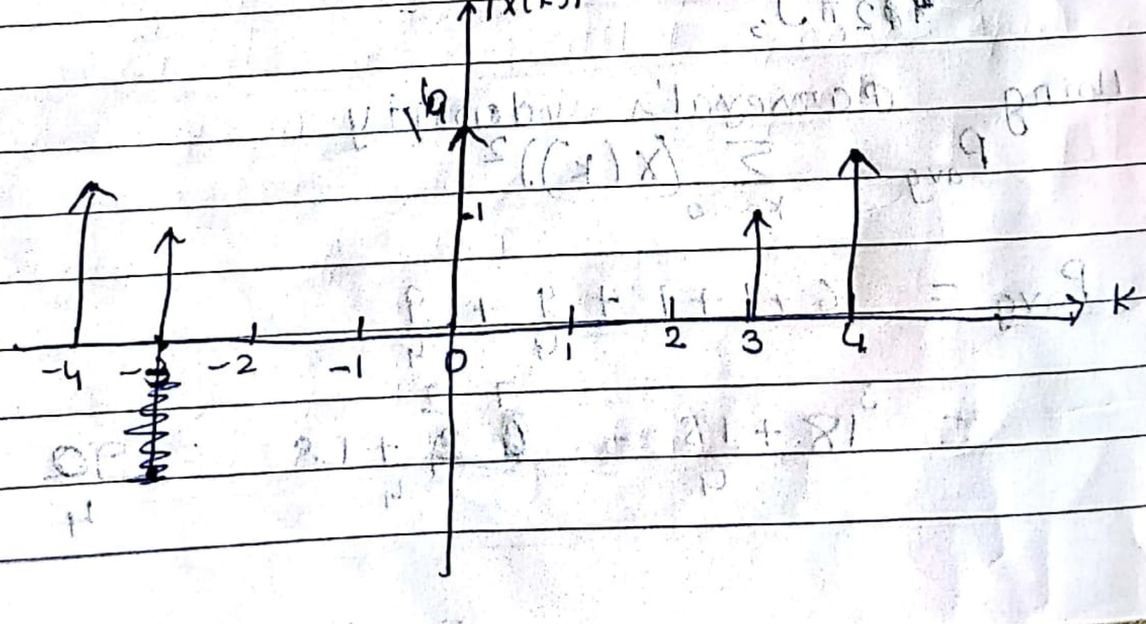
$X(3) = 1$

$X(-3) = 1$

$X(4) = -3/2\, j$

$X(-4) = 3/2\, j$

iv) Power spectrum $|X(k)|^2 \rightarrow k$



Page No.

19/10/18 Date

Saathi

Fourier Transform

Both fourier series and fourier transform are mathematical tool for representing signal in frequency domain.

Fourier series is mainly used for periodic signals whereas F.T is used for aperiodic signals

1) Defⁿ :- The fourier transform of a continuous time signal $x(t)$ is given by

$$x(t) \xrightarrow{F.T} X(j\omega) \doteq \overset{\text{rad/sec}}{X(\omega)} = \overset{\text{Hz}}{X(f)}$$

where $X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega_0 t} \, dt$

this equation is known as analysis eqⁿ used to transform time domain to frequency domain signal.

2) Defⁿ :- The inverse fourier transform is given by

$$X(j\omega) \longrightarrow x(t)$$

where

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot e^{+j\omega_0 t} \, d\omega$$

this formula is known as synthesis eqⁿ and used to transform frequency domain to time domain signal.

In general defⁿ 1 & 2 together known as fourier transform pair and we reprⁿ it as

$$x(t) \underset{\longleftarrow}{\xrightarrow{F.T}} X(j\omega)$$

Date __ / __

NOTE:-

1) Fourier transform of a signal is a complex number. It has both magnitude and angle. Hence we can represent

$$X(j\omega) = |X(j\omega)| \angle X(j\omega).$$

Convergence / Existance of F.T

F.T. of $X(j\omega)$ for a cont^n time signal $x(t)$ exist if following conditions are satisfied:-

i) $x(t)$ is absolutely integral.

$$\int_{-\infty}^{\infty} |f(t)| \, dt = \text{finite}$$

ii) $x(t)$ has finite no of maxima and minima over finite interval of time.

iii) $x(t)$ has finite no of discont^n over any finite interval of time.

Magnitude and Phase spectra

The graph of $X(j\omega)$ is called spectrum of the signal. The plot of $|X(j\omega)|$ v/s ω is called magnitude spectra and that of $\angle X(j\omega)$ v/s ω is called phase spectra of the given aperiodic signal.

Properties of F.T:-

i) Linearity property:-

if $x(t) \xleftrightarrow{FT} X(\omega)$

Saathi

Date ___/___/___

and $y(t) \xleftrightarrow{F.T} Y(\omega)$

then

$$z(t) = ax(t) + by(t) \longleftrightarrow aX(\omega) + bY(\omega)$$

ii) Time shift property :-

If

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

then $y(t) = x(t - t_0) \xleftrightarrow{F.T} e^{-j\omega t_0} X(\omega)$

iii) Frequency shift property :-

If

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

then $y(t) = e^{j\beta t} x(t) \xleftrightarrow{F.T} Y(\omega) = X(\omega - \beta)$

iv) Time scaling property

If

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

then $y(t) = x(at) \xleftrightarrow{F.T} \frac{1}{|a|} X\left(\frac{\omega}{|a|}\right)$

iv) Time differentiation property

If

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

then $\frac{dx(t)}{dt} \xleftrightarrow{F.T} j\omega X(\omega)$

v) Time reversal property

If

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

then $x(-t) \xleftrightarrow{F.T} X(-\omega)$

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Date ___ / ___ / ___

vi) Frequency diffirentration:

If
$$x(t) \xleftrightarrow{F.T} X(\omega)$$
then $$-jt\,x(t) \xleftrightarrow{F.T} \frac{d}{d\omega} X(\omega).$$

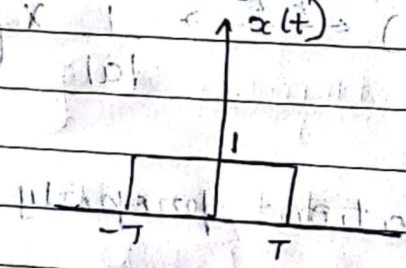
or
$$t\,x(t) \xleftrightarrow{F.T} j\frac{d}{d\omega} X(\omega).$$

vii)

vii) Duality or similarity theorem

If $x(t) \xleftrightarrow{F.T} X(\omega)$,
then
$$X(t) \xleftrightarrow{F.T} 2\pi\, x(-\omega)$$ = (obtained from inverse)

1) Find the fourier transform of rectangular pulse signal given by the fig → Draw the spectrum, magnitude and phase spectrum



Sol:- The given rectangular signal in terms of mathematical expⁿ:-

$$x(t) = 1 \qquad -T < t < T \quad \text{or} \quad |x| < T$$
$$= 0 \qquad \text{otherwise}$$

By defⁿ of F.T of $x(t)$, we have
$$x(t) \xleftrightarrow{F.T} X(j\omega)$$
$$X(j\omega) = X(\omega) = \int_{-\infty}^{\infty} x(t).\, e^{-j\omega t}\, dt$$

$\frac{\sin\theta}{\theta} = sa(\theta) \rightarrow$ sa~~ch~~ funⁿ | sampling funⁿ

If ω is converted to π, $\sin\tau$ (~~sink funⁿ~~)

Date ___/___/___

$$= \int_{-T}^{T} 1\, e^{-j\omega t}\, dt + 0$$

$$= \left.\frac{e^{-j\omega t}}{(-j\omega)}\right]_{-T}^{T}$$

$$= \frac{e^{-j\omega T} - e^{j\omega T}}{(-j\omega)}$$

$$= \frac{2}{\omega}\left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)$$

$$= \frac{2}{\omega}\sin\omega T$$

This is for all except $\omega = 0$.

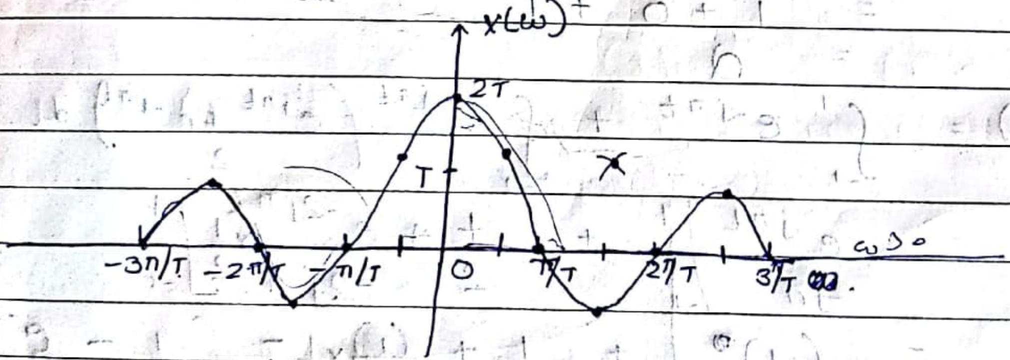
$X(0) = \int_{-\infty}^{\infty} x(t)\, dt$	or $X(0) = 2T\frac{\sin\omega T}{\omega T}$
$= T + T$	$= 2T \lim_{\omega \to 0} \frac{\sin\omega T}{\omega t}$
$= 2T$	$= 2T$

$\Rightarrow$

$X(j\omega) = 2/\omega \sin\omega T \quad \forall\omega$ except $\omega = 0$

$= 2T \quad \omega = 0$

The graph of $X(\omega)$ is called spectrum.



ω	0	$\pi/2T$	π/T	$3\pi/2T$	$2\pi/T$	$5\pi/2T$
$X(\omega)$	$2T$	$4T/\pi$	0	$-T(0.4244)$	0	0.256

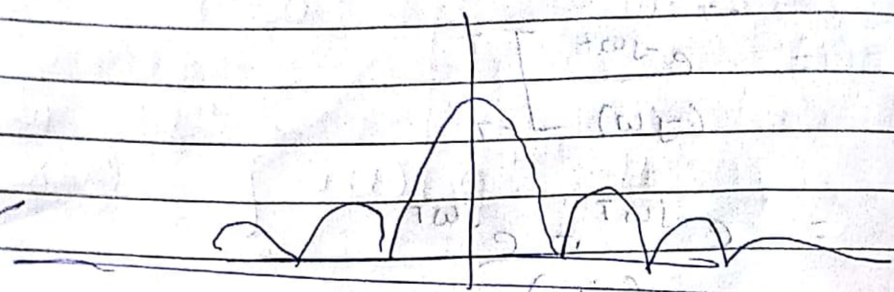
$\frac{2 \times 2T}{3\pi} =$

Page No.

Magnitude spectra $|X(\omega)| \longrightarrow \omega$

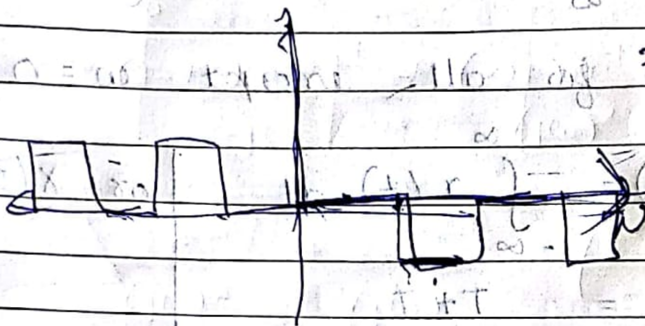
Date __ /__ /____

$$\left|\frac{2}{\omega}\sin\omega a\right| \longrightarrow \omega$$



Phase spectra $\angle X(\omega) = \tan^{-1}\left(\frac{0}{\frac{2}{\omega}\sin\omega a}\right)$

$$\angle X(\omega) = \frac{0}{\left(\frac{2}{\omega}\sin\omega a > 0\right)} = \pm\pi$$



$\omega > 0$, $-\pi$

$\omega < 0$, π

$$X(0) = \int_{-1}^{1} 1 + \cos\pi t \; dt$$

$$= \left[t + \frac{\sin\pi t}{t} \right]_{-1}^{1}$$

$$= 1 + 0 + 1$$

$$= 2$$

$$X(\pi) = \int_{-1}^{1} e^{-j\pi t} + \int_{-1}^{1} e^{-j\pi t}\left(\frac{e^{j\pi t} + e^{-j\pi t}}{2}\right) dt$$

$$= \left.\frac{e^{-j\pi t}}{-j\pi}\right]_{-1}^{1} + \left[\frac{t}{2} + \frac{e^{-2j\pi t}}{-2j\pi} \times \frac{1}{2}\right]_{-1}^{1}$$

$$= \frac{(-1)}{-j\pi} + \frac{1}{2} + \frac{(1)\times 1}{-2j\pi \cdot 2} - \left[\frac{-1}{2} - \frac{e^{-2j\pi}}{2j\pi}\times\frac{1}{2}\right]$$

$$= \frac{1}{j\pi} + 1 - \frac{2}{2j\pi}$$

$$= \frac{2 + 4 - 2}{}$$

cont^n last

Saathi

Date ___/___/______

Q 5)

i) $x(t) = \begin{cases} 1 + \cos\pi t & |t| \le 1 \\ 0 & |t| > 1 \end{cases}$

sol:- By definition of fourier T of sig $x(t)$ we have

$$X(\omega) = \int_{-\infty}^{\infty} x(t) . e^{-j\omega t}\, dt.$$

$$= \int_{-1}^{1} x(t) . e^{-j\omega t}\, dt$$

$$= \int_{-1}^{1} (1 + \cos\pi t) . e^{-j\omega t}\, dt.$$

$$= \int_{-1}^{1} e^{-j\omega t} + \int_{-1} e^{-j\omega t} \cos\pi t \;\; dt.$$

$$= \left. \frac{e^{-j\omega t}}{-j\omega} \right]_{-1}^{1} + \int_{-1}^{1} e^{-j\omega t} \left(\frac{e^{j\pi t} + e^{-j\pi t}}{2} \right) dt$$

$$= -\left. \frac{e^{-j\omega t}}{-j\omega} \right]_{-1}^{1} + \frac{1}{2} \int_{-1}^{1} e^{j(\pi t - \omega t)} + e^{-jt(\omega + \pi)} \; dt.$$

$$= \frac{e^{-j\omega} - e^{+j\omega}}{-j\omega} + \frac{1}{2} \left[\frac{e^{j(\pi - \omega)t}}{j(\pi - \omega)} + \frac{e^{-jt(\omega + \pi)}}{-j(\pi + \omega)} \right]_{-1}^{1} dt.$$

$$X(\omega) = \frac{e^{-j\omega} - e^{j\omega}}{-j\omega} + \frac{1}{2} \left[\frac{e^{j(\pi - \omega)}}{j(\pi - \omega)} + \frac{e^{-j(\omega + \pi)}}{-j(\pi + \omega)} - \frac{e^{-j(\pi - \omega)}}{j(\pi - \omega)} - \frac{e^{j(\omega + \pi)}}{-j(\pi + \omega)} \right]$$

$$X(\omega) = \frac{e^{j\omega} - e^{-j\omega}}{j\omega} + \frac{1}{2} \left[\frac{e^{j(\pi - \omega)} - e^{-j(\pi - \omega)}}{j(\pi - \omega)} + \frac{e^{j(\omega + \pi)} - e^{-j(\omega + \pi)}}{j(\pi + \omega)} \right]$$

$$= \frac{2 \sin\omega}{\omega} + \frac{\sin(\pi - \omega)}{(\pi - \omega)} + \frac{\sin(\pi + \omega)}{\pi + \omega}$$

$$X(\omega) = \frac{2 \sin\omega}{\omega} + \frac{\sin\omega}{\pi - \omega} - \frac{\sin\omega}{\pi + \omega} \qquad \forall\; \omega \neq 0, \pi, -\pi$$

Page No.

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$(-j)^2$ $(-1)j$

Date ___ / ___ / ___

ii) $x(t) = \begin{cases} 1 - t^2 & ; \quad 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$

Sol.

By def^n.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(\omega) = \int_0^1 x(t) \cdot e^{-j\omega t}\, dt$$

$$= \int_0^1 (1-t^2)\, e^{-j\omega t}\, dt$$

$$= \left[\frac{(1-t^2)\, e^{-j\omega t}}{-j\omega} - \frac{(-2t)\, e^{-j\omega t}}{(j\omega)^2} + \frac{(-2) \cdot e^{-j\omega t}}{-(j\omega)^3} \right]_0^1$$

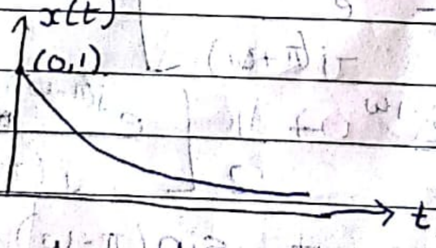
$$= \frac{2e^{-j\omega}}{(j\omega)^2} + \frac{2e^{-j\omega}}{(j\omega)^3} - \left(\frac{1}{-j\omega} + \frac{2}{(j\omega)^3} \right)$$

$$= \frac{2e^{-j\omega}}{-\omega^2} + \frac{2e^{-j\omega}}{-j\omega^3} + \frac{1}{j\omega} - \frac{2}{-j\omega^3}$$

$$= \frac{1}{j\omega} - \frac{2e^{-j\omega}}{\omega^2} + \frac{2}{j\omega^3}\left(1 - e^{-j\omega}\right).$$

L.P 1) $x(t) = e^{-at} \cdot u(t)$

Sol: The given signal $x(t) = e^{-at}\, u(t)$



Now, F.T for given signal

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

4

Date ___/___/______

$$X(\omega) = \int_0^{\infty} x(t) \cdot e^{-j\omega t}\, dt$$

$$= \int_0^{\infty} e^{-at}\, e^{-j\omega t}\, dt \qquad u(t) = 1$$

$$= \int_0^{\infty} e^{+t(-a-j\omega)}\, dt$$

$$= \left. \frac{e^{t(-a-j\omega)}}{-a-j\omega} \right]_0^{\infty}$$

$$= 0 - \frac{1}{-a-j\omega}$$

$$X(\omega) = \frac{1}{a+j\omega}$$

$$\boxed{x(t) = e^{-at} \xrightarrow{F.T} \frac{1}{a+j\omega} = X(\omega)} \quad \text{Imp}$$

Magnitude spectra

$$X(\omega) = \frac{1}{a+j\omega} = \frac{a-j\omega}{a^2+\omega^2} = \frac{a}{a^2+\omega^2} - \frac{j\omega}{a^2+\omega^2}$$

$$|X(\omega)| = \frac{\sqrt{a^2+\omega^2}}{a^2+\omega^2} \qquad \text{If } y = \frac{z_1}{z_2}$$

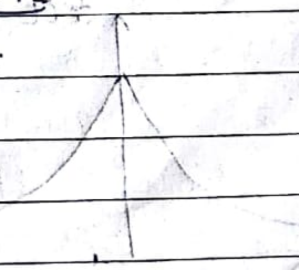
$$= \sqrt{\frac{1}{a^2+\omega^2}} \qquad |y| = \frac{|z_1|}{|z_2|}$$

$$= 1/\sqrt{a^2+\omega^2}$$

ω	0	a	2a
$\lvert X(\omega)\rvert$	$1/a$	$1/a\sqrt{2}$	$1/\sqrt{5}a$
		$\frac{0.7071}{a}$	$\frac{0.4472}{a}$

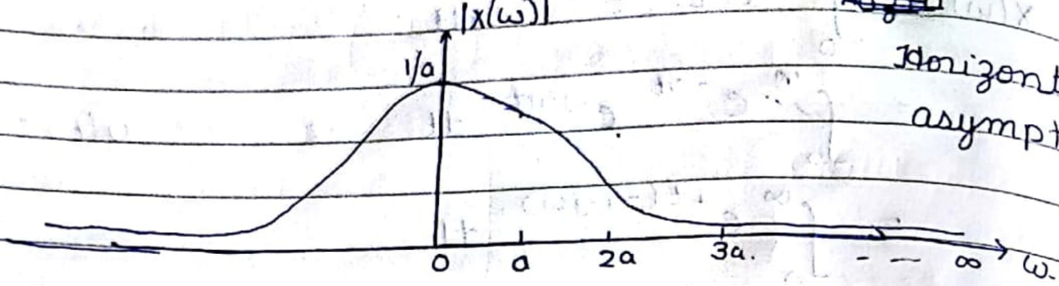
$\omega \to \infty$

$|X(\omega)| \to 0.$



Page No.

Date ___ / ___ / ______



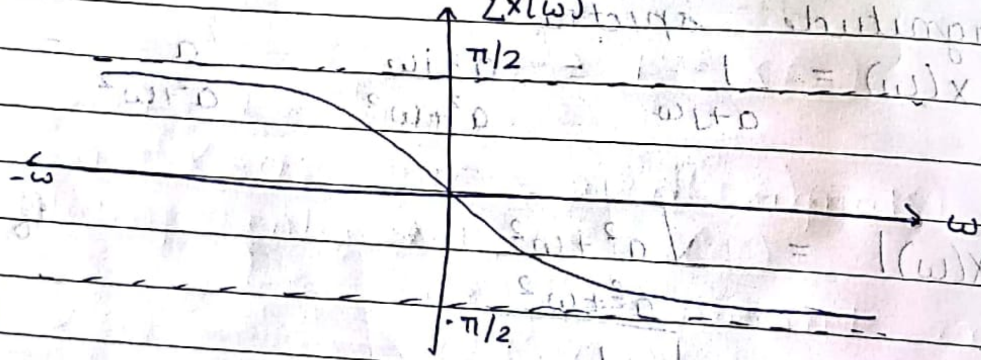
Phase spectra :-

$\angle X(\omega) = \tan^{-1}\left(\frac{-\omega}{a}\right)$.

ω	0	a	−a	2a	> 2a	$\omega \to \infty$
$\angle X(\omega)$.	0.	$-\pi/4$	$\pi/4$.	−1.1071	1.1071	$\angle X(\omega) \to -\pi/2$
		−0.785	0.785			

$\omega \to -\infty$

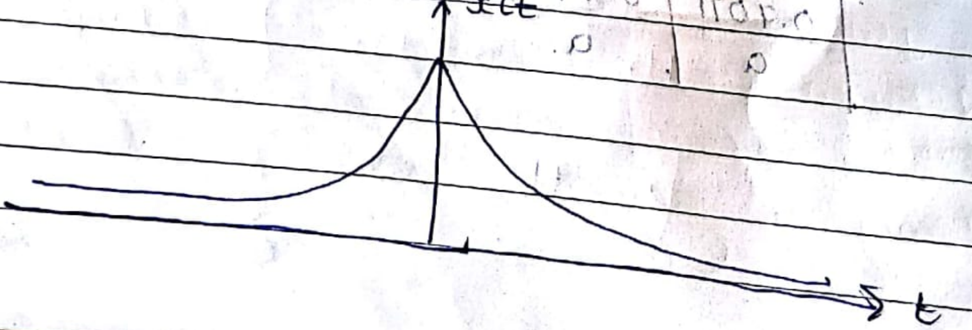
$\angle X(\omega) \to \pi/2$



L.P. 2). $x(t) = e^{-a|t|}$.

$x(t) = e^{-at}$ $\quad t > 0$

$\quad\;\; = e^{at}$ $\quad t < 0$.



Saathi

Date __/__/__

F.T of given signal is

$$X(\omega) = \int_{-\infty}^{\infty} x(t).\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{at}.\, e^{-j\omega t} + \int_{0}^{\infty} e^{-at}.\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{t(a-j\omega)}\, dt + \int_{0}^{\infty} e^{t(-a-j\omega)}\, dt$$

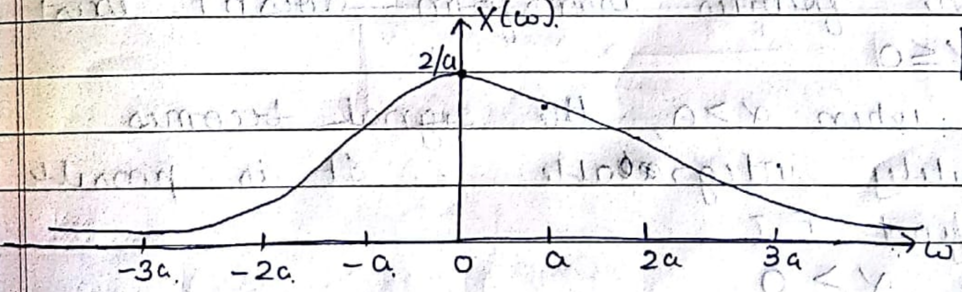
$$= \left[\frac{e^{t(a-j\omega)}}{a-j\omega}\right]_{-\infty}^{0} + \left[\frac{e^{t(-a-j\omega)}}{-a-j\omega}\right]_{0}^{\infty}$$

$$= \frac{1}{a-j\omega} - 0 + \frac{e^{-t(a+j\omega)}}{-a-j\omega} + \frac{1}{a+j\omega}$$

$$= \frac{a+j\omega + a-j\omega}{a^2+\omega^2}$$

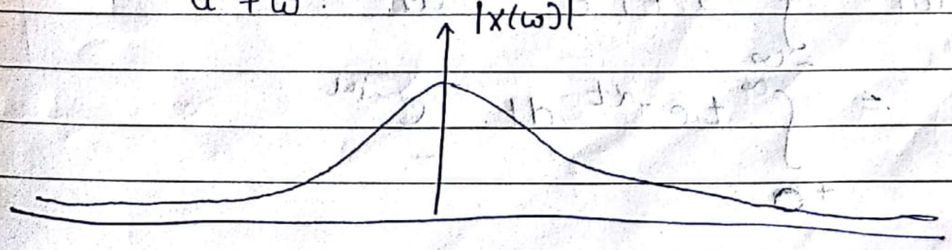
$$X(\omega) = \frac{2a}{a^2+\omega^2}$$

ω	0	a	2a	3a	∞
$X(\omega)$	2/a	1/a	2/5a	2/10a	0



$\boxed{\omega \text{ is H.A}}$

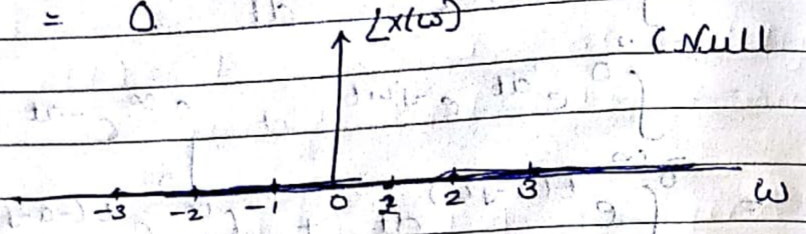
$$|X(\omega)| = \frac{2a}{a^2+\omega^2}$$



Date ___/___/___

$$\angle X(\omega) = \tan^{-1}\left(\frac{0}{\frac{2a}{a^2+\omega^2}}\right)$$

$$= 0$$



(Null spectrum)

$\alpha = \tan^{-1} \frac{b}{a}$

If $a, b > 0$ $\quad \theta = \alpha$
If $a < 0,\ b > 0$ $\quad \theta = \pi - \alpha$
If $a < 0,\ b < 0$ $\quad \theta = -\pi + \alpha$
If $a > 0,\ b < 0$ $\quad \theta = -\alpha$

Q 3). $\quad x(t) = t e^{-\alpha t} u(t)$.

Sol:- When $\alpha \leq 0$, the given signal is not absolutely integrable.
$\therefore$ The fourier transform doesn't exist for $\alpha \leq 0$

But when $\alpha > 0$, the signal becomes absolutely integrable $\therefore$ It is possible to find F.T when $\alpha > 0$

$$x(t) = t e^{-\alpha t} u(t).$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} t e^{-\alpha t}\, dt \cdot e^{-j\omega t}$$

Date __/__/__

$$= \left[\frac{t e^{-t(\alpha+\omega j)}}{-(\alpha+\omega j)} - \frac{e^{-t(\alpha+j\omega)}}{(\alpha+j\omega)^2}\right]_0^{\infty}$$

$$= 0 + 0 - \left[\frac{-1}{(\alpha+j\omega)^2}\right]$$

$$= \frac{1}{(\alpha+j\omega)^2}$$

$$|X(\omega)| = \frac{1}{\sqrt{(\alpha^2-\omega^2)^2 + 4\omega^2\alpha^2}} \qquad \angle X(\omega) = -\tan^{-1}\left(\frac{2\omega\alpha}{\alpha^2-\omega^2}\right)$$

I.P 8)

i) $x(t) = \delta(t)$

NOTE Impulse related signals are absolutely integrable signal we can find F.T. by using direct def^n of F.T.

By def^n of F.T,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t-0) . e^{-j\omega t}\, dt$$

$$= e^{-j\omega t}\Big|_{t=0}$$

$$= \underline{1}$$

Imp

$$\boxed{x(t) = \delta(t) \xrightarrow{F.T} 1}$$

Date ___ /___ /______

ii) $x(t) = 1$

Sol:- $\int_{-\infty}^{\infty} x(t)\,dt = \infty$.

Here $x(t) = 1$ is not absolutely integrable signal, we cannot find its fourier T with the help of def^n.

For such signal we try to find F.T using properities.

If $x(t) = \delta(t) \xleftrightarrow{F.T} X(\omega) = 1$

using duality property

$$x(t) \xrightarrow{F.T} X(\omega)$$
$$X(t) \longrightarrow 2\pi\, x(-\omega)$$

$$x(t) = \delta(t) \xrightarrow{F.T} 1 = X(\omega)$$

$$X(t) = 1 \xrightarrow{F.T} 2\pi\, \delta(-\omega)$$
$$X(t) = 1 \xleftrightarrow{F.T} 2\pi\, \delta(\omega)$$ δ is even fun^n.

$$\therefore \boxed{1 \xleftrightarrow{F.T} 2\pi\, \delta(\omega)}$$

NOTE:-

1) In general, $x(t) = A$ (dc value)

$$x(t) = A \xrightarrow{F.T} 2\pi A\, \delta(\omega)$$

* $x(t) = e^{-j\omega_0 t}$.

Sol: $\int_{-\infty}^{\infty} e^{-j\omega_0 t} = \left.\frac{e^{-j\omega_0 t}}{-j\omega_0}\right]_{-\infty}^{\infty} = \infty$

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Date ___/___/___

It is not absolutely integrable

$\therefore$ We find F.T using properties of F.T

we have

$$x(t) = 1 \xleftrightarrow{F.T} 2\pi\delta(\omega) \quad \text{— (1)}$$

By frequency shifting property

If

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

$$e^{j\omega_0 t}x(t) \longleftrightarrow X(\omega-\omega_0)$$

Multiply (1) by $e^{-j\omega_0 t}$

$$x(t) = e^{-j\omega_0 t} \longleftrightarrow X(\omega+\omega_0)$$

$$= 2\pi\delta(\omega+\omega_0)$$

P6 * $x(t) = \cos\omega_0 t$. Find F.T, M.S, P.S.

$$= \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$= \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

we have, if $x(t) = e^{j\omega_0 t}$

$$x(t) \xrightarrow{F.T} 2\pi\delta(\omega-\omega_0)$$

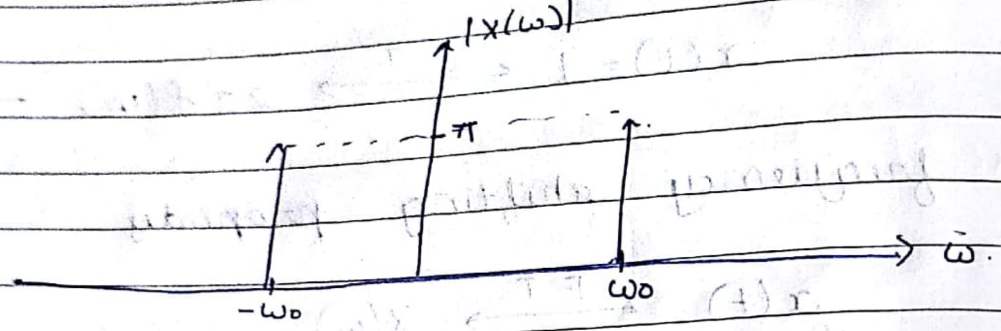
Now applying linearity property

$$x(t) = \cos\omega_0 t \xleftrightarrow{F.T} \frac{1}{2}(2\pi\delta(\omega-\omega_0)) + \frac{1}{2}(2\pi\delta(\omega+\omega_0))$$

$$\cos\omega_0 t \xleftrightarrow{F.T} \pi(\delta(\omega-\omega_0) + \delta(\omega+\omega_0))$$

Magnitude spectrum

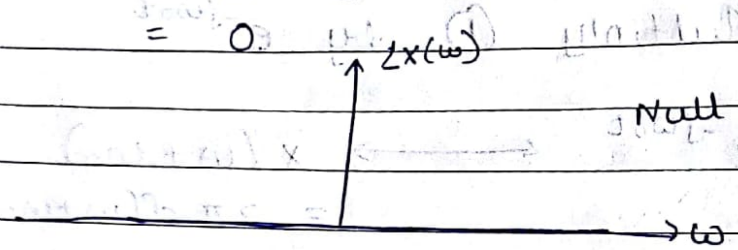
$|X(\omega)| = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$



Phase spectrum

$\angle X(\omega) = \tan^{-1} 0$

$= 0$



* $x(t) = \sin\omega_0 t.$

Sol:- $x(t) = \dfrac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$

Using

$e^{j\omega_0 t} \xrightarrow{F.T} 2\pi\delta(\omega-\omega_0)$

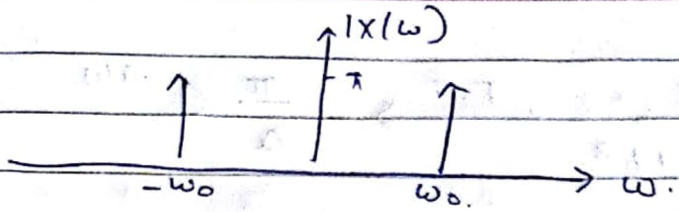
$\therefore$

$x(t) = \sin\omega_0 t \xrightarrow{F.T} \dfrac{1}{2j}\left(2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right)$

$\xleftrightarrow{F.T} \dfrac{\pi}{j}\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)$

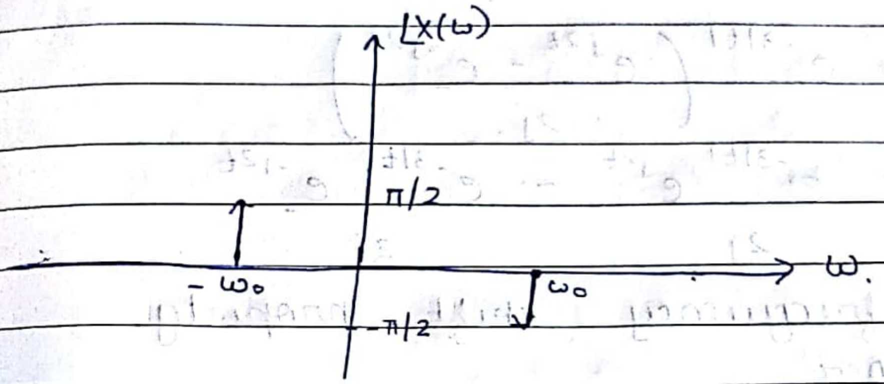
$\therefore |X(\omega)| = \pi\delta(\omega-\omega_0) - \pi\delta(\omega+\omega_0).$

Date ___/___/___



$$\angle X(\omega) = -\tan^{-1}\left(\frac{\pi(\delta(\omega-\omega_0)-\delta(\omega+\omega_0))}{0}\right)$$

$$= -\tan^{-1}(\pm\infty)$$



Q 8)

iii) $x(t) = \dfrac{1}{a^2+t^2}$

Sol:- $x(t)$ is absolutely integrable.

$$X(\omega) = \int_{-\infty}^{\infty} \frac{1}{a^2+t^2} \, e^{-j\omega t} \, dt$$

we have:-

if $x(t) = e^{-a|t|} \xleftrightarrow{F.T} \dfrac{2a}{a^2+\omega^2}$

Divide both by $2a$

$$x(t) = \frac{e^{-a|t|}}{2a} \xleftrightarrow{F.T} \frac{1}{a^2+\omega^2} = X(\omega)$$

Now applying the duality property we have,

$\cancel{x(t)}$

$$X(t) = \frac{1}{a^2+t^2} \xleftrightarrow{F.T} 2\pi \cdot \frac{e^{-a|-\omega|}}{2a}$$

$$= \pi e^{-a|\omega|}$$

Date ___/___/___

$$\therefore \frac{1}{a^2+t^2} \xleftrightarrow{F.T} \frac{\pi}{a} e^{-a\omega}.$$

L.P 9)

i) $x(t) = e^{-3|t|} \sin 2t.$

Sol:- $x(t) = e^{-3|t|} \left(\frac{e^{j2t} - e^{-j2t}}{2j} \right).$

$$= \frac{e^{-3|t|} e^{j2t}}{2j} - \frac{e^{-3|t|} e^{-j2t}}{2j}$$

By frequency shift property we have

$$e^{-a|t|} \xrightarrow{F.F} \frac{2a}{a^2+\omega^2}$$

$$e^{-3|t|} \xrightarrow{F.T} \frac{6}{9+\omega^2}$$

By F.S.P

$$e^{-3|t|} . e^{j2t} \xrightarrow{F-T} \frac{6}{9+(\omega-2)^2}$$

$$e^{-3|t|} . e^{-j2t} \xrightarrow{F.T} \frac{6}{9+(\omega+2)^2}$$

$$X(\omega) = \frac{1}{2j} \left[\frac{6}{9+(\omega-2)^2} - \frac{6}{9+(\omega+2)^2} \right]$$

ii) $x(t) = e^{-2t} \sin \pi t u(t)$

Sol $x(t) = e^{-2t} u(t) . \left(\frac{e^{j\pi t} - e^{-j\pi t}}{2j} \right)$

$$= \frac{1}{2j} \left[e^{-2t} . e^{j\pi t} u(t) - e^{-2t} . e^{-j\pi t} u(t) \right)$$

Date ___/___/___

we have

$$y(t) = e^{-2t} u(t) \xleftrightarrow{F.T} \frac{1}{2+j\omega} = X(\omega)$$

By frequency shift

$$e^{j\beta t} x(t) \xrightarrow{F.T} Y(\omega - \beta).$$

$$e^{-2t} u(t) . e^{j\pi t} \xrightarrow{F.T} \frac{1}{2+j(\omega-\pi)}$$

$$e^{-2t} u(t)\, e^{-j\pi t} \xrightarrow{F.T} \frac{1}{2+j(\omega+\pi)}$$

By applying linearity property.

$$x(t) = \frac{1}{2j}\left[\frac{1}{2+j(\omega-\pi)} - \frac{1}{2+j(\omega+\pi)}\right].$$

iii) $x(t) = e^{-3|t-2|}$.

Sol. Given $x(t) = e^{-3|t-2|}$.

we have $e^{-a|t|} \xrightarrow{F.T} \frac{2a}{a^2+\omega^2}$

$$e^{-3|t|} \xrightarrow{F.T} \frac{2(3)}{9+\omega^2}$$

$$e^{-3|t|} \xrightarrow{F.T} \frac{6}{9+\omega^2}$$

By time shifting property

$$e^{-3|t-2|} \xrightarrow{F.T} e^{-j\omega 2} . \frac{6}{9+\omega^2}$$

Date

iii) $x(t) = \frac{2}{1+t^2}$

Sol. WKT

$$\frac{1}{a^2+t^2} \xrightarrow{F.T} \frac{\pi}{a} e^{-a\omega}$$

$$\frac{1}{1+t^2} \xrightarrow{FT} \pi e^{-a\omega}$$

Multiplying by 2

$$\frac{2}{1+t^2} \xrightarrow{F.T} 2\pi e^{-\omega}.$$

OR

$$e^{-at} \xrightarrow{F.T} \frac{2a}{a^2+\omega^2}$$

If $a = 1$

$$e^{-|t|} \xrightarrow{FT} \frac{2}{1+\omega^2} \quad — ①$$

$$\frac{2}{1+\omega^2} \xrightarrow{F.T} e^{-|\omega|}$$ By duality prope

iv) $x(t) = e^{-j|a|}$ $\quad |t| < \pi$

$= 0 \quad |t| > \pi$

sol Using rectangular signal

$x(t) = 1 \quad |t| < a$

$= 0 \quad$ otherwise

$$x(t) \xrightarrow{F.T} = \frac{2}{\omega} \sin \omega a$$

Put $a = \pi$ $\quad = 2a$

$$x(t) \xrightarrow{F.T} \frac{2}{\omega} \sin \pi\omega. \qquad \omega = 0$$

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Date ___/___/___

By frequency shifting property, we have

$x(t) = e^{-j10t} \cdot (1) \xrightarrow{F.T} \frac{2}{\omega+10} \sin\pi(\omega+10)$ provided $\omega+10 \neq 0$, $\omega \neq -10$

$= 2\pi$ $\quad |\omega = -10$

Inverse fourier Transform

The three main approaches to find inverse fourier transform of $X(\omega)$ i.e $x(t)$ are :-

i) Using synthesis eq^n :-

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

ii) From table values

iii) Using partial fraction.

Table of inverse fourier transform

	Time domain $x(t)$	Freq domain $X(\omega)$	
1)	1	$2\pi\,\delta(\omega)$	
2)	DC value $= A$	$2\pi A\,\delta(\omega)$	
3)	$\delta(t)$	1	
4)	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$	$(a>0)$
5)	$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2+\omega^2}$	$(a>0)$
6)	$t.e^{-at}u(t)$	$\frac{1}{(a+j\omega)^2}$	$(a>0)$
7)	$t^n e^{-at}u(t)$	$\frac{n!}{(a+j\omega)^{n+1}}$	$(a>0)$

Date ___/___/___

8) $e^{j\omega_0 t}$ — $2\pi\delta(\omega-\omega_0)$

9) $\cos\omega_0 t$ — $\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$

10) $\sin\omega_0 t$ — $-j\pi(\delta(\omega-\omega_0) - \delta(\omega+\omega_0))$

11) $e^{-at}\cos\omega_0 t\, u(t)$ — $\dfrac{j\omega+a}{(j\omega+a)^2+\omega_0^2}$ $(a>0)$

12) $e^{-at}\sin\omega_0 t\, u(t)$ — $\dfrac{\omega_0}{(j\omega+a)^2+\omega_0^2}$ $(a>0)$

Find the time domain signal corresponding to foll^n fourier transform

i) $X(\omega) = 2\pi\delta(\omega)$

Sol. Let $x(t)$ be its frequency domain fun^n

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty} \cancel{2\pi}\,\delta(\omega)\, e^{j\omega t}\, d\omega$$

$$= e^{j\omega t}\Big|_{\omega\to 0}$$

$$= e^{0}$$

$$= 1$$

I.P 10)

1. $X(j\omega) = e^{-2\omega}u(\omega)$

Sol:

$$\int_{-\infty}^{\infty} |e^{-2\omega}\, u(\omega)\, d\omega$$

$$\int_{0}^{\infty} |e^{-2\omega}\, d\omega < \infty$$

Date ___/___/___

Since given signal is absolutely integrable

$$\therefore x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot e^{j\omega t}\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2\omega} u(\omega) e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} e^{-\omega(-jt+2)}\, d\omega.$$

$$= \frac{1}{2\pi}\left[\frac{e^{-\omega(2-jt)}}{-(2-jt)}\right]_0^{\infty}$$

$$= \frac{1}{2\pi}\left[0 - \frac{1}{-(2-jt)}\right]$$

$$= \frac{1}{2\pi} \times \frac{1}{2-jt}$$

2) $X(j\omega) = \cos\left(\frac{\omega}{2}\right) + j\sin\left(\frac{\omega}{2}\right) \qquad |\omega| < \pi.$

$= 0 \qquad$ otherwise

Sol. $X(j\omega) = e^{j\omega/2} \qquad |\omega| < \pi$

$$|X(j\omega)| = \sqrt{\cos^2\omega/2 + \sin^2\omega/2}$$

$$= 1$$

$X(j\omega)$ is absolutely integrable

$$\therefore x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot e^{j\omega t}\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega/2}\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\omega(j/2+jt)}\, d\omega.$$

$=$

$(-1)^k = \cos \pi k$

$$= \frac{1}{2\pi}\left[\frac{e^{\omega(j/2+jt)}}{j/2+jt}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\pi/2}\cdot e^{j\pi t}}{j/2+jt} - \frac{e^{-j\pi/2}\, e^{-j\pi t}}{j/2+jt}\right]$$

$$= \frac{1}{2\pi}\left[\frac{j((-1)^k)}{j/2+jt} - \frac{(-j)(-1)^k}{j/2+jt}\right]$$

$$= \cancel{\frac{1}{2\pi}}\left[\cancel{\frac{2(-1)^k}{1+t}} + \cancel{\frac{2(-1)^k}{1+t}}\right]$$

$$= \cdot(-1)^k$$

$$\cancel{= \frac{1}{2\pi}\left[\frac{2(e^{j\pi t} - e^{j\pi t})}{1+t}\right]}$$

$$= \cancel{\frac{(-1)^k}{\pi(1+t)}}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\frac{\omega}{2}}\cdot e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\frac{1}{2}+t)\omega}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j(\frac{1}{2}+t)\omega}}{j(\frac{1}{2}+t)}\right]_{-\pi}^{\pi}$$

$$= \frac{\sin(\frac{1}{2}+t)\pi}{\pi(\frac{1}{2}+t)}$$

$$= \frac{\sin(\pi/2 + \pi t)}{\pi(1/2+t)}$$

$$x(t) = \frac{\cos \pi t}{\pi(1/2+t)}$$

$$t = -1/2$$

$$x\left(-\frac{1}{2}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega/2}\, e^{-j\omega/2}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\, d\omega$$

$$= \frac{2\pi}{2\pi}$$

$$= 1.$$

Page No.

Date ___/___/___

Required time domain

$$x(t) = \frac{\cos \pi t}{\pi(1/2 + t)} \quad \forall\ t \neq -1/2$$

$$= 1 \qquad t = -1/2$$

12) i) $X(\omega) = \dfrac{j\omega + 12}{(j\omega)^2 + 5j\omega + 6}$

Sol:- Put $j\omega = s$

$$X(\omega) = \frac{s+12}{s^2+5s+6}$$

$$\frac{s+12}{s^2+5s+6} = \cancel{A}$$

$$\frac{s+12}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2}$$

$$s+12 = A(s+2) + B(s+3)$$

$$s+12 = s(A+B) + 2A + 3B$$

$$A + B = 1$$

$$2A + 3B = 12$$

$$A = -9$$

$$B = 10$$

$$X(\omega) = \frac{-9}{s+3} + \frac{10}{s+2}$$

$$= \frac{-9}{j\omega+3} + \frac{10}{j\omega+2}$$

Taking inverse both sides

given

$$x(t) = -9e^{-3t}u(t) + 10e^{-2t}u(t)$$

$$e^{-at}u(t) \xrightarrow{F.T} \frac{1}{a+j\omega}.$$

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ii) $X(\omega) = \dfrac{j\omega}{(j\omega+2)^2}$

Sol. $X(\omega) = \dfrac{s}{(s+2)^2} = \dfrac{A}{(s+2)} + \dfrac{B}{(s+2)^2}$

$$s = A(s+2) + B$$

$$1 = A$$
$$0 = 2A + B$$
$$= 2 + B$$
$$B = -2$$

$$X(\omega) = \frac{1}{s+2} - \frac{2}{(s+2)^2}$$
$$= \frac{1}{j\omega+2} - \frac{2}{(j\omega+2)^2}$$

$t \cdot e^{-at} u(t) \longrightarrow \dfrac{1}{(a+j\omega)^2}$

$$x(t) = e^{-2t} u(t) - 2te^{-2t} u(t)$$

iii) $X(\omega) = \dfrac{-j\omega}{(j\omega)^2 + 3j\omega + 2}$

Sol. $X(\omega) = \dfrac{-s}{s^2+3s+2}$

$$= \frac{-s}{s^2+s+2s+s}$$
$$= \frac{-s}{(s+2)(s+1)}$$

$$\frac{A}{s+2} + \frac{B}{s+1} = -s$$

$$A(s+1) + B(s+2) = -s$$

Date __/__/__

$A + B = -1$

$A + 2B = 0$

$A = -2$

$B = 1$

$$X(\omega) = \frac{-2}{s+2} + \frac{1}{s+1}$$

$$= \frac{1}{j\omega+1} - \frac{2}{j\omega+2}$$

$$\Rightarrow e^{-t}u(t) - 2e^{-2t}u(t)$$

$$X(\omega) = \frac{2j\omega + 24}{(j\omega)^2 + 4(j\omega) + 29}$$

Sol: $$X(\omega) = \frac{2s + 24}{s^2 + 4s + 29}$$

$$= \frac{2s + 24}{s^2 + 4s + 4 + 25}$$

$$= \frac{2s + 24}{(s+2)^2 + 25}$$

$$= \frac{2(s + 2 + 10)}{(s+2)^2 + 25}$$

$$= 2\left[\frac{s+2}{(s+2)^2 + 25} + \frac{10}{(s+2)^2 + 25}\right]$$

$$= 2\left[\frac{s+2}{(s+2)^2 + 25} + \frac{2.5}{(s+2)^2 + 25}\right]$$

$$= 2\left[\frac{j\omega + 2}{(j\omega+2)^2 + 25} + \frac{2(5)}{(j\omega+2)^2 + 25}\right]$$

$$x(t) = 2e^{-2t}\cos 5t\, u(t) + 2e^{-2t}\sin 5t\, u(t)$$

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Parsevals energy theorem for fourier transform

For an energy signal $x(t)$, the foll^n relationship holds good.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

Example:-

1) Calculate the energy for $x(t)$

i) $x(t) = e^{-at} u(t)$

a) In time domain

b) In frequency domain

c) Verify Parsevals theorem

d) What is the energy of the signal in frequency $|\omega| \leq 0.5$ rad/sec

Sol:- The given signal is

$$x(t) = e^{-at} u(t)$$

Energy in time domain

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{-\infty}^{\infty} |e^{-at} u(t)|^2 \, dt$$

$$= \int_{0}^{\infty} |e^{-at} u(t)|^2 \, dt$$

$$= \int_{0}^{\infty} |e^{-at}|^2 \, dt$$

$$= \frac{1}{2a} \quad \text{—} (1)$$

Date __/__/__

Saathi

Energy in frequency domain.

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

$=$

we have $x(t) = e^{-at}u(t) \longrightarrow \frac{1}{a+j\omega}$.

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

$$|X(j\omega)|^2 = \frac{1}{a^2+\omega^2}$$

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{a^2+\omega^2} \, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{1}{a}\tan^{-1}\frac{\omega}{a}\right]_{-\infty}^{\infty}$$

$$= \frac{1}{2\pi}\left[\frac{1}{a}\left(\tan^{-1}\infty - \tan^{-1}-\infty\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{a}\,\frac{2\pi}{2}\right]$$

$$= \frac{1}{2a} \quad - (2)$$

From ~~&~~ (1) and (2) parsevals thm is verified.

ii) Energy in frequency band $|\omega| \le 0.5$ rad/sec

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

$$= \frac{1}{2\pi}\int_{-0.5}^{0.5} |X(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-0.5}^{0.5} \frac{d\omega}{a^2+\omega^2}$$

Date ___ / ___ / ______

$$= \frac{1}{2\pi a}\left[\tan^{-1}\frac{\omega}{a}\right]_{-0.5}^{0.5}$$

$$= \frac{1}{2\pi a}\left[\tan^{-1}\frac{0.5}{a} - \tan^{-1}\frac{-0.5}{a}\right]$$

$$= \frac{1}{2\pi a}\left[2\tan^{-1}\frac{0.5}{a}\right]$$

$$= \frac{1}{\pi a}\left(\tan^{-1}\frac{0.5}{a}\right)$$

$$X(-\pi) = \left[\frac{e^{j\pi t}}{j\pi}\right]_{-1}^{1} + \frac{t}{2} + \left[\frac{e^{2j\pi t}}{2(2j\pi)}\right]_{-1}^{1}$$

$$= \frac{e^{j\pi} - e^{-j\pi}}{j\pi} + \frac{1}{2} + \frac{e^{2j\pi t} - e^{-2j\pi t}}{4j\pi} + \frac{1}{2}$$

$$= 2\sin\pi + 1 + \frac{\sin 2\pi}{2}$$

$$= 1.$$

Date 10-10-18

Regression

Curve fitting:-
It's a method of finding a specific relation connecting the dependent and independent variables for a given data so as to satisfy the data as accurately as possible. Such a curve is called cure of best fit. One method to find a curve of best fit is a method of least square.

Suppose $Y = f(x)$ is an appropriate relation that fits given set (x_i, y_i) for $i = 1$ to n. Here Y_i's are called observed values and $Y_i = f(x_i)$ are called expected values. Their difference $E_i = Y_i - y_i$ are called errors/residuals. Method of least square provides a relationship $(Y = f(x))$ such that sum of squares of residuals is minimum.

Fitting a straight line - "$y = a + bx$"
Here we consider the eq^n in form and these values where a & b are parameters and are obtained from normal eq^n in least square sense, here normal eq^n is given by

$$y = a + bx$$

$$\Sigma Y = na + b\Sigma x$$

$$\Sigma Y = a\Sigma x + b\Sigma x^2$$

Fitting a quadratic/parabolic/2nd degree equation.

Page No.

Date __/__/__

Here we consider eq^n in form $y = a + bx + cx^2$
where a, b, c are obtained from normal equations

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$
$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

Fitting exponential type:-
For fitting " curve $y = ax^b$
Taking log both sides:-

$$\ln(y) = \ln(a) + b\ln(x)$$
$$\Rightarrow Y = A + bX$$ which is a linear eq^n in x and y

∴ normal eq^n w.r.t x and y are

$$\Sigma y = nA + b\Sigma X$$
$$\Sigma xy = A\Sigma x + b\Sigma X^2$$

Solving these expressions for A and b:
since $A = \ln(a)$
$$a = e^A$$
substituting these values in main equation
i,e $y = ax^b$ we have required curve fit.

L.P Q1) Given:- W & d
required line is $d = aW + b$ where a and b are constants to be determined.
From normal eq^n's using least square method:-

$$\Sigma d = a\Sigma W + nb$$
$$\Sigma Wd = a\Sigma W^2 + b\Sigma W$$
n = no. of observations = 10

Date __/__/__

W	d	Wd	W^2
10			
20			
30			
40			
50			
60			
70			
80			
90			
100			
550	862	59830	38500

substituting these values in normal eq^n:-

$$862 = a(550) + 10b$$

$$59830 = a(38500) + b(550)$$

$a = 1.505$

$b = 3.4$

substituting these values $\Rightarrow$ $d = 1.505W + 3.4$

I.P Q3) Required quadratic $eq^n \Rightarrow R = a + bV + cV^2$

normal eq^n's $\Rightarrow$

$$\Sigma R = na + b\Sigma V + c\Sigma V^2$$

$$\Sigma VR = a\Sigma V + b\Sigma V^2 + c\Sigma V^3$$

$$\Sigma V^2R = a\Sigma V^2 + b\Sigma V^3 + c\Sigma V^4$$

V	R
20	5.5
40	9.1
60	14.9
80	22.8
100	33.8
120	46.0

$\Sigma VR = 12042$

$\Sigma R^2 = 4079.8$

$\Sigma R = 131.6$

$\Sigma V = 420$

$\Sigma V^2 = 36400$

$\Sigma V^3 = 3528000$

Page No.

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Date ___/___/___

$\Sigma V^2 R = 1211720$

$\Sigma V^4 = 364000000$

$n = 6.$

substituting in normal eq^n's :-

$131.6 = 6a + b(420) + c(36400)$

$131.6 = 6a + 420b + 36400c$ — ①

$12042 = a(420) + b(36400) + c(3528000)$

$12042 = 420a + 36400b + 3528000c$ — ②

$1211720 = a(36400) + b(3528000) + c(364000000)$ — ③

solving ①, ②, ③

$a = \frac{87}{20} = 4.35$

$(b = 2.41 \times 10^{-3})$

$C = 2.87 \times 10^{-3}$

L.P 9 8) $x = 3.75 \Rightarrow y = ?$

$Y = ab^x$

x	Y
1	2.98
2	4.26
3	5.21
4	6.10
5	6.80
6	7.50

Normal eq^n:-

$\Sigma Y = nA + B\Sigma x$

$\Sigma xY = A\Sigma x + B\Sigma x^2$

$\Sigma Y = 32.85$

$\Sigma x = 21$

$n = 6$

$\Sigma xY = 130.53$

Page No.

Date __/__/__

$\Sigma X^2 = 91$

substituting in normal eq^n:

$32.85 = 6A + B(21)$

$130.53 = 21A + 91B$

$A = 2.822$

$b = 1.19$

Required:

$Y = (2.822)(1.19)^x$

at $x = 3.75$

$Y = (2.822)(1.19)$

$= 5.41$

Date

Independent random variables

The discrete random variables X and Y are said to independent random variables if $P_{11} = P_1 q_1$

in general $P_{ij} = P_i q_j \quad \forall\ i, j$

Expectation, co variance, correlation

If X and Y are 2 discrete random variables having joint probability dist^n $f(x, y)$, the expectation of X and ~~x/xy/xy~~ X, Y, XY

$$E(X) = \mu_x = \sum x\, p(x)$$

$$E(Y) = \mu_y = \sum y\, p(y)$$

$$E(XY) = \mu_{xy} = \sum_{xy} xy\, f(x, y)$$
$$= \sum xy\, P(x, y)$$

Co-variance of X and Y denoted by

cov(X, Y) is ~~denoted~~ given by

$$cov(X, Y) = \cancel{E(X)} \; E(XY) - E(X)E(Y)$$

correlation of X and Y denoted by $\rho(X, Y)$ and given by

$$\rho(X, Y) = \frac{cov(X, Y)}{\sigma_x \sigma_y}$$

Page No.

Date ___/___/___

where $\sigma_x = \sqrt{var(X)} = \sqrt{\sum x^2 p(x) - \mu^2}$
$= \sqrt{E(X^2) - [E(X)]^2}$

If X and Y are independent random variables then expected values of
1) $E(XY) = E(X).E(Y)$
2) $cov(X,Y) = 0$
3) $\rho(X,Y) = 0$

Ex:-

1) A fair coin is tossed thrice, the random variables X and Y are defined as follows

$X = 0$ or 1 according as head or tail occurs on the 1^{st} toss

$Y =$ number of heads

i) Determine the joint distribution of X and Y
ii) Determine marginal " " of X and Y
iii) Obtain expectations of X, Y, XY
iv) Find std. deviation of X, Y
v) compute covariance, correlation of X, Y
vi) are X and Y independent?

Sol:- Sample space

S	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
X	0	0	0	0	1	1	1	1
Y	3	2	2	1	2	1	1	0

$X = \{0, 1\}$
$Y = \{3, 1, 2, 0\}$

Date

X \ Y	0	1	2	3	Total
0	0	1/8	1/4	1/8	4/8 = 1/2
1	1/8	2/8	1/8	0	4/8 = 1/2
Total	1/8	3/8	3/8	1/8	1

$P_{11} = P(X=0, Y=0)$

$= \frac{0}{8} = 0$

$P_{14} = P(X=0, Y=3)$

$P_{12} = P(X=0, Y=1)$

$= \frac{1}{8}$

$P_{13} = P(X=0, Y=2)$

$= \frac{2}{8} = \frac{1}{4}$

Marginal distribution of X :

X	0	1	Total
P(x)	1/2	1/2	1

Marginal distribution of Y :

Y	0	1	2	3	Total
P(Y)	1/8	3/8	3/8	1/8	1

Expectations $E(x) = \sum x P(x)$

$= 0 \times P(0) + 1 \times P(1)$

$= 1/2$

$E(Y) = \sum y P(x)$

$= 0 + 1 \times 3/8 + 2 \times 3/8 + 3 \times 1/8$

$= \frac{3}{8} + \frac{6}{8} + \frac{3}{8}$

$= \frac{12}{8} = \frac{3}{2}$

Saathi

Date ___/___/___

$E(XY) = \sum_{xy} xy\, P(x,y)$

$= 0.0\, P(0,0) + 0.1 (P(0,1) + 0.2\, P(0,2)$
$+ 0.3\, P(0,3) + 1.0\, P(1,0) + 1.1\, P(1,1)$
$1.2\, P(1,2) + 1.3\, P(1,3)$

$= 0+0+0+0+ 0\times\frac{1}{8} + 1\times\frac{2}{8} + 2\times\frac{1}{8} + 0$

$= \frac{2}{8} + \frac{2}{8} = \frac{4}{8} = \frac{1}{2}$.

=

$E(x^2) = \sum x^2 p(x)$

$= 0^2 p(0) + 1^2 \cdot p(1)$

$= 0 + 1\times 1/2 = 1/2$

$\sigma_x = \sqrt{\text{var}\, x}$

$= \sqrt{1/2 - (1/2)^2}$

$= \sqrt{1/4}$

$1/2$

$E(Y^2) = 0^2 P(0) + 1^2 P(1) + 2^2 (P(2) + 3^2 P(3))$

$= 1\times\frac{3}{8} + 4\times\frac{3}{8} + 9\times\frac{1}{8}$

$= \frac{3}{8} + \frac{12}{8} + \frac{9}{8}$

$= \frac{24}{8} = 3$

$\sigma_Y = \sqrt{3 - (3/2)^2}$

$= \sqrt{3\times 4 - 9/4}$

$= \sqrt{12 - 9/4}$

$= \frac{\sqrt{3}}{2}$

Date ___ / ___ / ___

$$\text{cov}(X, Y) = E(XY) - E(X) \cdot E(Y)$$

$$= \frac{1}{2} - \frac{1}{2} \times \frac{3}{2}$$

$$= \frac{1}{2} - \frac{3}{4}$$

$$= \frac{2-3}{4} = \frac{-1}{4}$$

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_x \cdot \sigma_y}$$

$$= \frac{-1/4}{1/2 \cdot \sqrt{3}/2}$$

$$= \frac{-1}{\sqrt{3}}$$

Since $p_{ij} \neq p_i q_i$

$\therefore$ X and Y are dependent

2) An urn contains 3 red balls and 5 white balls. 3 balls are drawn at random with replacement. Obtain a bivariate distribution, where random variables are defined as follows.

$X \rightarrow$ no. of red balls

$Y \rightarrow$ no. of white balls

RWW + WRW + WWR

Sol:-

$X = 3 \quad 2 \quad 1 \quad 0$

$Y = 0 \quad 1 \quad 2 \quad 3$

X \ Y	0	1	2	3	Total
0	0	0	0	$5^3/8^3$ (0.2441)	0.2441
1	0	0	$3 \times \left(\frac{3}{8} \cdot \frac{5^2}{8^2}\right)$ 0.146	0	0.1460 × 3
2	0	$3 \times \left(\frac{3^2}{8^2} \cdot \frac{5}{8}\right)$ 0.087	0	0	0.087 × 3
3	$\frac{3^3}{8^3}$ 0.0527	0	0	0	0.053

Date ___/___/___

Probability of taking white ball one time = 5/8

" " red " " " = 3/8

$p(x=0, y=0) = 0$

$p(x=1, y=0) = 0$ $\quad p(x=2, y=0) = 0$

$p(x=0, y=3) = 5/8 \times 5/8 \times 5/8$

Q.2 Given probability funⁿ is

$f(x,y) = k(2x+y)$

where X and Y are discrete random variables, given by integers

$0 \le x \le 2$

$0 \le y \le 3$

$X = 0, 1, 2$

$Y = 0, 1, 2, 3$

X \ Y	0	1	2	3	Total
0	0	k	2k	3k	6k
1	2k	3k	4k	5k	14k
2	4k	5k	6k	7k	22k
	6k	9k	12k	15k	

42k

$42k = 1$

$k = \frac{1}{42}$

Marginal distribution of X

X	0	1	2
P(X)	$6\left(\frac{1}{42}\right)$	$14\left(\frac{1}{42}\right)$	$22\left(\frac{1}{42}\right)$
	0.142	0.333	0.5238

Page No.

Date ___/___/___

Marginal distribution of Y

Y	0	1	2	3
P(Y)	$6\left(\frac{1}{42}\right)$	$8\left(\frac{1}{42}\right)$	$12\left(\frac{1}{42}\right)$	$15\left(\frac{1}{42}\right)$
	0.142	0.190	0.285	0.3571

$$E(X) = \sum x\,(P(x))$$
$$= 0 \times \frac{6}{42} + 1 \times \frac{14}{42} + 2 \times \frac{22}{42}$$
$$= \frac{29}{21}$$
$$= 1.380$$

$$E(Y) = \sum y\, P(y)$$
$$= 0 \times \frac{6}{42} + 1 \times \frac{8}{42} + 2 \times \frac{12}{42} + 3 \times \frac{15}{42}$$
$$= \frac{11}{6}$$
$$= 1.833$$

$$E(XY) = \sum xy\, P(x,y)$$
$$= 0.0\, P(0,0) + 0.1\, P(0,1) + 0.2\, P(0,2)$$
$$+ 0.3\, P(0,3) + 1.0\, P(1,0) + 1.1\, P(1,1)$$
$$+ 1.2\, P(1,2) + 1.3\, P(1,3)$$
$$+ 2.0\, P(2,0) + 2.1\, P(2,1) + 2.2\, P(2,2)$$
$$+ 2.3\, P(2,3)$$
$$= 0 + 0 + 3K + 2(4K) + 3(5K)$$
$$+ 0 + 2(5K) + 4(6K) + 6(7K)$$

$= \frac{102}{42}$

$= \frac{17}{7} = 2.428$

$cov(X, Y) = E(XY) - E(X).E(Y)$

$= \frac{17}{7} - \left(\frac{29}{21}\right)\left(\frac{11}{6}\right)$

$= -\frac{13}{126}$

$= -0.103$

$E(X^2) = \Sigma x^2 P(x)$

$= 0 + 1 \cdot \frac{14}{42} + 4 \cdot \frac{22}{42}$

$= \frac{14}{42} + \frac{88}{42}$

$= \frac{17}{7}$

$\sigma_x = \sqrt{\frac{17}{7} - \left(\frac{29}{21}\right)^2}$

$= \frac{\sqrt{230}}{21}$

$E(Y^2) = 0 + 1 \times \frac{8}{42} + 4 \times \frac{12}{42} + 9 \times \frac{15}{42}$

$= \frac{191}{42}$

$\sigma_y = \sqrt{\frac{191}{42} - \left(\frac{11}{6}\right)^2}$

Date ___/___/___

for the above distribution table.
find probability of

i) $P(x=1, y=2)$
ii) $P(x \geq 1, y \leq 2)$
iii) $P(x+y > 2)$
iv) are X and Y independent?

Sol. $P(x=1, Y=2) = \frac{4}{42} = \frac{2}{21}$

$P(x \geq 1, Y \leq 2) = P(x=1, y \leq 2) + P(x=2, Y \leq 2)$
$= P(x=1, Y=0) + P(x=1, Y=1)$
$+ P(x=1, Y=2) + P(x=2, Y=0)$
$+ P(x=2, Y=1) + P(x=2, Y=2)$

$= 9K + 15K$
$= 24K$
$= \frac{24}{42} = \frac{12}{21} = \frac{4}{7}$

Page No.

Date ___ / ___ / ___

iii) $P(x+y>2) = P(0+3>2) + P(1+2>2)$
$+ P(2+1>2) + \cancel{P(\dots)}$
$+ P(1+3>2) + P(2+2>2) + P(2+3>2)$

$= 5k + 5k + 6k + 7k + 3k + 4k$

$= \frac{30}{42} = \frac{5}{7}$

iv) No x and y are not independent
$E(XY) \neq E(X)E(Y)$

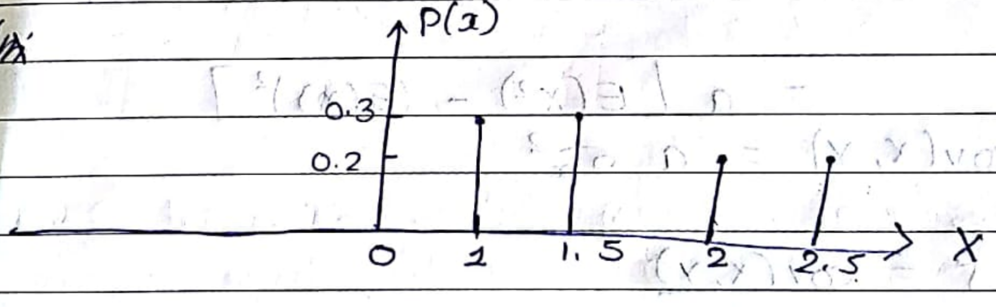
4) M.D of X

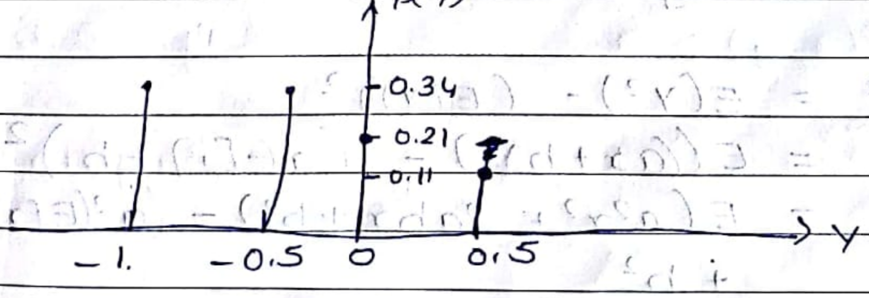
X	1	1.5	2	2.5	Total
$P(X)$	0.3	0.3	0.2	0.2	1

M.D of Y

Y	-1	-0.5	0	0.5	Total
$P(Y)$	0.34	0.34	0.21	0.11	1

~~E/X~~





Date

If marginal distribution is given and the condition is X and Y are independent. Draw the table.

8) $Y = aX + b$

Let X, Y be the random variable and its relation is given by

~~Let E(x)~~ $Y = aX + b$

Let $E(X)$, $E(Y)$ denoted the expected value of X and Y resp.

$E(Y)$ can be written as

$$E(Y) = E(aX+b)$$
$$= aE(X) + b$$

$$E(XY) = E(X(aX+b))$$
$$= E(aX^2 + bX)$$
$$= aE(X^2) + bE(X)$$

$$\operatorname{cov}(X, Y) = -E(x)E(Y) + E(XY)$$
$$= -E(X)\cdot[aE(x)+b] + aE(x^2) + bE(x)$$

$$= aE(x^2) + \cancel{bE(x)} - a[E(x)]^2 - \cancel{bE(x)}$$

$$= a[E(x^2) - (E(x))^2]$$

$$\operatorname{cov}(X, Y) = a\sigma_x^2$$

$$\rho = \frac{\operatorname{cov}(X,Y)}{\sigma_x \sigma_y}$$

$$\sigma_y^2 = E(Y^2) - (E(Y))^2$$
$$= E((ax+b)^2) - (aE[x]+b)^2$$
$$= E(a^2x^2 + 2abx + b^2) - (a^2(E(x))^2 + 2abE(x) + b^2$$

Pace No.

Date ___/___/___

$$= a^2E(x^2) + \cancel{2abE(x)} + \cancel{b^2} - a^2(E(x))^2 - \cancel{2abE(x)} - \cancel{b^2}$$

$$= a^2\left(E(x^2) - (E(x))^2\right)$$

$$\sigma_y^2 = a^2\sigma_x^2$$

$$\sigma_y = \pm a\sigma_x \qquad \sigma_y = |a|\sigma_x$$

$$\therefore \rho = \frac{a\,\sigma_x^2}{\sigma_x\,|a|\,\sigma_x}$$

$$= \frac{a}{|a|}$$

$$= 1 \qquad a > 0$$

$$= -1 \qquad a < 0$$

$$\therefore \rho(X, Y) = 1 \qquad a > 0$$
$$= -1 \qquad a < 0$$
} ~~b any value.~~

Joint Probability density distribution:-

If X and Y are two continuous random variables, then funⁿ $f(x, y)$ is called joint probability density funⁿ of (X, Y) if it satisfies the follⁿ conditions

i) $f(x, y) \geq 0 \quad \forall x, y$

ii) $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{+\infty} f(x, y)\, dx\, dy = 1$

Marginal distⁿ of x and y

In case of jointly contⁿ random variables, marginal distⁿ of x and y are respectively given by

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy.$$

Date ___/___/___

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

Independent random variables

In case of jointly cont^n random variables x & y are said to be independent random variables.

$$f(x,y) = g(x)\, h(y)$$

where $f(x,y)$ is density fun^n. $g(x)$ and $h(y)$ are marginal dist^n of x and y resp.

Cummulative dist^n fun^n:-

cummulative distribution fun^n of jointly distributed random variables denoted by F_{xy}, it gives the probability of an event of which

$$F_{XY} = P(X \leq x, Y \leq y)$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f(x,y)\, dx\, dy$$

7)

a) $f_{XY}(x,y) = 3xy \quad 0<x<1$ and $0<y<b$

$= 0$ elsewhere

Sol:- Based on the conditions:-

$f(x,y) \geq 0 \quad \forall\, x, y \Rightarrow b \geq 0$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = \int_{0}^{b} \int_{0}^{1} 3xy\, dx\, dy$$

Page No.

Date ___/___/___

$$= \int_0^b 3y \frac{x^2}{2}\Big|_0^1 \, dy$$

$$= \int_0^b \frac{3y}{2} \, dy$$

$$= \frac{3y^2}{4}\Big|_0^b$$

$$= \frac{3b^2}{4} = 1$$

$$b^2 = \frac{4}{3}$$

$$b = \pm \frac{2}{\sqrt{3}}$$

$$\therefore \boxed{b = \frac{2}{\sqrt{3}}}$$

b) $f_{XY}(x,y) = xb(1-y) \quad 0 < x < 0.5 \;\&\; 0 < y < 1$

$= 0$ elsewhere

Base on the conditions

$f(x,y) \geq 0$

$\therefore b > 0$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^{0.5} x b(1-y)\,dx\,dy = 1$$

$$= \int_0^1 \frac{x^2}{2} b(1-y)\Big|_0^{0.5} dy = 1$$

$$\frac{0.25}{2} b \left[y - \frac{(y)^2}{2} \right]_0^1 = 1$$

$$\frac{0.25}{2} b \left(\frac{1}{2}\right) = 1$$

$$b = \frac{4}{0.25} = \frac{400}{25} = 16$$

Date __/__/__

The joint density distⁿ funⁿ of two random variable x and y is given by

$$f(x,y) = c(x^2+y^2) \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1 \end{cases}$$

$$= 0 \text{ otherwise}$$

i) determine the constant c.
ii) Find expected value of $E(X)$, $E(Y)$ & $E(XY)$
iii) Find marginal distribution of X and Y
iv) Are X and Y independent
v) Find $P(X < 1/2, Y > 1/2)$
$P(1/4 < x < 3/4)$
$P(y < 1/2)$

Sol.
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_0^1\int_0^1 c(x^2+y^2)\,dx\,dy$$

$$c\left[\frac{x^3}{3}\right|_0^1 + c\left(\frac{y^3}{3}\right|_0^1$$

$$= c\left[\frac{1}{3}+\frac{1}{3}\right]$$

$$\frac{2}{3}c = 1$$

$$\therefore c = 3/2.$$

$$E(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\,dx\,dy$$

Date __/__/__

$$= \int_0^1\int_0^1 x\left(\frac{3}{2}\right)\left(x^2+y^2\right) dx\, dy$$

$$= \frac{3}{2}\int_0^1\int_0^1 x^3 + xy^2 \; dx\, dy$$

$$= \frac{3}{2}\int_0^1 \frac{x^4}{4} + \frac{x^2}{2}y^2 \Big|_0^1$$

$$= \frac{3}{2}\int_0^1 \frac{1}{4} + \frac{y^2}{2} \; dy$$

$$= \frac{3}{2}\left[\frac{y}{4} + \frac{y^3}{6}\Big|_0^1\right.$$

$$= \frac{3}{2}\left[\frac{1}{4} + \frac{1}{6}\right]$$

$$= \frac{3}{2}\left[\frac{4+6}{24}\right]$$

$$= \frac{3}{2}\left[\frac{5}{12}\right]$$

$$= \frac{15}{24} = \frac{5}{8}$$

similary $E(Y)$

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\, dx\, dy = \frac{5}{8}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

$$= c\int_0^1\int_0^1 xy\left(x^2+y^2\right) dx\, dy$$

$$= c\int_0^1\int_0^1 x^3y + xy^3 \; dx\, dy$$

$$= c\int_0^1 \left[\frac{x^4y}{4} + \frac{x^2y^3}{2}\right]_0^1 dy$$

Date

$$= c\int_0^1 \frac{y}{4} + \frac{y^3}{2}\,dy$$

$$= c\left[\frac{y^2}{8} + \frac{y^4}{8}\right]_0^1$$

$$= \frac{3}{2}\left(\frac{1}{8} + \frac{1}{8}\right)$$

$$= \frac{3}{2} \times \frac{2}{8}$$

$$= \frac{3}{8}$$

Marginal distribution of X

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^1 c(x^2+y^2)\,dy$$

$$= c\left(x^2y + \frac{y^3}{3}\right)_0^1$$

$$= c\left(x^2 + \frac{1}{3}\right)$$

$$= \frac{3}{2}\left(x^2 + \frac{1}{3}\right)$$

$$= \frac{3}{2}x^2 + \frac{1}{2}$$

Marginal dist^n of Y.

$$h(y) = \int_0^1 c(x^2+y^2)\,dx$$

$$= \frac{3}{2}\left(\frac{x^3}{3} + xy^2\right)_0^1$$

$$= \frac{3}{2}\left(\frac{1}{3} + y^2\right)$$

$$= \frac{3y^2}{2} + \frac{1}{2}$$

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Date __ / __ / __

Saathi

consider $g(x).h(y) = \frac{1}{4}(3x^2+1)(3y^2+1)$

It is not equal to $\frac{3}{2}(x^2+y^2) = f(x,y)$.

$\therefore$ x and y are not independent.

v) $P(x \leq \frac{1}{2}, y > \frac{1}{2}) = \int_{-\infty}^{1/2}\int_{1/2}^{\infty} \frac{3}{2}(x^2+y^2)\,dy\,dx$

$= \int_{0}^{1/2}\int_{1/2}^{1} \frac{3}{2}(x^2+y^2)\,dy\,dx$

$= \int_0^{1/2} \frac{3}{2}\left(\frac{y^3}{3} + x^2y\right)_{1/2}^{1}$

$= \int_0^{1/2} \frac{3}{2}\left(\frac{1}{3} + x^2 - \frac{1}{24} - \frac{x^2}{2}\right)$

$= \int_0^{1/2} \frac{3}{2}\left(\frac{7}{24} + \frac{x^2}{2}\right)$

$= \frac{3}{2}\left(\frac{7x}{24} - \frac{x^3}{6}\right)_0^{1/2}$

$= \frac{3}{2}\left(\frac{7}{48} + \frac{1}{8\times 6}\right)$

$= \frac{3}{2}\left(\frac{7}{48} + \frac{1}{48}\right) = \frac{3}{2}\left(\frac{\cancel{8}^{1}}{\cancel{48}_{6}}\right) =$

$= \frac{3}{12} = \frac{1}{4}$

$P(\frac{1}{4} < x < \frac{3}{4}) = \int_{1/4}^{3/4}\int_0^1 \frac{3}{2}(x^2+y^2)\,dy\,dx$

$= \int_{1/4}^{3/4} \frac{3}{2}\left(x^2y + \frac{y^3}{3}\right)_0^1 dx$

$= \int_{1/4}^{3/4} \frac{3}{2}\left(x^2 + \frac{1}{3}\right)dx =$

Date ___ / ___ / ___

$$= \int \frac{3}{2}\left(\frac{x^3}{3} + \frac{x}{3}\right)\Big|_{1/4}^{3/4}$$

$$= \frac{3}{2\times 3}\left(\frac{27}{64} + \frac{3}{4} - \frac{1}{64} - \frac{1}{4}\right)$$

$$= \frac{3}{6}\left(\frac{26}{64} + \frac{2\times 16}{4}\right)$$

$$= \frac{3}{6}\left(\frac{26+32}{64}\right)$$

$$= \frac{3}{6}\left(\frac{58}{64}\right)^{29}_{32}$$

$$= \frac{29}{64}.$$

$$P(y < 1/2) = \int_0^1 \int_0^{1/2} \frac{3}{2}(x^2 + y^2)\, dy\, dx$$

$$= \int_0^1 \left[\frac{3}{2}\left(x^2 y + \frac{y^3}{3}\right)\right]_0^{1/2} dx$$

$$= \int_0^1 \frac{3}{2}\left(\frac{x^2}{2} + \frac{1}{24}\right) dx$$

$$= \frac{3}{2}\left[\frac{x^3}{6} + \frac{x}{24}\right]_0^1$$

$$= \frac{3}{2}\left(\frac{1}{6} + \frac{1}{24}\right)$$

$$= \frac{3}{2} \times \frac{5}{24_8}$$

$$= \frac{5}{16}$$

$f_{XY}(x,y) = \frac{1}{4} e^{-|x|-|y|}$

Let $g(x)$ denotes marginal distⁿ of x

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{4} e^{-|x|-|y|}\, dy$$

$$= \frac{1}{4} \int_{-\infty}^{\infty} e^{-|x|} e^{-|y|}\, dy$$

$$= \frac{1}{4} e^{-|x|} \int_{-\infty}^{\infty} e^{-|y|}\, dy$$

$$= \frac{1}{4} e^{-|x|} \left[\int_{-\infty}^{0} e^{+y}\, dy + \int_{0}^{\infty} e^{-y}\, dy \right]$$

$$= \frac{1}{4} e^{-|x|} \left| e^{y} \right|_{-\infty}^{0} + \frac{e^{-y}}{-1} \Big|_{0}^{\infty}$$

$$= \frac{1}{4} e^{-|x|} \left[(1) + \left(\frac{-1}{-1} \right) \right]$$

$$g(x) = \frac{e^{-|x|}}{2} \quad (1)$$

$$h(y) = \frac{1}{4} e^{-|y|} \int_{-\infty}^{\infty} e^{-|x|}\, dx$$

$$= \frac{1}{4} e^{-|y|} \int_{-\infty}^{0} e^{x} + \int_{0}^{\infty} e^{-x}\, dx$$

$$= \frac{1}{4} e^{-|y|} (2)$$

$$= \frac{e^{-|y|}}{2}$$

$$g(x)\, h(y) = \frac{e^{-|x|-|y|}}{4}$$

$$= f(x,y)$$

$\therefore$ The random variables are independent

b) $P(X \le 0, Y \le 0) = \int_{-\infty}^{0}\int_{-\infty}^{0} \frac{1}{4} e^{-|x|-|y|}\, dx\, dy$

$= \int_{-\infty}^{0} \frac{1}{4} e^{-|y|} \left[\int_{-\infty}^{0} e^{x}\, dx\right]$

$= \int_{-\infty}^{0} \frac{1}{4} e^{-|y|}\, e^{x}\Big|_{-\infty}^{0}$

$= \int_{-\infty}^{0} \frac{1}{4} e^{-|y|}\, dy$

$= \frac{1}{4} e^{-y}\Big|_{-\infty}^{0}$

$= \frac{1}{4}.$

6). Let $f(x)$ denote density funn of random variable X

By defn of density funn

$$\int_{-\infty}^{\infty} f(x)\,dx = 1. \quad — \quad (1)$$

Let $F_X(x_0)$ denotes commulative funn of random variable denotes

$$F_X(x_0) = P(x \le x_0) = \int_{-\infty}^{x_0} f(x)\,dx \quad - (2)$$

given $g(x) = 1 \quad x \ge x_0$

$\qquad = 0 \quad x < x_0$

$$E(g(x)) = \int_{-\infty}^{\infty} g(x).\, f(x)\, dx$$

$$= \int_{x_0}^{\infty} f(x)\, dx. \quad — \quad (3).$$

Since x_0 is a real number betn $-\infty < x_0 < \infty$.

$\therefore$ eqn (1) can be written as

$$\int_{-\infty}^{x_0} f(x)\,dx + \int_{x_0}^{\infty} f(x)\,dx = 1$$

$$\int_{x_0}^{\infty} f(x)\,dx = 1 - \int_{-\infty}^{x_0} f(x)\,dx$$

$$E(g(x)) = 1 - F_X(x_0) \quad \text{(from eq}^n\text{ 2 \& 3)}$$