Intro to DSP: "Digital Signal Processing"

* All sigs in nature are Analog sig
* But to process we require Digital sigs to do real tym sig processing

Analog sig $x(t)$ → ADC → digital sig $x(n)$

continuous Time domain — by sampling — discrete Time domain

Digital sig → DSP unit → digital → DAC → o/p sig

We can perform DFT, Spectral analysis (FFT, fast fourier Transform) tool

We can use filters to remove noise sig.

Filters → FIR (Finite impulse response filters)
IIR (Infinite " " ")

Levels of sig processing:

1] System level ⇒ can use 'C' programming, python, any image processing tool, Audio processing tool or matlab, Simulink and so on.

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2] Board level ⟹ FPGA , Transistor level → (CMOS)
(chip level)
↓
Ex: mobile

* Analog slgs are captured by sensors/hardware

Prerequities : signals & systems , Analog low pass/
Fourier Transforms high pass filters
Linear algebra

* Applications of DSP:

In various fields we use DSP
① Stock market analysis: to know when profits are peak
② Biomedical : ECG, EEG analysis
③ Radar slg processing : to detect targets
④ Sonar slg " : in sea to navigate
⑤ Communicatn slgs : Image /Audio processing
⑥ Audio, Vedio : Speaker Recognitn , Ceptral analysis
⑦ Seismic
⑧ Satellite /Rocket testing

2 types of analysis
→ Time domain analysis
→ Frequency " "
→ Noise removement

Filters
→ IIR → Butterwort
→ Chebyshev
→ FIR → windows filter

Unit I : ① DFT (Discrete fourier Transform) , IDFT & ⟹ properties → Gorzel & Chirp-z algorithm
DFT in linear filtering , overlap add, Overlapp metho
convolutn mechanism
② FFT (fast fourier transform) → 2 types of algorithm, DIT FFT, DIF FFT, DIF IFF

Unit II : Filters
chapter ③ :- Design of digital filters (unit II)
FIR filters : types , methods of designing filters
④ IIR filters →

Unit III : chapter ⑤ Digital Realizatns for FIR filters
" ⑥ " " " IIR filters

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Time and frequency domain Signals:

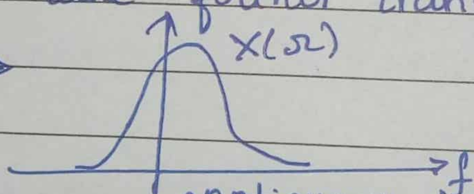
$x(t) \rightarrow$ analog domain
$x(n) \rightarrow$ digital domain

$$\underbrace{x(t)}_{\text{time domain}} \xrightarrow{\text{sampling}} x(n) \longleftrightarrow \underbrace{x(K)}_{\text{frequency domain}}$$

→ Freq domain Sampling and Reconstructⁿ of discrete time signals.

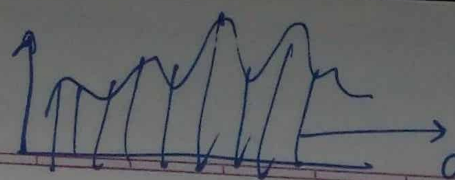
→ Why DFT (Discrete fourier Transform) ?
We cannot process a signal directly through digital signal processor
So we will take a part of slg in tym domain $x(t)$ w is continuous in time and Aperiodic in nature.
We will now take fourier transform of that

All continuous aperiodic slg $\xrightarrow{\text{CTFT}}$



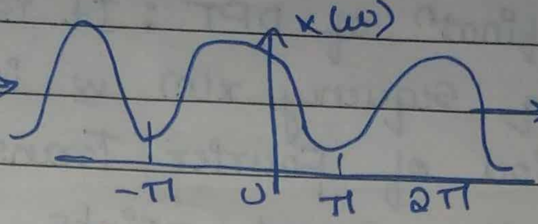
continuous in freq
aperiodic

$$\boxed{X(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, d\Omega}$$



Now we sample the sig with sampling freq using nyquist criteria

DT aperiodic $\xrightarrow{\text{DTFT}}$



$$X(\omega) = \sum_{-\infty}^{\infty} x(n) e^{-j\omega n}$$

$W = \frac{2\pi k}{N}$ N = total number of samples

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{n=N-1} x(n) e^{-j\frac{2\pi k n}{N}}$$

DFT
$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi k n}{N}}} \quad K = 0,1,2 \ldots N-1$$

↳ This can be matched to ~~discrete~~ DS processors

IDFT (Inverse discrete fourier Transform)

$$\boxed{x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi k n}{N}}}$$

This is known as N pt IDFT.

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* Definitⁿ of DFT and IDFT:

→ Definatⁿ of DFT ($\to X(K)$) : It is a finite duratⁿ discrete freq sequency $x(n)$ w is obtained by sampling one period of Fourier Transform sampling is done at N equally spaced points over the period from $w=0$ to 2π

DFT ⇒ $X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi nk}{N}}$ $K=0,1,2 \dots N-1$

$X(K)$ → freq domain, $x(n)$ → tym domain

$x(n) \underset{IDFT}{\overset{DFT}{\Longleftrightarrow}} X(K)$

IDFT ⇒ $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, e^{\frac{j2\pi nk}{N}}$, $n=0,1,\dots N-1$

Twiddle factor notation

$W_N = e^{\frac{-j2\pi}{N}}$

W_N → Twiddle factor

∴ DFT → $X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$ $K=0,1,2 \dots N-1$

IDFT → $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, W_N^{-Kn}$ ~~K= 0,1,2 ---- N-1~~ $n=0,1,2 \dots N-1$

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* Twiddle factors and Zero Padding :

$W_N = e^{\frac{-j2\pi}{N}}$ is twiddle factor

$N = 2, 4, 8, 16 \ldots\ldots$

W.K.T $e^{-j\theta} = \cos\theta - j\sin\theta$

For $N = 2$

$W_N = e^{-j2\pi/2} = e^{-j\pi}$

$= \cos\pi - j\sin\pi$

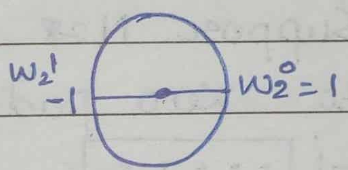
$W_2 = -1$

~~WN~~ $\boxed{W_N^K = e^{\frac{-j2\pi K}{N}}}$

$\longrightarrow W_2^0 = e^{\frac{-j2\pi 0}{2}} = 1$

$\longrightarrow W_2^1 = e^{\frac{-j2\pi 1}{2}} = e^{-j\pi} = -1$

Euler's diagram is



when $N = 4$ $W_4^K = e^{\frac{-j2\pi K}{4}} = e^{-j\pi K/2}$

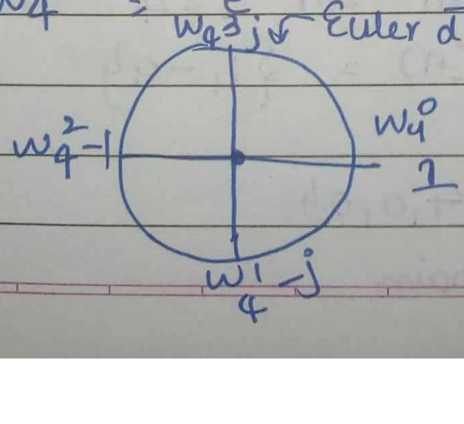
$W_4^0 = e^{-j\pi 0} = 1$

$W_4^1 = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2 = -j$

$W_4^2 = e^{-j\pi} = -1$

$W_4^3 = e^{-j\pi 3/2} = \cos 3\pi/2 - j\sin 3\pi/2 = j$

Euler diagram



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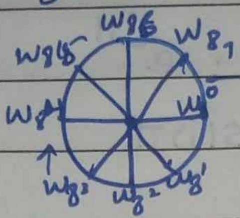
When $N=8$, $W_8^K = e^{-j2\pi k/8} = e^{\frac{-j2\pi k}{4}}$

$W_8^0 = 1$ $\qquad W_8^4 = -1$

$W_8^1 = 0.707 - j0.707$ $\qquad W_8^5 = -0.707 + j0.707 \rightarrow \frac{1}{\sqrt{2}}$

$W_8^2 = -j$ $\qquad W_8^6 = +j$

$W_8^3 = -0.707 - j0.707$ $\qquad W_8^7 = 0.707 + j\,0.707$



→ Euler's diagram

→ **Zero Padding:** Used in signal processing applicatn

↳ Adding sigs of '0' Amplitude in the time domain to end of sequence

$x(kn)$

↳ time index

Suppose $N=8$ & we have only 1,1,-1 then we are going to zero pad remaining 5

i.e, 1,1,-1 $\boxed{\cdot\ \cdot\ \cdot\ \cdot\ \cdot}$

↳ going to end sig with '0' Amplitude, not adding any info only extending the sig with 0 Amplitude

→ If I have to zero pad for $N=4$

given sig is $x(n) = \{1, -4\}$

Ans is $x(n) = \{1, -4, 0, 0\}$

↳ Time domain

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* DFT as a linear Transformation: (Matrix method)

Let i/p slg be $x_N = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}_{N\times 1}$

$X_K = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}_{N\times 1}$ → freq domain

By defn

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}$$

$$W_N = e^{-j2\pi/N}$$

$$\therefore X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

Let $N = 4$, $\therefore x(n) = [x(0), x(1), x(2) \; \& \; x(3)]$ → Time domain

In freq domain $\{X(0), X(1), X(2), X(3)\}$

$W_4^0 = 1$ $W_4^1 = -j$, $W_4^2 = -1$ $W_4^3 = j$ (W.KT)

$X(k) = [W_N]\, x_N$

DFT

$$\underset{x(k)}{\cancel{[W_N]}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} x(n)$$

↑ complex conjugate

IDFT $x(n) = \frac{1}{N}\left[W_N^{*}\right] X(k)$

$= \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} X(k)$

here $N=4$

$\therefore \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & j \end{bmatrix} X(K)$

PROPERTIES OF DFT:

① * Linearity & Periodicity Property

1] LINEARITY :-

Suppose we have slg $x_1(n)$ & its DFT is $X_1(K)$

$x_1(n) \xleftrightarrow{DFT} X_1(K)$

$x_2(n) \xleftrightarrow{DFT} X_2(K)$

then, property states

$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{DFT} a_1 X_1(K) + a_2 X_2(K)$//.

W.K.T $X_1(K) = \sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi nk}{N}}$

& $X_2(K) = \sum_{n=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi nk}{N}}$

DFT $\{a_1 x_1(n) + a_2 x_2(n)\} = \sum_{n=0}^{N-1}\left[a_1 x_1(n) + a_2 x_2(n)\right] e^{\frac{-j2\pi nk}{N}}$

$= \sum_{n=0}^{N-1} a_1 x_1(n)\, e^{\frac{-j2\pi nk}{N}} + \sum_{n=0}^{N-1} a_2 x_2(n)\, e^{\frac{-j2\pi nk}{N}}$

DFT $\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 X_1(K) + a_2 X_2(K)$ //

hence proved.

2] PERIODICITY:

$x(n) \xleftrightarrow{DFT} X(K)$

$x(n+N) \longleftrightarrow = x(n)$

$X(K+N) = X(K)$

$X(K+N) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n(K+N)}{N}}$

$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}\, e^{\frac{-j2\pi n \cancel{N}}{\cancel{N}}}$ $\quad \because e^{-j2\pi n} = 1$

$X(K+N) = X(K) \rightarrow$ hence prooved.

3] CIRCULAR TIME SHIFT PROPERTY:

If a discrete time signal is circularly shifted in time by m units then its DFT is multiplied by $e^{\frac{-j2\pi km}{N}}$

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If $x(n) \xleftrightarrow{DFT} X(K)$ then

$x(n-m)_N \xleftrightarrow{DFT} X(K)\, e^{-j2\pi km/N}$

$DFT\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N\, e^{\frac{-j2\pi kn}{N}}$

Let $p = n-m$

$n = p+m$

$DFT\{x((n-m))_N\} = \sum_{n=0}^{N-1} x(p)_N\, e^{\frac{-j2\pi k(p+m)}{N}}$

$= \sum_{n=0}^{N-1} x(p)_N\, e^{\frac{-j2\pi kp}{N}} \cdot e^{\frac{-j2\pi km}{N}}$

$DFT\{x((n-m))_N\} = X(K)\, e^{-j2\pi km/N}$ //

hence prooved.

4] TIME REVERSAL:

Reversing N point sequence in time is equivalent to reversing the DFT Sequence

If $DFt\{x(n)\} = X(K)$

$Dft\{x(N-n)\} = X(N-K)$ //

proof > $dft\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)\, e^{\frac{-j2\pi nk}{N}}$

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Let

~~$m = n-N$~~ $m = N-n$

$n = N-m$

$= \sum_{n=0}^{N-1} x(m)\, e^{\frac{-j2\pi k(N-m)}{N}}$ $\left[e^{-j2\pi k} = 1\right]$

$Dft\{x(N-n)\} = \sum_{n=0}^{N-1} x(m)\, e^{\frac{-j2\pi k\cancel{N}}{\cancel{N}}}\, e^{\frac{j2\pi km}{N}}$

$= \sum_{n=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}}\, e^{-j2\pi m}$ $\left[\because e^{-j2\pi m} = 1\right]$

$Dft\{x(N-n)\} = \sum_{n=0}^{N-1} x(m)\, e^{-j\frac{2\pi m}{N}(N-k)}$

$\therefore Dft\{x(N-n)\} = X(N-K)$ // hence prooved.

5] Conjugation:

Let $x(n)$ be a complex N point discrete sequence & $x^*(n)$ be complex conjugate. Now if

$DFT\, x(n) = X(K)$ then

$DFT\{x^*(n)\} = X^*(N-K)$

$\rightarrow dft(x^*(n)) = \sum_{n=0}^{N-1} x^*(n)\, e^{\frac{-j2\pi nk}{N}}$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}}\right]^* \quad \text{W.K.T } e^{j2\pi n} = 1$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}} e^{-j2\pi n}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}} e^{\frac{-j2\pi nN}{N}}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi n(N-K)}{N}}\right]^*$$

$$= \{X(N-K)\}^*$$

$$= X^*(N-K) \text{ // hence proved.}$$

6] CIRCULAR FREQUENCY SHIFT PROPERTY:

The circular frequency shift property of DFT says that if a discrete time signal is multiplied by $e^{\frac{j2\pi mn}{N}} = X((k-m))_N$

proof:

→ if $DFT\{x(n)\} = X(k)$ then

$DFT\{x(n) e^{\frac{j2\pi mn}{N}}\} = X((k-m))_N$

$$DFT\{x(n) e^{\frac{j2\pi mn}{N}}\} = \sum_{n=0}^{N-1} x(n) e^{\frac{+j2\pi mn}{N}} e^{\frac{-j2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi(k-m)n}{N}}$$

$$= X((k-m))_N \text{ // hence proved}$$

DTFT TO DFT :-

→ Sampling and reconstructⁿ of discrete time signals:-

W.K.T Aperiodic finite energy signals have continuous spectra

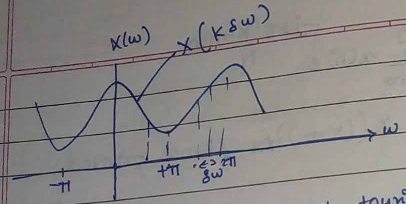
Let us considere Aperiodic discrete time sig $x(n)$ with fourier tranform. $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

↳ We will sample periodically in frequency domain and we will keep spacing bt 2 samples and $\delta\omega$ radians.

W.K.T $X(\omega)$ is periodic, period is 2π

If we take only samples of the fundamental freq range That is enough

Let us take N equidistant samples in the interval $0 \leq \omega < 2\pi$ with spacing $\delta\omega = \frac{2\pi}{N}$



Freq domain sampling of fourier transform

consider N number of samples in freq domain

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Evaluate this @ $\omega = \frac{2\pi k}{N}$

$$\cancel{X(\omega)} = X\left(\frac{2\pi k}{N}\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

↳ subdivids RHS summatn

each should have only N terms

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=-N}^{-1} x(n)\, e^{\frac{-j2\pi kn}{N}} + \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}} +$$

$$\sum_{n=N}^{2N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} + \cdots$$

$$\boxed{X\left(\frac{2\pi k}{N}\right) = \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}}$$

change $n \Rightarrow n - lN$ & interchange the summat's

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x(n-lN)\, e^{\frac{-j2\pi k(n-lN)}{N}}$$

$$\boxed{\text{let } x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)}$$

fourier series expansion of $x_p(n)$ is

w.k.t $x_p(n) = \sum_{k=0}^{N-1} C_k\, e^{j2\pi nk/N}$

C_k = fourier coefficient

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi kn/N}$$

$$\therefore\ x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) e^{\frac{j2\pi nk}{N}}$$

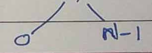
↳ It provides reconstructn of periodic s/g $x_p(n)$ from $X(\omega)$

let Relatnship b/t $x_p(n)$ and $x(n)$.

for $x(n)$ $N \geq L \rightarrow$ no aliasing

$N < L \rightarrow$ aliasing

<u>conclusion</u>: spectrum of an aperiodic discrete time signal with finite duratn (L) can be exactly recovered from samples $\omega_k = \frac{2\pi nk}{N}$ if $\underline{\underline{N \geq L}}$



CIRCULAR CONVOLUTION

$$x_1(n) \circledast x_2(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N$$

property

If $DFT\{x_1(n)\} = X_1(K)$ & $DFT\{x_2(n)\} = X_2(K)$ then

$$DFT\{x_1(n) \circledast x_2(n)\} = X_1(K)X_2(K)$$

proof: $X_1(K) = \sum_{n=0}^{N-1} x_1(n)\, e^{-\frac{j2\pi nk}{N}}$

change n to m

$$X_1(K) = \sum_{m=0}^{N-1} x_1(m)\, e^{-\frac{2j\pi mk}{N}}$$

$$X_2(K) = \sum_{n=0}^{N-1} x_2(n)\, e^{-\frac{j2\pi nk}{N}} \quad \text{change } n \to p$$

$$X_2(K) = \sum_{n=0}^{N-1} x_2(p)\, e^{-\frac{j2\pi pk}{N}}$$

IDFT :- $DFT^{-1}\{X_1(K)X_2(K)\} = \frac{1}{N}\sum_{K=0}^{N-1} X_1(K)X_2(K)\, e^{j2\pi nk/N}$

$$= \frac{1}{N}\sum_{K=0}^{N-1}\left[\sum_{m=0}^{N-1} x_1(m)\, e^{-\frac{2j\pi mk}{N}}\right]\left[\sum_{n=0}^{N-1} x_2(p)\, e^{-\frac{j2\pi pk}{N}}\right] e^{j2\pi nk/N}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x_1(m) \sum_{p=0}^{N-1} x_2(p) \sum_{K=0}^{N-1} e^{\frac{j2\pi K(n-m-p)}{N}}$$

Let $n-m-p = qN$, q is integer

$$\therefore \sum_{K=0}^{N-1} e^{\frac{j2\pi k(n-m-p)}{N}} = \sum_{k=0}^{N-1} e^{\frac{j2\pi kqN}{N}} = \sum_{k=0}^{N-1}\left(e^{j2\pi q}\right)^k = \sum_{k=0}^{N-1} 1^k$$

and $\sum_{p=0}^{N-1} x_2(p) = \sum_{m=0}^{N-1} x_2(n-m-qN) = \sum_{m=0}^{N-1} x_2(n-m), \bmod N$

$$= \sum_{m=0}^{N-1} x_2((n-m))_N$$

$$\therefore DFT\{X_1(K)X_2(K)\} = \frac{1}{N}\sum_{m=0}^{N-1} x_1(m) \sum_{m=0}^{N-1} x_2((n-m))_N \cdot N$$

$$= \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N$$

$$= x_1(n) \circledast x_2(n)$$

$$\therefore X_1(K)X_2(K) = DFT\{x_1(n) \circledast x_2(n)\} \;//.$$

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MULTIPLICATION PROPERTY OF DFT:

If
$\rightarrow$ $DFT\{x_1(n)\} = X(K)$ & then

$$DFT\{x_1(n)x_2(n)\} = \frac{1}{N}\left[X_1(K) \circledast X_2(K)\right]$$

$$x_1(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_1(K) e^{j2\pi kn/N} = \frac{1}{N}\sum_{m=0}^{N-1} X_1(m) e^{\frac{j2\pi mn}{N}} \quad \because K=m$$

$$DFT\{x_1(n)x_2(n)\} = \sum_{n=0}^{N-1} x_1(n)x_2(n) e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{m=0}^{N-1} X_1(m) e^{j2\pi mn/N}\right] x_2(n) e^{-j2\pi kn/N}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n) e^{-j2\pi kn/N} e^{j2\pi mn/N}\right]$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n) e^{\frac{-j2\pi(K-m)n}{N}}\right]$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\, X_2((K-m))_N = \frac{1}{N}\left[X_1(K) \circledast X_2(K)\right]$$

$$\therefore DFT\{x_1(n)x_2(n)\} = \frac{1}{N}\left[X_1(K) * X_2(K)\right]$$

hence proved

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CIRCULAR CORRELATION:

If we have 2 i/p sigs $x(n)$ and $y(n)$ then my circular correlatn property

$$\boxed{\bar{r}_{xy}(m) = \sum_{n=0}^{N-1} x(n) y^*((n-m))_N}$$

If $DFT\{x(n)\} = X(K)$ & $DFT\{y(n)\} = Y(K)$

By correlation property

$$DFT\{\bar{r}_{xy}(m)\} = DFT\left\{\sum_{n=0}^{N-1} x(n)\, y^*((n-m))_N\right\}$$

$$= X(K)\, Y^*(K)$$

proof: $X(K) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$ and $Y(k) = \sum_{n=0}^{N-1} y(n) e^{-j2\pi nk/N}$

$n \rightarrow p$

$$Y(K) = \sum_{p=0}^{N-1} y(p) e^{-j2\pi pk/N}$$

$$DFT^{-1}\{X(K)Y^*(K)\} = \frac{1}{N}\sum_{K=0}^{N-1} X(K)Y^*(K) e^{j2\pi nK/N}$$

$$= \frac{1}{N}\sum_{K=0}^{N-1}\left[\sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}}\right]\left[\sum_{p=0}^{N-1} y(p) e^{\frac{-j2\pi pk}{N}}\right]^* e^{\frac{j2\pi nk}{N}}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n) \sum_{p=0}^{N-1} y^*(p) \underbrace{\sum_{k=0}^{N-1} e^{\frac{j2\pi K(m-n+p)}{N}}}$$

$\hookrightarrow$ Let us consider this part.

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consider $\sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n+p)}{N}}$, let $m-n+p = qN$ (q → integer)

$$= \sum_{k=0}^{N-1} e^{j2\pi k qN/N} = \sum_{k=0}^{N-1} e^{(j2\pi q)^k}$$

$$= \sum_{k=0}^{N-1} 1^k = N$$

∴ let us take $\sum_{p=0}^{N-1} y^*(p)$ from that eqn

let $m-n+p = qN$

$p = n-m+qN$

$$\therefore \sum_{p=0}^{N-1} y^*(p) = \sum_{n=0}^{N-1} y^*(n-m+qN)$$

$$= \sum_{n=0}^{N-1} y^*(n-m), \text{ mod } N$$

$$\sum_{p=0}^{N-1} y^*(p) = \sum_{n=0}^{N-1} y^*((n-m))_N$$

$$\therefore DFT^{-1}\{X(K)\, Y^*(K)\} = \frac{1}{N}\sum_{n=0}^{N-1} x(n) \sum_{n=0}^{N-1} y^*((n-m))_N \cdot N$$

$$= \sum_{n=0}^{N-1} x(n)\, y^*((n-m))_N$$

$$= \bar{r}_{xy}(m)$$

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$$\therefore X(K)\, y^*(K) = DFT\{\bar{r}_{xy}(m)\} \;//$$

PARSEVAL'S THEOREM (energy)

- If $DFT\{x_1(n)\} = X_1(K)$ & $DFT\{x_2(n)\} = X_2(K)$ then

$$\sum_{n=0}^{N-1} x_1(n)\, x_2^*(n) = \frac{1}{N}\sum_{K=0}^{N-1} X_1(K)\, X_2^*(K)$$

proof: $X_1(K) = \sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi nK/N}$

$$x_2(n) = \frac{1}{N}\sum_{K=0}^{N-1} X_2(K)\, e^{j2\pi nK/N}$$

$$\frac{1}{N}\sum_{K=0}^{N-1} X_1(K)\, X_2^*(K) = \frac{1}{N}\sum_{K=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi nK}{N}}\right] X_2^*(K)$$

Rearrange $= \sum_{n=0}^{N-1} x_1(n) \left[\frac{1}{N}\sum_{K=0}^{N-1} X_2^*(K)\, e^{\frac{-j2\pi nK}{N}}\right]$

$$= \sum_{n=0}^{N-1} x_1(n) \underbrace{\left[\frac{1}{N}\sum_{K=0}^{N-1} X_2(K)\, e^{\frac{+j2\pi nK}{N}}\right]^*}$$

$$\frac{1}{N}\sum_{K=0}^{N-1} X_1(K)\, X_2^*(K) = \sum_{n=0}^{N-1} x_1(n)\, x_2^*(n) \quad // \text{ proved}$$

Problems on Circular Convolutⁿ

1] Time domain (matrix method)

If given $x(n)$ and $h(n)$.

Let us use matrix method of convolution.

If $h(n) = \{h(0), h(1), h(2), h(3)\}$ and

$x(n) = \{x(0), x(1), x(2), x(3)\}$

$$\begin{bmatrix} h(0) & h(3) & h(2) & h(1) \\ h(1) & h(0) & h(3) & h(2) \\ h(2) & h(1) & h(0) & h(3) \\ h(3) & h(2) & h(1) & h(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} x(0)h(0) + x(1)h(3) - - - - \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix}$$

$y(n) = \{y(0), y(1), y(2), y(3)\}$

2] Frequency domain $x(n)$ & $h(n)$ (matrix method)

Step 1: Find $X(K)$ and $H(K)$

" 2: $X(K)H(K)$ (element by element mul)

& say $Y(K) = X(K)H(K)$

Step 3: Take IDFT of $Y(K)$ to obtain $y(n)$.

Ex :

$$\underset{DFT}{X(K)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}$$

$$\underset{DFT}{H(K)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ h(3) \end{bmatrix} = \begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \end{bmatrix}$$

$$Y(K) = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} \begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \end{bmatrix} = \begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{bmatrix}$$

$$y(n) = IDFT(Y(K)) = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & j \end{bmatrix} \begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix}$$

$\boxed{\text{here } N = 4}$

$y(n) = \{y(0), y(1), y(2), y(3)\}$.

Both by time domain and by frequency domain $y(n)$ should be same. (verificatⁿ).

Problems on Circular Convolutⁿ

1] Time domain (matrix method)

If given $x(n)$ and $h(n)$.

Let us use matrix method of convolution.

If $h(n) = \{h(0), h(1), h(2), h(3)\}$ and

$x(n) = \{x(0), x(1), x(2), x(3)\}$

$$\begin{bmatrix} h(0) & h(3) & h(2) & h(1) \\ h(1) & h(0) & h(3) & h(2) \\ h(2) & h(1) & h(0) & h(3) \\ h(3) & h(2) & h(1) & h(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} x(0)h(0) + x(1)h(3) - - - \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix}$$

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Step 1: Find $X(K)$ and $H(K)$

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Ex :

$$X(K)_{DFT} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}$$

$$H(K)_{DFT} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ h(3) \end{bmatrix} = \begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \end{bmatrix}$$

$$Y(K) = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} \begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \end{bmatrix} = \begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{bmatrix}$$

$$y(n) = IDFT(Y(K)) = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix}$$

$\boxed{\text{here } N = 4}$

$y(n) = \{y(0), y(1), y(2), y(3)\}$.

Both by time domain and by frequency domain $y(n)$ should be same. (verificatⁿ).

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: Overlap Add Method :

↳ for finding linear convolutⁿ, segmenting given sequence

Suppse $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7), x(8), x(9)\}$.

$h(n) = \{h(0), h(1), h(2)\} \rightarrow M = 3 \quad M-1 = 2.$

choose segment length $L_s = 5$ (we can vary)

1st Segment $[x(0), x(1), x(2)\ 0\ 0]$

2nd Segment $[x(3), x(4)\ x(5)\ 0\ 0]$

3rd " $[x(6), x(7), x(8), 0\ 0]$

4th " $[x(9), 0, 0, 0, 0]$

$$\begin{bmatrix} h(0) & 0 & 0 & h(2) & h(1) \\ h(1) & h(0) & 0 & 0 & h(2) \\ h(2) & h(1) & h(0) & 0 & 0 \\ 0 & h(2) & h(1) & h(0) & 0 \\ 0 & 0 & h(2) & h(1) & h(0) \end{bmatrix} \underbrace{\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ 0 \\ 0 \end{bmatrix}}_{\text{1st segment}} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} \quad ① \ y(n)$$

$$\begin{bmatrix} h(0) & 0 & 0 & h(2) & h(1) \\ h(1) & h(0) & 0 & 0 & h(2) \\ h(2) & h(1) & h(0) & 0 & 0 \\ 0 & h(2) & h(1) & h(0) & 0 \\ 0 & 0 & h(2) & h(1) & h(0) \end{bmatrix} \begin{bmatrix} x(3) \\ x(4) \\ x(5) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f \\ g \\ h \\ i \\ j \end{bmatrix} \quad ② \ y(n)$$

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$$\begin{bmatrix} \text{same as previous} \end{bmatrix} \underbrace{\begin{bmatrix} x(6) \\ x(7) \\ x(8) \\ 0 \\ 0 \end{bmatrix}}_{\text{3rd segment}} = \begin{bmatrix} k \\ l \\ m \\ n \\ o \end{bmatrix} \quad ③$$

$$\begin{bmatrix} \text{same as previous} \end{bmatrix} \underbrace{\begin{bmatrix} x(9) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{4th segment}} = \begin{bmatrix} p \\ q \\ r \\ s \\ t \end{bmatrix} \quad ④$$

2 overlapping as $M-1 = 2$

①	a	b	c	d	e														
②				f	g	h	i	j											
③							k	l	m	n	o								
④										p	q	r	s	t					
	a	b	c	(d+f)	(e+g)	h	(i+k)	(j+l)	m	(n+p)	(o+q)	r	s	t					

$y(n) = \{a, b, c, (d+f)\ (e+g)\ h\ (i+k)\ (j+l)\ m\ (n+p)\ (o+q)\ r\ s\ t$

" " " mul are 4x4 } " " mul N^2

: OVERLAP SAVE METHOD: (Segmented method)

↳ To find linear convolutⁿ.

$x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6) \ldots x(10)\}$

$h(n) = \{h(0), h(1), h(2)\} \rightarrow M = 3,$

$M - 1 = 2.$

$y(n) = ?$

$L_s = 5$ (segment length)

~~y(0)~~

M-1

0	0	x(0)	x(1)	x(2)

zeros

~~x(3) x(4) x(5)~~

x(1)	x(2)	x(3)	x(4)	x(5)

x(4)	x(5)	x(6)	x(7)	x(8)

x(7)	x(8)	x(9)	x(10)	0

↳ rule for segmentatⁿ

$$\begin{bmatrix} h(0) & 0 & 0 & h(2) & h(1) \\ h(1) & h(0) & 0 & 0 & h(2) \\ h(2) & h(1) & h(0) & 0 & 0 \\ 0 & h(2) & h(1) & h(0) & 0 \\ 0 & 0 & h(2) & h(1) & h(0) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ x(0) \\ x(1) \\ x(2) \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$$

$$\begin{bmatrix} h(0) \\ \text{same as previous segment} \end{bmatrix} \begin{bmatrix} x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \end{bmatrix} = \begin{bmatrix} f \\ g \\ h \\ i \\ j \end{bmatrix}$$

$$\begin{bmatrix} \text{same as 1st one segment} \end{bmatrix} \begin{bmatrix} x(4) \\ x(5) \\ x(6) \\ x(7) \\ x(8) \end{bmatrix} = \begin{bmatrix} k \\ l \\ m \\ n \\ o \end{bmatrix}$$

$$\begin{bmatrix} \text{same as 1st segment} \end{bmatrix} \begin{bmatrix} x(7) \\ x(8) \\ x(9) \\ x(10) \\ 0 \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \\ s \\ t \end{bmatrix}$$

as $M-1 = 2$, discard 1st 2 element of all matrix

i.e. (a, b, f, g, k, l, p, q)

& $y(n) = \{c, d, e, h, i, j, m, n, o, r, s, t\}$

$y(n) = \{y(0), y(1), y(2), y(3), y(4), y(5), y(6), y(7), y(8), y(9), y(10), y(11)\}$ //.

RELATIONSHIP OF DFT TO OTHER TRANSFORMS.

1] DTFT to DFT

By defⁿ

DTFT is $X(e^{jw}) = \sum_{n=0}^{N-1} x(n) e^{-jwn}$ $\quad n = 0, 1 \ldots N-1$

DFT is $X(K) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}}$

$X(K) = X(w)$ @ $w = \frac{2\pi K}{N}$, $K = 0, 1, 2, 3 \ldots N-1$

∴ If we want to find DFT, we have to evaluate DTFT @ ~~K~~ $w = \frac{2\pi K}{N}$

2] Z Transform to DFT:

$\hookrightarrow X(z) = \sum_{n=0}^{N-1} x(n) z^{-n} \longrightarrow ①$

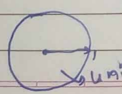
DFT $X(K) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}} \longrightarrow ②$

from ① & ②

w.k.T $|e^{j2\pi K/N}| = 1 \qquad \angle e^{j2\pi K/N} = \frac{2\pi K}{N}$

∴ On comparing 1 & 2

$\boxed{\therefore\ z = e^{j2\pi K/N}}$



unit ○ᶜᵉ

$X(K) = X(z)$ @ $z = e^{j2\pi K/N}$

∴ On unit ○ᶜᵉ evaluatⁿ is performed. N values are equally spaced

3] FS to DFT:

W.K.T

$x_p(n) = \sum_{K=0}^{N-1} C_K e^{j2\pi nK/N} \longrightarrow$ periodic sig $\hookrightarrow ①$

where C_K are fourier coeff

$C_K = \frac{1}{N} \sum_{K=0}^{N-1} x_p(n) e^{-j2\pi nK/N} \longrightarrow ②$

$x(n) = x_p(n) \longrightarrow 0 \le n \le N-1$

$= 0 \longrightarrow$ otherwise.

becoz of this rlnship

$C_K = \frac{1}{N} X(K) \longrightarrow ③$

$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nK/N}$

compare eq 1 & 2

$\hookrightarrow \boxed{X(K) = N C_K}$ //

CHAPTER - 2

FAST FOURIER - TRANSFORM (FFT) ALGORITHMS

→ In this we have 2 algorithms:

DIT : Decimation in time

DIF : Decimation in frequency

→ FFT is tool to find discrete fourier transform (DFT) efficiently

→ In this the symmetry property of the twiddle factors is used

* Symmetry property is:

1) $W_N^{K+N/2} = -W_N^K$

* Periodicity property is:

$W_N^{K+N} = W_N^K$ $\qquad W_N = e^{\frac{-j2\pi}{N}}$

→ In FFT, the number of complex multiplicatn are $\frac{N}{2}\log_2 N$

→ In FFT, the number of complex additns, are $N\log_2 N$.

$2\log_2 N$

2 3

DIT (Decimation In Time): FFT algorithms.

→ We use following properties to to DIT

1] Symmetry:

$$W_N^{K+N/2} = -W_N^K$$

2] Periodicity

$$W_N^{K+N} = W_N^K$$

→ Radix 2 DITFFT Algorithm $\quad N = 2^M$

Ex: 4, 8, 16, - - - - 2048

In case of DITFFT

x(n)

even odd

$$x_e(n) = x(2n)$$

$$x_o(n) = x(2n+1)$$

→ Derivatn of DITFFT

W.K.T
$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} x(n) W_N^{nk},$$

$$= \sum_{n=0,\ \text{even}}^{N-1} x(n) W_N^{nk} + \sum_{n=0,\ \text{odd}}^{N-1} x(n) W_N^{nk}$$

$$= \sum_{n=0}^{N-1} x(2n) W_N^{2nk} + \sum_{n=0}^{N-1} x(2n+1) W_N^{(2n+1)k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2nk} + W_N^K \sum_{n=0}^{N-1} x(2n+1) W_n^{2nk}$$

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x_e(n) W_N^{2nk} + W_N^K \sum_{n=0}^{\frac{N}{2}-1} x_o(n) W_n^{2nK}$$

Suppose N = 4, $W_4^0, W_4^1, W_4^2, W_4^3$
$\quad 1, \quad -j, \quad -1, \quad j$

$$W_N^2 = \left(e^{\frac{-j2\pi}{N}}\right)^2 = W_{N/2}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x_e(n) W_{N/2}^{nk} + W_N^k \sum_{n=0}^{\frac{N}{2}-1} x_o(n) W_{N/2}^{nk}$$

$$X(k) = \underbrace{X_e(k)}_{\frac{N}{2}\text{pt DFT}} + W_N^k \underbrace{X_o(k)}_{\frac{N}{2}\text{pt DFT}}$$

Let us symbolise X(k) as

$$X(k) = \underbrace{G(k)}_{x(n)\text{even}} + W_N^k \underbrace{H(k)}_{x(n)\text{odd}} \qquad W_N^{k+N/2} = -W_N^k$$

$$X(k) = G(k) + W_N^k H(k)$$

$$\cancel{X(k)}\ X\left(k+\frac{N}{2}\right) = G\left(k+\frac{N}{2}\right) + W_N^{k+N/2} H\left(k+\frac{N}{2}\right)$$

Break G(k) & H(k) into $\frac{N}{4}$pt (Decimat^n)

$$G(k) = \sum_{r=0}^{\frac{N}{2}-1} g(r) W_{N/2}^{rk}$$

even 2l, odd 2l+1

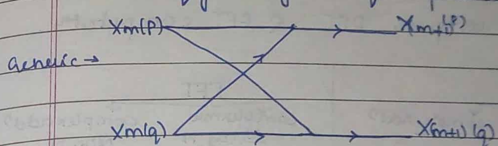
$$G(k) = \sum_{l=0}^{\frac{N}{4}-1} g(2l) W_{N/4}^{lk} + W_{N/2}^k \sum_{l=0}^{\frac{N}{4}-1} g(2l+1) W_{N/4}^{lk}$$

similarly

$$H(k) = \sum_{r=0}^{\frac{N}{2}-1} h(r) W_{N/2}^{rk}$$

$$H(k) = \sum_{l=0}^{\frac{N}{4}-1} h(2l) W_{N/4}^{lk} + W_{N/2}^k \sum_{l=0}^{\frac{N}{4}-1} h(2l+1) W_{N/4}^{lk}$$

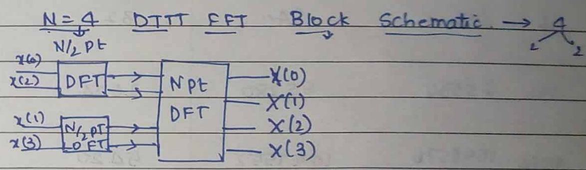
Butterfly diagram for previous eqn



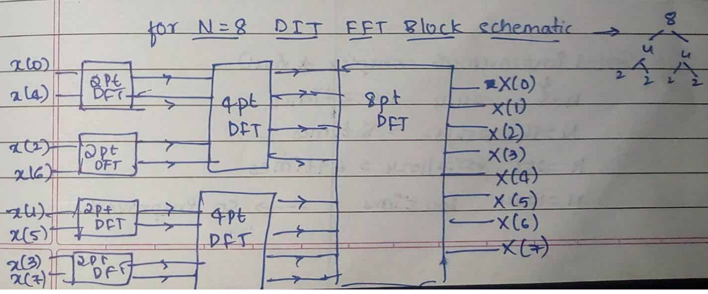
$X_{m+1}(p) \qquad X_{m+1}(q)$

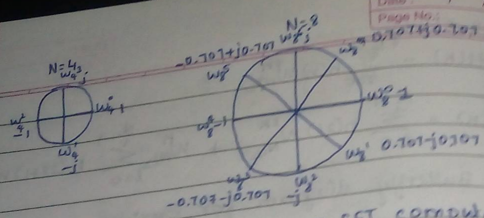
$$X_{m+1}(p) = X_m(p) + X_m(q)$$
$$X_{m+1}(q) = [X_m(p) - X_m(q)] W_N^r$$

N = 4 ($N/2$ pt) DIT FFT Block Schematic →



for N = 8 DIT FFT Block schematic →





Now we compare DFT & FFT computation.

N	DFT complex mul (*) N^2	DFT complex add $N(N-1)$	FFT complex mul $\frac{N}{2}\log_2 N$	FFT complex add $N\log_2 N$
8	64	52	12	24
16	256	240	32	64
256	65536	65,280	1024	2048
1024	1048576	1047552	5120	10240

Speed improvement complex * (mul)

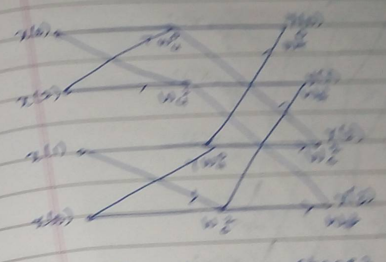
N=8, 64/12 = 5.3 times

N=16 256/32 = 8 times

N=256 65536/1024 = 64 times

N=1024 204 times ⟶ so improved.

Signal flow graph N=4 ...



Stage 1

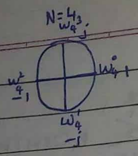
N=4 = 2^2

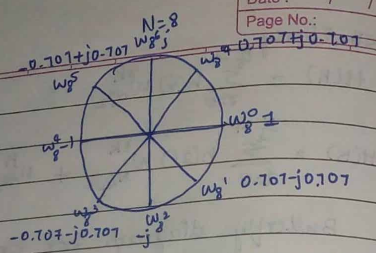
2 stage

Stage 2

each stage

2 butterflies





Now we compare DFT & FFT computation.

N	DFT complex mul (*) N^2	DFT complex add'n $N(N-1)$	FFT complex mul $\frac{N}{2}\log_2 N$	FFT complex add'n $N\log_2 N$
8	64	52	12	24
16	256	240	32	64
256	65536	65,280	1024	2048
1024	1048576	1047552	5120	10240

Speed improvement complex * (mul)

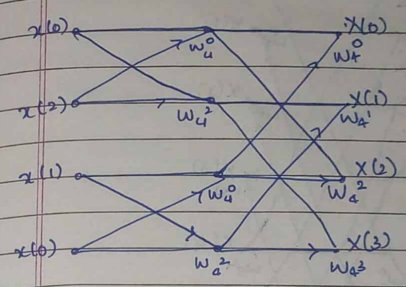
N=8, 64/12 = 5.3 times

N=16 256/32 = 8 times

N=256 65536/1024 = 64 times

N=1024 204 times $\longrightarrow$ so improved.

Signal flow graph N=4 pt DITFFT



Stage 1 stage 2

$N = 4 = 2^2$ Each stage

2 stage 2 butterflies

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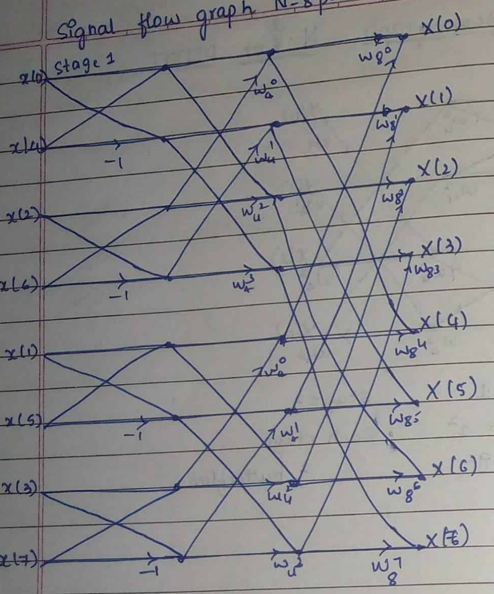
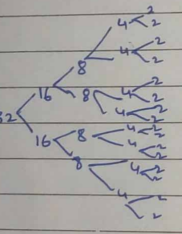
Signal flow graph N=8pt DITFFT



Week 3

Decimatⁿ :-



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Decimation in Frequency (DIF) FFT Algorithm:

Input values are in order

Output values are bit reversed.

→ Derivatⁿ

By defn $X(K) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=N/2}^{N-1} x(n) W_N^{kn}$$

Let $r = n - \frac{N}{2}$

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{r=0}^{\frac{N}{2}-1} x\left(r+\frac{N}{2}\right) W_N^{K(r+N/2)}$$

r is dummy variable ∴ r is replaced by n

$$\therefore X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=0}^{N/2-1} x\left(n+\frac{N}{2}\right) W_N^{K(n+N/2)}$$

WKN $W_N^{KN/2} = (-1)^K$

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^K x\left(n+\frac{N}{2}\right)\right] W_N^{kn}$$

even $K = 2r$, $r = 0, 1, 2, \ldots N/2 - 1$

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2r} x\left(n+\frac{N}{2}\right)\right] W_N^{2rn}$$

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$$= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x\left(n+\frac{N}{2}\right)\right] W_{N/2}^{rn}$$

<u>odd</u> $k = 2r+1$, $r = 0, 1, \ldots \frac{N}{2}-1$

$$X(2r+1) = \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^{2r+1} x\left(n+\frac{N}{2}\right)\right] W_N^{n} W_N^{2rn}$$

even $e(n) = x(n) + x\left(n+\frac{N}{2}\right)$

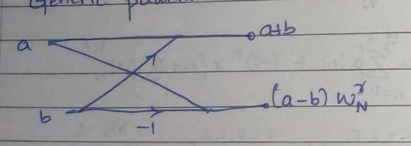
odd $o(n) = \left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^{n}$

$$\therefore X(2r) = \sum_{n=0}^{N/2-1} e(n) W_{N/2}^{rn}$$

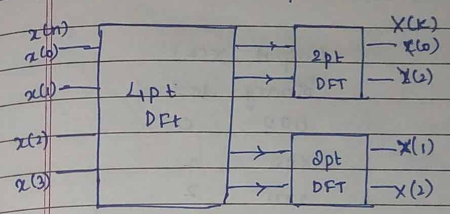
$$X(2r+1) = \sum_{n=0}^{N/2-1} o(n) W_{N/2}^{rn} \qquad r = 0, 1, \ldots N/2-1$$

DIFFFT Butterfly pattern

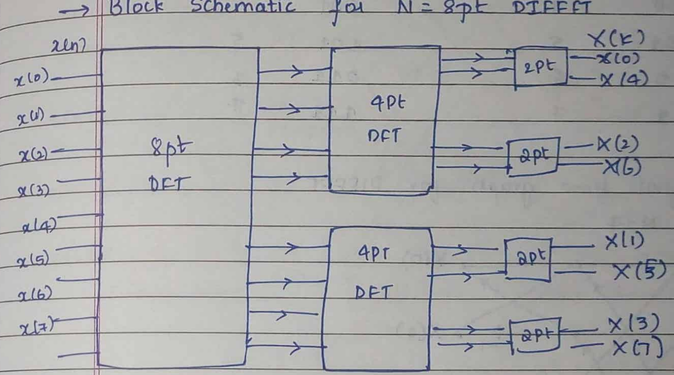
Generic pattern is



→ Block Schematic: for N = 4pt DIFFFT



→ Block Schematic for N = 8pt DIFFFT



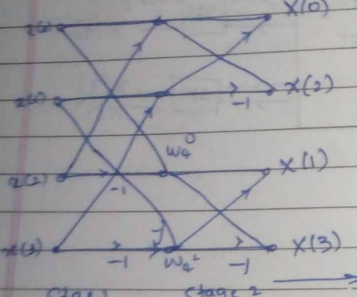
In place computat^n:

Bit Reversed pattern

Input $x(n)$ Binary	Input $x(n)$ decimal	Output $X(k)$ Binary	Output $X(k)$ decimal
000	0	000	0
001	1	100	4
010	2	010	2
011	3	110	6
100	4	001	1
101	5	101	5
110	6	011	3
111	7	111	7

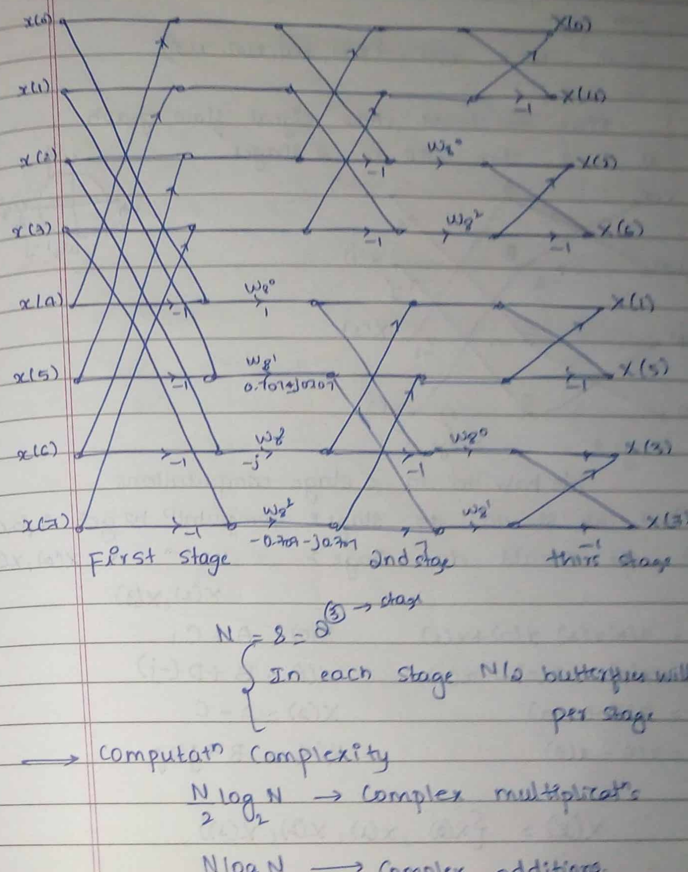
Signal flow graph for DIFFFT

N=4



$\rightarrow N=4=2^{(2)} \rightarrow$ stage

$\rightarrow$ for N = 8



$N=8=2^{(3)} \rightarrow$ stage

In each stage N/2 butterflies will be ... per stage

$\longrightarrow$ Computat^n Complexity

$\frac{N}{2}\log_2 N \rightarrow$ Complex multiplicat^{n}s

$N\log_2 N \longrightarrow$ Complex additions.

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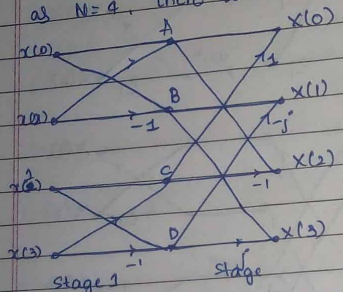
Four Point DIT FFT and DIF FFT Algorithms :-

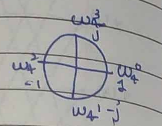
→ DIT FFT

→ Given, $N = 4$, $x(n) = \{x(0), x(1), x(2), x(3)\}$

To solve,

First we should draw signal flow graph as $N = 4$, there will be 2 stages.





We have to do 2 stage computations

First we should do stage 1 computat^n to get A, B, C, D & then we should do stage 2 " " " X(0), X(1), X(2), X(3)

$A = \cancel{x(0)x(2)}\ x(0) + x(2)$ $\qquad X(0) = A + C$

$B = \cancel{x(2)}\ x(0) - x(2)$ $\qquad X(1) = B + D(-j)$

$C = x(1) + x(3)$ $\qquad X(2) = A - C$

$D = x(1) - x(3)$ $\qquad X(3) = B + jD$ //

$X(k) = \{X(0), X(1), X(2), X(3)\}$

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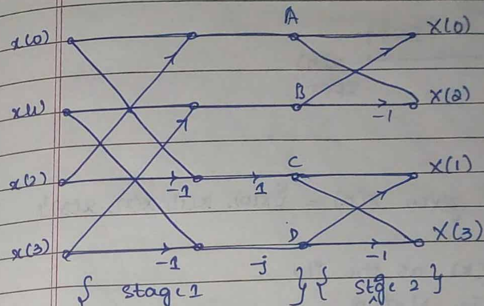
→ DIF FFT 4 pts



$A = x(2) + x(0)$ $\qquad X(0) = A + B$

$B = x(3) + x(1)$ $\qquad X(2) = A - B$

$C = x(0) - x(2)$ $\qquad X(1) = C + D$

$D = x(1) - x(3)(-j)$ $\qquad X(3) = C - D$

$X(K) = \{X(0), X(1), X(2), X(3)\}$ //

Four Point DIT IFFT and Four Point DIF IFFT ALGORithm :

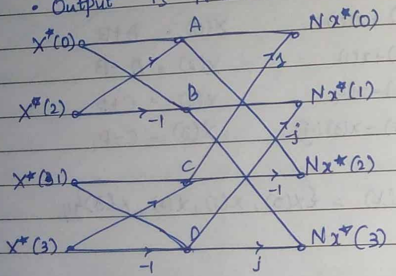
DIT
$X(K) \xrightarrow{IFFT} x(n)$
DIF IFFT

DIT IFFT :

u will be given $x(K) = \{x(0), x(1), x(2), x(3)\}$

procedure:

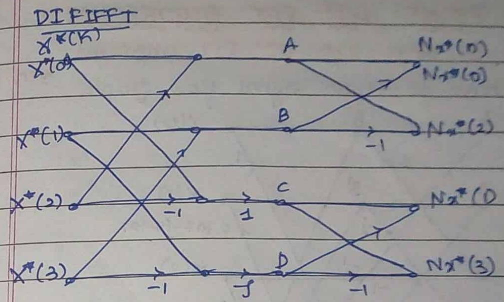
- Take $X^*(K)$ as the i/p
- Output is $N x^*(n)$



$A = X^*(0) + X^*(2)$
$B = X^*(0) - X^*(2)$
$C = X^*(1) + X^*(3)$
$D = X^*(1) - X^*(3)$

- Take o/p $x(n) \rightarrow$ complex conjugate & divide by N to get $x(n)$

DIF IFFT



A =
B =
C =
D =

procedure:

- Take $X^*(K)$ as the inputs
- Output is $N x^*(n)$
- Take o/p $x(n) \rightarrow$ complex conjugate and divide by N to get $x(n)$.

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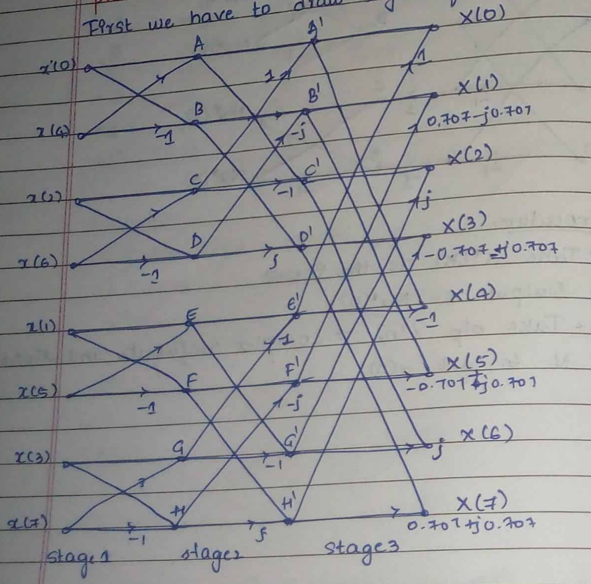
: 8 POINT DIT FFT ALGORITHM :

→ procedure to solve 8pt DIT FFT problems

First we have to draw signal flow graph   x(n) will be given



→ stage1 cal^n s: $A = x(0) + x(4)$ $F = x(1) - x(5)$

$B = x(0) - x(4)$ $G = x(3) + x(7)$

$C = x(2) + x(6)$ $H = x(3) - x(7)$

$D = x(2) - x(6)$

$E = x(1) + x(5)$

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→ stage 2 cal^n s:

$A' = A + C(1)$ $F' = F + H(-j)$

$B' = B + D(-j)$ $G' = E + G(-1)$

$C' = A + C(-1)$ $H' = F + H(j)$.

$D' = B + D(j)$

$E' = E + G$

→ Stage 3 cal^n s

$X(0) = A' + E'$ $X(4) = E' - A'$

$X(1) = B' + F'(0.707 - j0.707)$ $X(5) = F'(-0.707 + j0.707) + B'$

$X(2) = C' + G'(-j)$ $X(6) = G'(j) + C'$

$X(3) = D' + H'(-0.707 - j0.707)$ $X(7) = D' + H'(0.707 + j0.707)$

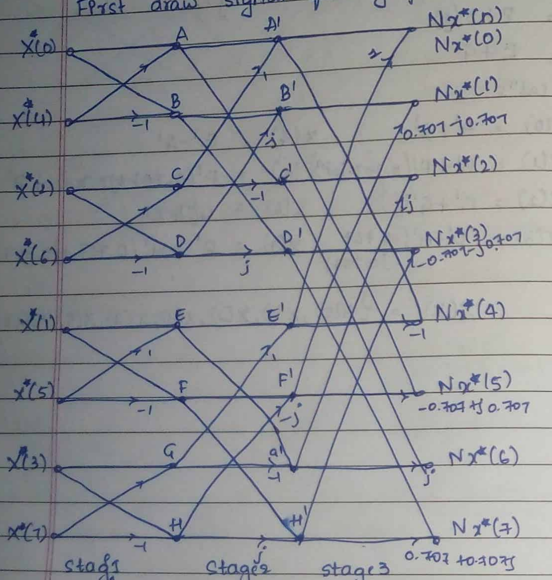
$X(K) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$

8 POINT DIT IFFT ALGORITHM:

→ procedure to solve 8 point DIT IFFT problems

X(K) will be given, we ~~wi~~ should find x(n).

First draw signal flow graph.



procedure

1) Take $X^*(K)$ as inputs

2) Find A, B, C, D, E, F, G, H

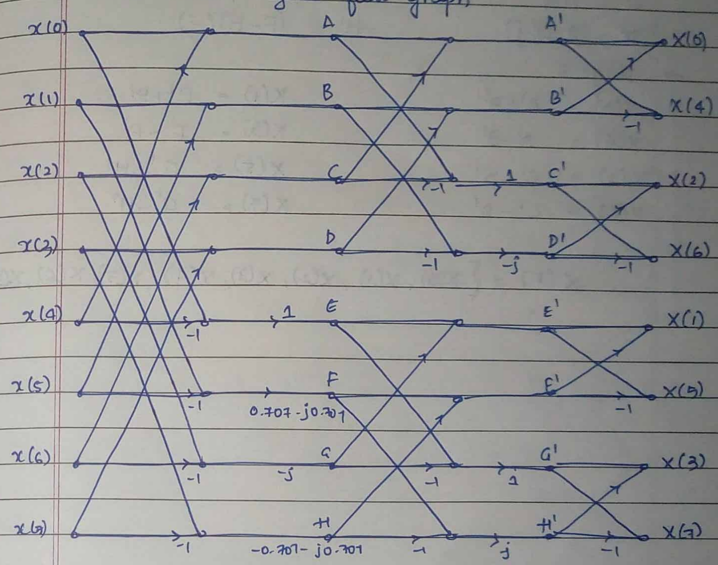
3) Find A', B' C' D' E' F' G' H'

4) Find o/ps $Nx^*(n)$ ⟶ Find x(n) by taking complex conjugate & divide by N (8).

8 POINT DIF FFT ALGORITHM:

→ Procedure to solve 8pt DIF FFT problems

First draw signal flow graph



→ Find A B C D E F G H

→ " A' B' C' D' E' F' G' H'

$A = x(0) + x(4)$

$B = x(1) + x(5)$

$C = x(2) + x(6)$

$D = x(3) + x(7)$

$E = [x(0) - x(4)]1$

$F = [x(1) - x(5)](0.707 - j0.707)$

$G = (x(2) - x(6))\,{-j}$

$H = [x(3) - x(7)][-0.707 - j0.707]$

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Stage 2

$A' = A+C$ $\qquad$ $E' = E+G$

$B' = B+D$ $\qquad$ $F' = F+H$

$C' = A-C$ $\qquad$ $G' = E-G$

$D' = (B-D)(-j)$ $\qquad$ $H' = (F-H)(-j)$

→ Stage 3

$X(0) = A'+B'$ $\qquad$ $X(1) = E'+F'$

$X(4) = A'-B'$ $\qquad$ $X(3) = E'-F'$

$X(2) = C'+D'$ $\qquad$ $X(5) = G'+H'$

$X(6) = C'-D'$ $\qquad$ $X(7) = G'-H'$

$X(K) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$

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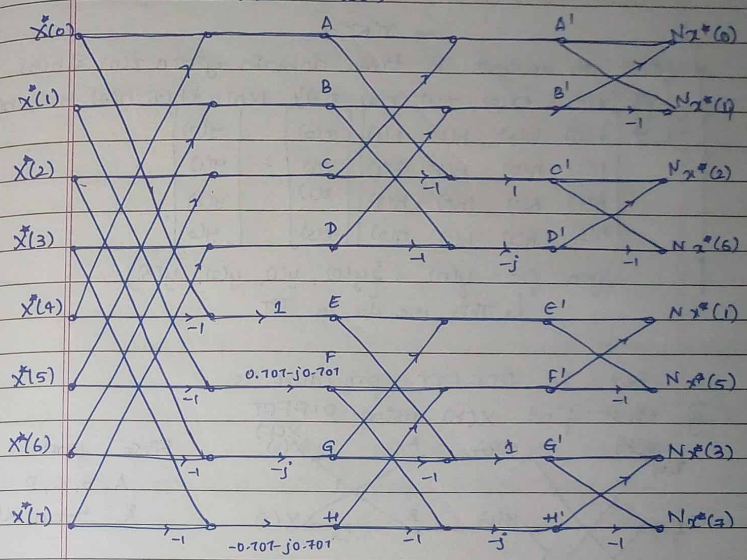
8 POINT DIF IFFT ALGORITHM:

→ Procedure to solve 8 point DIF IFFT problems:

$X(K) \rightarrow x(n)$



→ Take $X^*(K)$ as i/p

→ Find A, B, C, D, E, F, G, H & A' B' C' D' E' F' G' H'

→ Find o/p $Nx^*(n)$ ⟶ Find o/p $x(n)$ by find complex conjugating and dividing by N (=8)

$x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$.

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CIRCULAR CONVOLUTION USING FFT & IFFT

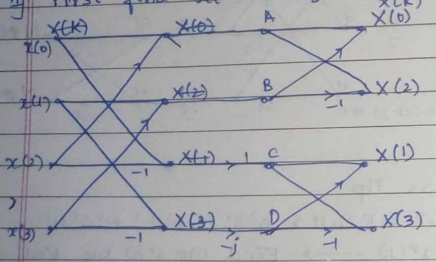
→ Circular convolution using DIF FFT
→ $Y(K)$

* First do analysis in time domain given $x(n)$ & $h(n)$
If $x(n) = \{x(0), x(1), x(2), x(3)\}$ $h(n) = \{h(0), h(1), h(2), h(3)\}$

$$\begin{bmatrix} h(0) & h(3) & h(2) & h(1) \\ h(1) & h(0) & h(3) & h(2) \\ h(2) & h(1) & h(0) & h(3) \\ h(3) & h(2) & h(1) & h(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix}$$

you gets $y(n) = \{y(0), y(1), y(2), y(3)\}$
↳ This we do in FT

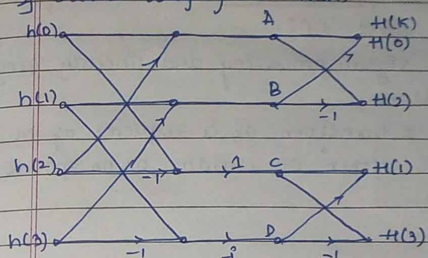
But in DIFFFT procedure is
1] First find $X(K)$ using DIFFFT



First find A, B, C, D & then $X(K)$

$X(K) = \{X(0), X(1), X(2), X(3)\}$

2] Same way find $H(K)$



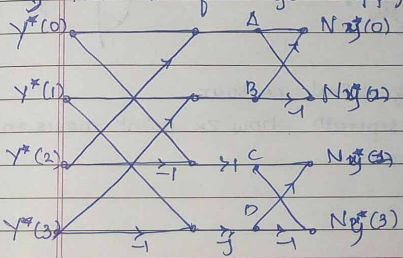
Find A, B, C, D and then $H(K)$

$H(K) = \{H(0), H(1), H(2), H(3)\}$

3] Find $Y(K) = X(K)\,H(K)$

$$Y(K) = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} \begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \end{bmatrix}$$ → not matrix mul, do element by element multi-plicatⁿ

4] Now to find $y(n)$ apply DIF IFFT to $Y(K)$
↳ by taking complex conj of $Y(K)$



here we get $Ny^*(n)$
To get $y(n)$ take complex conjugate and divide by $N(4)$

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CHIRP - Z TRANSFORM (CZT Algorithm transform)

↳ to find FFT

→ Used is radar signal processing and linearly varying frequency applications.

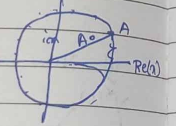
→ For evaluating z transform of a sequence of M pts in zplane w lie either on circular or on spiral contour in zplane

$$X(Z_K) = \sum_{n=0}^{N-1} x(n) Z_K^{-n} \qquad K = 0, 1, 2 \ldots M-1$$

If this reduces to DFT

$$Z_K = e^{j2\pi k/N} = W_N^{k} \quad , K = 0, 1, \ldots N-1$$

Let Z_K be a point of on the spiral contour centered about the origin



→ $Z_K = AB^{-K}$

$A \to$ pt @ w spiral starts

$A = A_0 e^{j\theta_0}$

$B = B_0 e^{-j\phi_0}$

$B \to$ rate of spiral contour

↳ angular seperatn, how Z_K point moves on spiral

Case 1: $B_0 < 1$

$\{Z_K\} \to$ It spirals towards the origin

Case 2: $B_0 > 1$

$\{Z_K\} \to$ spirals away from the origin

Case 3: $|B_0| = 1$

$\{Z_K\} \to$ will be on circle A_0

* Special case

If $A_0 = 1$ and $B = e^{-j2\pi/N}$

→ $M = N$

→ It corresponds to DFT & countour is a unit c

$$\rightarrow X(Z_K) = \sum_{n=0}^{N-1} x(n) A^{-n} B^{Kn} \quad , 0 \le K \le N-1$$

W.K.T $nK = \frac{1}{2}[n^2 + K^2 - (K-n)^2]$

$$\therefore X(Z_K) = \sum_{n=0}^{N-1} x(n) A^{-n} B^{\frac{n^2}{2}} B^{K^2/2} B^{-\frac{(K-n)^2}{2}}$$

Let us consider 2 functn s $g(n) = x(n) A^{-n} B^{n^2/2}$

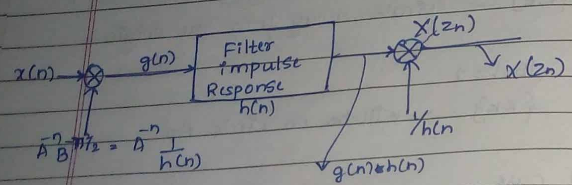
and $h(n) = B^{K^2/2} B^{-n^2/2}$

$$\therefore X(Z_K) = \sum_{n=0}^{N-1} g(n) \frac{1}{h(K)} (h(K-n)) \qquad 0 \le n \le N-1$$

Now consider

$$X(Z) = \frac{1}{h(n)}\left[g(n) * h(n)\right]$$

↳ This is computation of chip z ~~transfor~~ algorithm



↓

Generating Chirp Z Transform

$$h(n) = \left[e^{-j\phi_0}\right]^{\frac{n^2}{2}} = e^{\frac{j n^2 \phi_0}{2}} = e^{j\omega n} \qquad \omega = \frac{n\phi_0}{2}$$

↳ ex) complex exponential sequence Linearly increasing frequency is called "chirp".

GOERTZEL ALGORITHM :- To find DFT

→ This algorithm is linear filtering technique

→ Efficient tool to compute DFT when small number of frequency points are there.

→ This algorithm makes use of periodicity property of W_N^{-kN}

→ Let us take, given a signal $x(m)$

$$X(k) = \sum_{m=0}^{N-1} x(m)\, W_N^{km}\, W_N^{-kN}$$

$$X(k) = \sum_{m=0}^{N-1} x(m)\, W_N^{-k(N-m)}$$

$$X(k) = \sum_{m=-\infty}^{\infty} x(m) W_N^{-k(N-m)}$$

Let $y_k(n) = \sum_{m=-\infty}^{\infty} x(m) W_N^{-(n-m)k}$

$$X(k) = y_k(n)\Big|_{n=N}$$

$$y_k(n) = x(n) * W_N^{-kn}$$

Transfer func of Goertzel filer is

$$H_k(z) = \sum_{n=0}^{\infty} h_k(n) z^{-n} = \sum_{n=0}^{\infty} (W_N^{-k} z^{-1})^n$$

WKT $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a|<1$

$\therefore H_k(z) = \frac{1}{1 - W_N^{-k} z^{-1}}$

$\frac{Y_k(z)}{X(z)} = \frac{1}{1 - W_N^{-k} z^{-1}}$

cross multiply and take inverse z transform

$y_k(n) = W_N^{-k} y_k(n-1) = x(n)$

Take $H_k(z)$ & Multiply and divide by $1 - W_N^{-k} z^{-1}$

$H_k(z) = \frac{1 - W_N^{-k} z^{-1}}{(1 - W_N^{-k} z^{-1})(1 - W_N^{k} z^{-1})}$

$H_k(z) = \frac{1 - W_N^{-k} z^{-1}}{1 - W_N^{k} z^{-1} - W_N^{-k} z^{-1} + z^{-2}}$

$H_k(z) = \frac{1 - W_N^{-k} z^{-1}}{1 - e^{\frac{j2\pi k}{N}} z^{-1} - e^{\frac{-j2\pi k}{N}} z^{-1} + z^{-2}}$

$H_k(z) = \frac{1 - W_N^{-k} z^{-1}}{1 - 2\cos\left(\frac{2\pi k}{N}\right) z^{-1} + z^{-2}}$

$= H_1(z) H_2(z)$

Let $V_k(z) = X(z) H_1(z)$

$= \frac{1 \cdot X(z)}{1 - 2\cos\left(\frac{2\pi k}{N}\right) z^{-1} + z^{-2}}$

cross mul & then take invert z transform

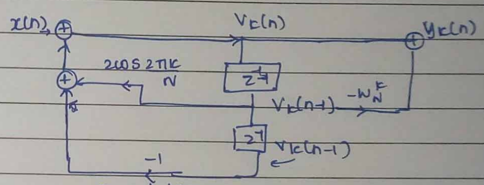
$V_k(n) = 2\cos\left(\frac{2\pi k}{N}\right) V_k(n-1) - V_k(n-2) + x(n)$

$Y_k(z) = H_2(z) V_k(z)$

$Y_k(z) = (1 - W_N^{-k} z^{-1}) V_k(z)$

$y_k(n) = V_k(n) - W_N^{k} V_k(n-1)$

$V_k(-1), V_k(-2) \Rightarrow$ zero



Direct form I Gorzetz filter.

→ Its an very efficient algorithm as Each iteratⁿ will have only one real multiplicatⁿ & two addⁿs.

$x(n) \Rightarrow$ real

N+1 real multiplicatⁿs only