Unit - 3

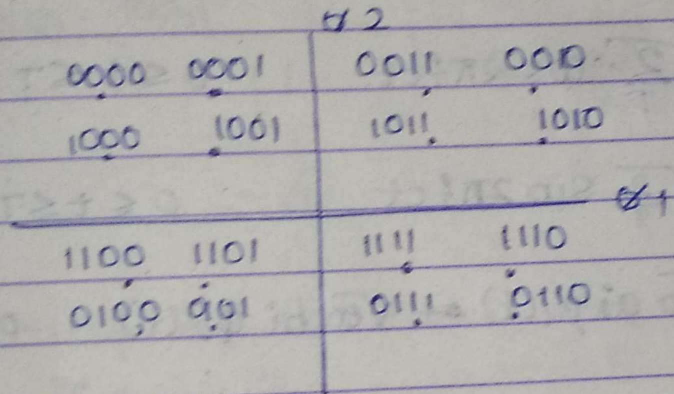
Chapter - 7 : Quadrature Digital Modulation techniques

* MQAM modulation / demodulation
* comparision of Binary and quaternary Modulation tech
* QPSK. modulation / demodulation

* M-QAM Modulation/Demodulation

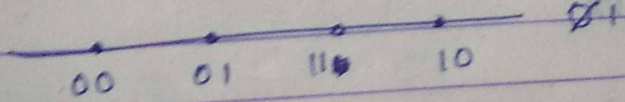
→ inphase and quadrature phase components are independent in M-QAM

→ amplitude and phase both changes in ~~QAM~~ carrier.

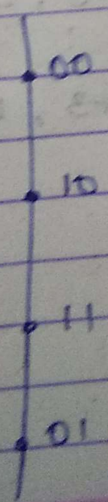


Constellation diagram for M=16

Msg points indicated as 4 bit gray codes.



inphase components



quadrature phase component

fig b and c ~~tracing~~ L-ary ASK with $L=4$
Thus M-QAM enables transmission of $M = L^2$ independent symbols over the channel

$$s_i(t) = \sqrt{\frac{2E_0}{T}}\, a_i \cos(2\pi f_c t) + \sqrt{\frac{2E_0}{T}}\, b_i \sin(2\pi f_c t) \quad \text{—①}$$

$$0 \le t \le T$$

$E_0 \rightarrow$ energy of the signal

In terms of basis function

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos 2\pi f_c t \qquad 0 \le t \le T$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin 2\pi f_c t \qquad 0 \le t \le T$$

$$s_i(t) = \sqrt{E_0}\, a_i\, \phi_1(t) + \sqrt{E_0}\, b_i\, \phi_2(t) \qquad 0 \le t \le T$$

The co-ordinates of the ith msg points are $a_i\sqrt{E_0}$ and $b_i\sqrt{E}$ where (a_i, b_i) are elements of $L \times L$ matrix

$$\{a_i, b_i\} = \begin{bmatrix} -L+1, L-1 & -L+3, L-1 & \cdots & L-1, L-1 \\ -L+1, L-3 & -L+3, L-3 & \cdots & L-1, L-3 \\ \vdots & \vdots & & \vdots \\ -L+1, L+1 & -L+3, L+1 & \cdots & L-1, L+1 \end{bmatrix}$$

$$\therefore L = \sqrt{M}$$

Probability of M-QAM

→ in phase and quadrature phase are independent ∴ the probability of correct detection of such scheme is

$$P_c = (1 - P_e')^2 \quad \text{--- (1)}$$

P_e' → probability of symbol error for either component.

→ in phase and quadrature phase components resembles discrete PAM

$$\therefore \quad P_e' = \left(1 - \frac{1}{L}\right) \operatorname{erfc}\left(\sqrt{\frac{E_0}{N_0}}\right) \quad \text{--- (2)}$$

$$L = \sqrt{M} \quad \text{--- (3)}$$

→ probability symbol error of QAM is given by

$$P_e = 1 - P_c \quad \text{--- (4)}$$

$$= 1 - (1 - P_e')^2$$

$$= \cancel{1 - 1 - 2P_e' + P_e'^2}$$

$$\simeq 2P_e' \quad \text{--- (5)}$$

∴ by (3) and (2)
eq (4) becomes

$$P_e \cong 2\left(1 - \frac{1}{\sqrt{M}}\right) \operatorname{erfc}\sqrt{\frac{E_0}{N_0}} \quad \text{--- (6)}$$

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assuming L amplitude levels of the inphase and quadrature phase are equal and expressing Pe in term of average value we have

$$E_{av} = 2\left[\frac{2E_0}{L}\sum_{i=1}^{L/2}(2i-1)^2\right] \quad \text{—— (7)}$$

accounts bcoz of equal contribution made by inphase & quadrature phase component

summing equation (7) we obtain

$$E_{av} = \frac{2(L^2-1)E_0}{3}$$

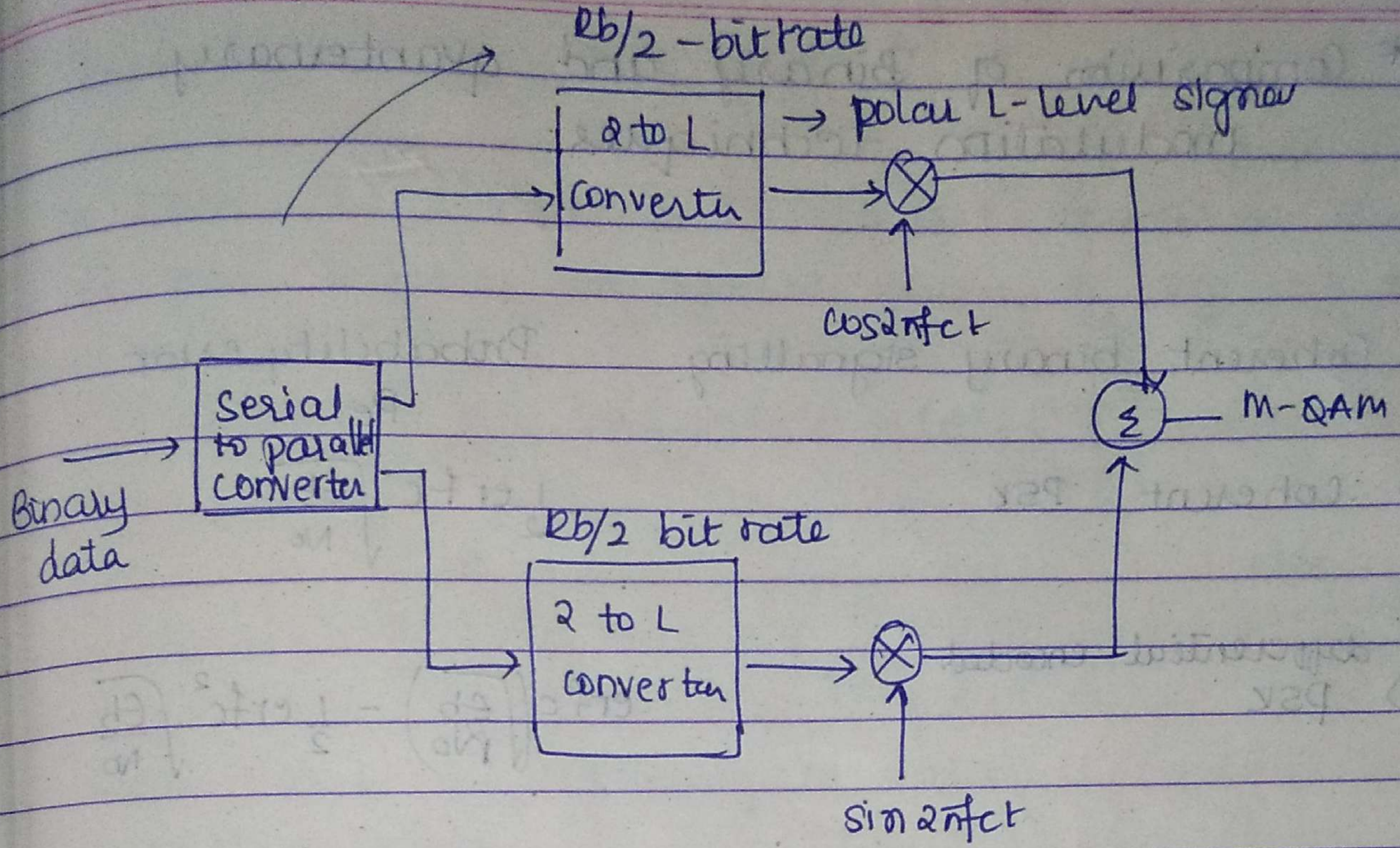
$$= \frac{2(M-1)E_0}{3}$$

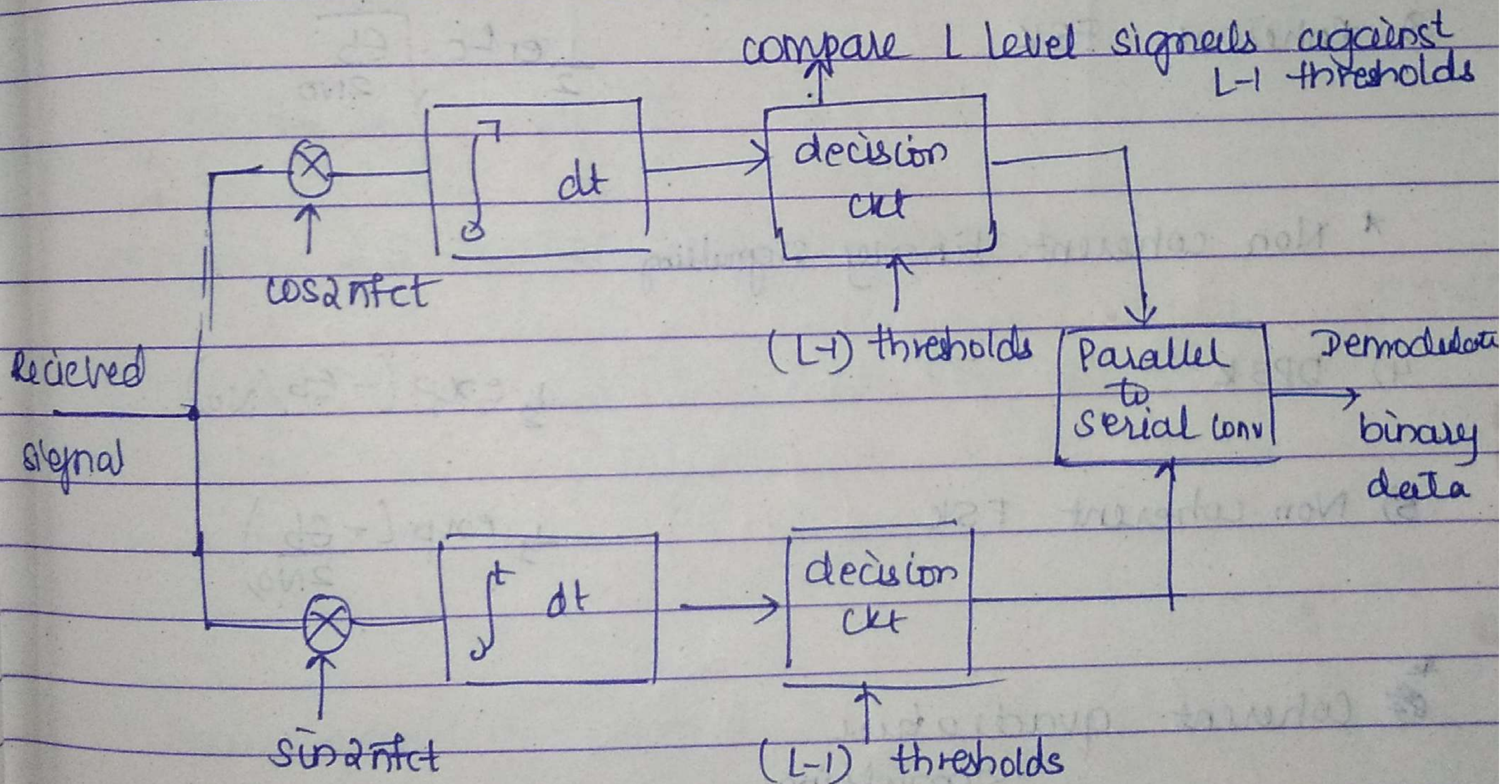
$L = \sqrt{m}$

$m^2 = L$

accordingly eqn (6) can be written as

$$P_e \simeq 2\left(1-\frac{1}{\sqrt{m}}\right)\operatorname{erfc}\left(\frac{3E_{av}}{2(M-1)N_0}\right)$$


Rb/2 - bit rate
2 to L converter
polar L-level signal
cos2πfct
Serial to parallel converter
Binary data
M-QAM
Rb/2 bit rate
2 to L converter
sin2πfct


compare L level signals against L-1 thresholds
decision ckt
dt
cos2πfct
(L-1) thresholds
Parallel to serial conv
Demodulated binary data
Recieved signal
dt
decision ckt
sin2πfct
(L-1) thresholds

the corresponding M-ary PSK system provided that the transmitted energy per symbol is increased by the following factor:

$$k(M) = \frac{\sin^2\left(\frac{\pi}{M}\right)}{2\sin^2\left(\frac{\pi}{2M}\right)} \qquad M \geq 4 \tag{7.112}$$

For example, $k(4) = 1.7$. That is, differential QPSK (which is noncoherent) is approximately 2.3 dB poorer in performance than coherent QPSK.

(2) M-ary QAM

In an M-ary PSK system, in-phase and quadrature components of the modulated signal are interrelated in such a way that the envelope is constrained to remain constant. This constraint manifests itself in a circular constellation for the message points. However, if this constraint is removed, and the in-phase and quadrature components are thereby permitted to be independent, we get a new modulation scheme called *M-ary quadrature amplitude modulation* (QAM). In this modulation scheme, the carrier experiences amplitude as well as phase modulation.

The signal constellation for M-ary QAM consists of a *square lattice* of message points, as illustrated in Fig. 7.24 for $M = 16$. The corresponding signal constellations for the in-phase and quadrature components of the amplitude-phase modulated wave are shown in Figs. 7.25*a* and 7.25*b*, respectively. The basic format of the signal constellations shown in the latter figures is recognized to be that of a *polar L-ary* ASK *signal* with $L = 4$. Thus, in general, an M-ary QAM scheme enables the transmission of $M = L^2$ independent symbols over the same channel bandwidth as that required for one polar L-ary ASK scheme.

The general form of M-ary QAM is defined by the transmitted signal

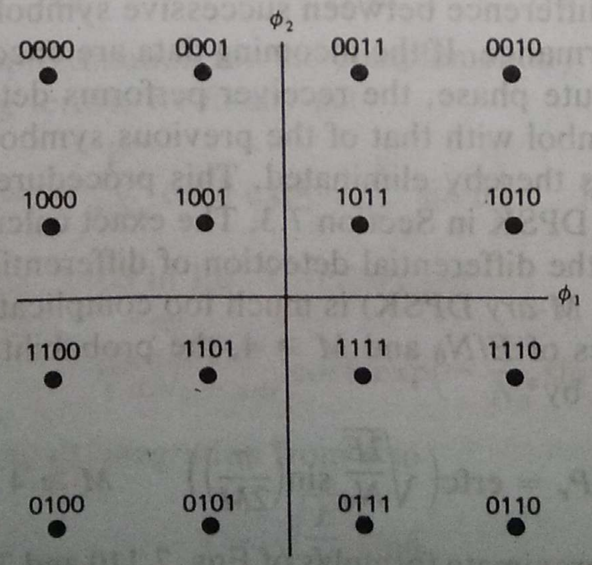
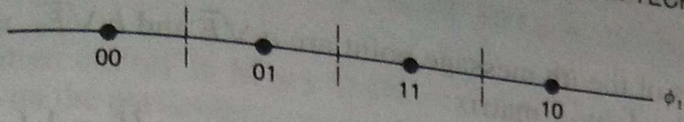
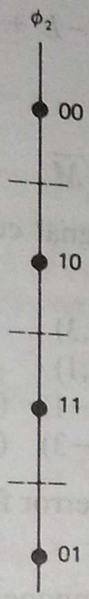


Figure 7.24 Signal-constellation of M-ary QAM for $M = 16$. (The message points are identified with 4-bit Gray codes for later discussion.)



(a)



(b)

Figure 7.25 Decomposition of signal constellation of M-ary QAM (for $M = 16$) into two signal-space diagrams for (a) in-phase comment $\phi_1(t)$, and (b) quadrature comment $\phi_2(t)$. (The message points are identified by 2-bit Gray codes for later discussion.)

$$s_i(t) = \sqrt{\frac{2E_0}{T}}\, a_i \cos(2\pi f_c t) + \sqrt{\frac{2E_0}{T}}\, b_i \sin(2\pi f_c t) \qquad 0 \le t \le T \tag{7.113}$$

where E_0 is the energy of the signal with the lowest amplitude, and a_i and b_i are a pair of independent integers chosen in accordance with the location of the pertinent message point. The signal $s_i(t)$ consists of two phase-quadrature carriers, each of which is modulated by a set of discrete amplitudes; hence, the name "quadrature amplitude modulation."

The signal $s_i(t)$ can be expanded in terms of a pair of basis functions:

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \qquad 0 \le t \le T \tag{7.114}$$

and

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t) \qquad 0 \le t \le T \tag{7.115}$$

The coordinates of the ith message point are $a_i\sqrt{E_0}$ and $b_i\sqrt{E_0}$, where (a_i, b_i) is an element of the L-by-L matrix:

$$\{a_i, b_i\} = \begin{bmatrix} (-L+1, L-1) & (-L+3, L-1) & \cdots & (L-1, L-1) \\ (-L+1, L-3) & (-L+3, L-3) & \cdots & (L-1, L-3) \\ \vdots & \vdots & & \vdots \\ (-L+1, -L+1) & (-L+3, -L+1) & \cdots & (L-1, -L+1) \end{bmatrix} \tag{7.116}$$

where

$$L = \sqrt{M} \tag{7.117}$$

For example, for the 16-QAM whose signal constellation is depicted in Fig. 7.24, where $L = 4$, we have the matrix

$$\{a_i, b_i\} = \begin{bmatrix} (-3,3) & (-1,3) & (1,3) & (3,3) \\ (-3,1) & (-1,1) & (1,1) & (3,1) \\ (-3,-1) & (-1,-1) & (1,-1) & (3,-1) \\ (-3,-3) & (-1,-3) & (1,-3) & (3,-3) \end{bmatrix} \tag{7.118}$$

To calculate the probability of symbol error for M-ary QAM, we proceed as follows:

1. Since the in-phase and quadrature components of M-ary QAM are independent, the probability of correct detection for such a scheme may be written as

$$P_c = (1 - P_e')^2 \tag{7.119}$$

   where P_e' is the probability of symbol error for either component.

2. The signal constellation for the in-phase or quadrature component has a geometry similar to that for discrete pulse-amplitude modulation (PAM) with a corresponding number of amplitude levels. We may therefore adapt the formula of Eq. 3.75 to fit the terminology of the signal constellations shown in Fig. 7.25, and so we write

$$P_e' = \left(1 - \frac{1}{L}\right) \text{erfc}\left(\sqrt{\frac{E_0}{N_0}}\right) \tag{7.120}$$

   where L is the square root of M.

3. The probability of symbol error for M-ary QAM is given by

$$\begin{aligned} P_e &= 1 - P_c \\ &= 1 - (1 - P_e')^2 \\ &\simeq 2P_e' \end{aligned} \tag{7.121}$$

   where it is assumed that P_e' is small compared to unity. Hence, using Eqs. 7.117 and 7.120 in Eq. 7.121, we find that the probability of symbol error for M-ary QAM is (for all practical purposes) given by

$$P_e \simeq 2\left(1 - \frac{1}{\sqrt{M}}\right) \text{erfc}\left(\sqrt{\frac{E_0}{N_0}}\right) \tag{7.122}$$

The transmitted energy in M-ary QAM is variable in that its instantaneous value depends on the particular symbol transmitted. It is logical to express P_e in terms of the *average* value of the transmitted energy rather than E_0. Assuming that the L amplitude levels of the in-phase or quadrature component are equally likely, we have

$$E_{av} = 2\left[\frac{2E_0}{L}\sum_{i=1}^{L/2}(2i-1)^2\right] \tag{7.123}$$

where the multiplying factor of 2 accounts for the equal contributions made by the in-phase and quadrature components. The limits of the summation take account of the symmetric nature of the pertinent amplitude levels around zero. Summing the series in Eq. 7.123, we get

$$\begin{aligned} E_{av} &= \frac{2(L^2-1)E_0}{3} \\ &= \frac{2(M-1)E_0}{3} \end{aligned} \tag{7.124}$$

Accordingly, we may rewrite Eq. 7.122 in terms of E_{av} as

$$P_e \simeq 2\left(1-\frac{1}{\sqrt{M}}\right)\operatorname{erfc}\left(\sqrt{\frac{3E_{av}}{2(M-1)N_0}}\right) \tag{7.125}$$

which is the desired result.

The case of $M = 4$ is of special interest. The signal constellation for this value of M is shown in Fig. 7.26, which is recognized to be the same as that for QPSK. Indeed, putting $M = 4$ in Eq. 7.27 and noting from Fig. 7.26 that E_{av} equals E, where E is the energy per symbol, we find that the resulting formula for the probability of symbol error becomes identical to that in Eq. 7.46, and so it should.

Figure 7.27*a* shows the block diagram of an M-ary QAM transmitter. The

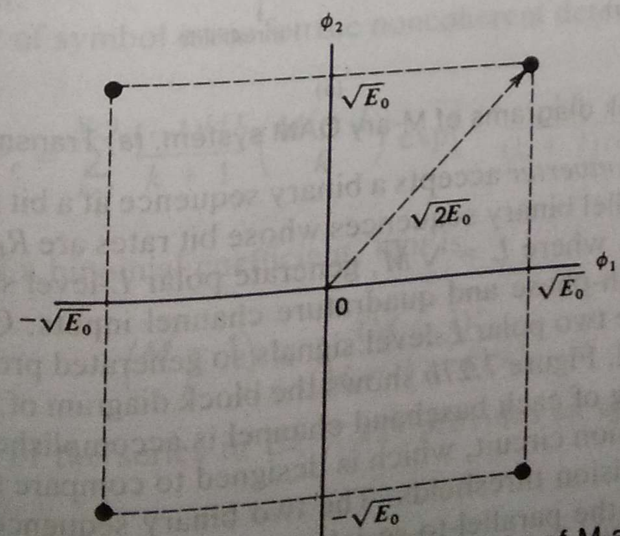


Figure 7.26 Signal constellation for the special case of M-ary QAM for $M = 4$.

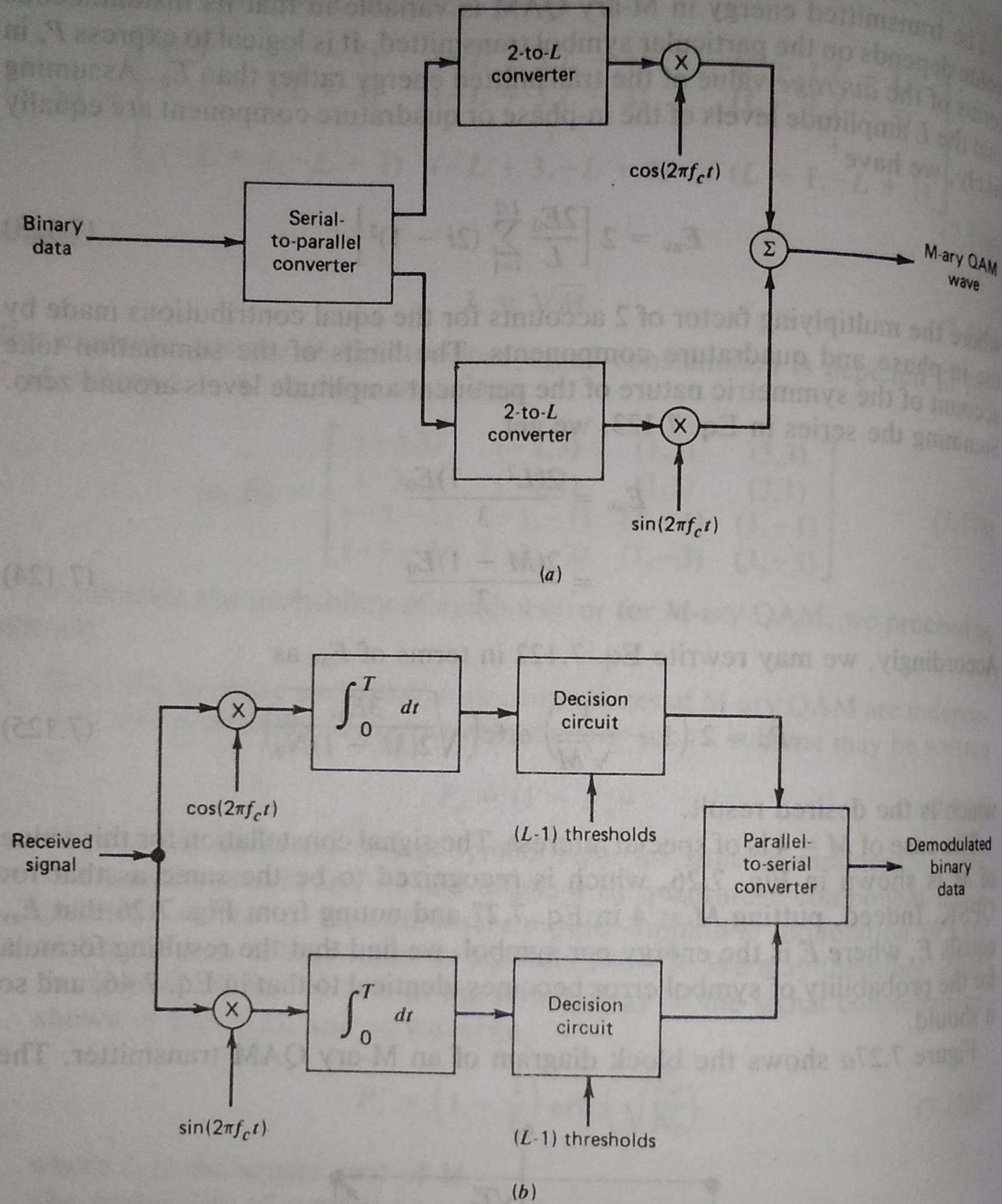


Figure 7.27 Block diagrams of M-ary QAM system. (*a*) Transmitter. (*b*) Receiver.

serial-to-parallel converter accepts a binary sequence at a bit rate $R_b = 1/T_b$ and produces two parallel binary sequences whose bit rates are $R_b/2$ each. The *2-to-L level converters*, where $L = \sqrt{M}$, generate polar L-level signals in response to the respective in-phase and quadrature channel inputs. Quadrature-carrier multiplexing of the two polar L-level signals so generated produces the desired M-ary QAM signal. Figure 7.27*b* shows the block diagram of the corresponding receiver. Decoding of each baseband channel is accomplished at the output of the pertinent decision circuit, which is designed to compare the L-level signals against $L - 1$ decision thresholds. The two binary sequences so detected are then combined in the parallel-to-serial converter to reproduce the original binary sequence.

difference between the ... intervals lies inside the range $-\pi/2$ to $\pi/2$, ... the phase symbol 1. If, on the other hand, the correlator output is negative, the phase difference lies outside the range $-\pi/2$ to $\pi/2$, modulo-2π, and the receiver decides in favor of symbol 0.

7.5 COMPARISON OF BINARY AND QUATERNARY MODULATION TECHNIQUES

Two systems having an unequal number of symbols may be compared in a meaningful way only if they use the same amount of energy to transmit each bit of information. It is the total amount of energy needed to transmit the complete message that represents the cost of the transmission, not the amount of energy needed to transmit a particular symbol satisfactorily. Accordingly, in comparing the different data transmission systems considered before, we will use, as the basis of our comparison, the average probability of symbol error expressed as a function of the bit energy-to-noise density ratio E_b/N_0.

In Table 7.4,* we have summarized the expressions for the average probability of symbol error P_e for the coherent PSK, conventional coherent FSK with one-bit decoding, DPSK, noncoherent FSK, QPSK, and MSK, when operating over an AWGN channel. In Fig. 7.20† we have used these expressions to plot P_e as a function of E_b/N_0.

Based on the performance curves shown in Fig. 7.20, the summary of formulas given in Table 7.4, and the defining equations for the pertinent modulation formats, we can make the following statements.

1. The error rates for all the systems decrease monotonically with increasing values of E_b/N_0.

* In Table 7.4, we have also included the expression for the average probability of symbol error for the *coherent detection of differentially encoded binary PSK*. Differential encoding is used to resolve the phase ambiguity problem that arises in synchronizing PSK receivers. This issue is discussed in Section 7.12.

† The average probability of error for the coherent detection of differentially encoded binary PSK is practically the same as that of QPSK and MSK; hence, the use of the same curve for these three modulation formats in Fig. 7.20.

Table 7.4 Summary of Formulas for the Symbol Error Probability for Different Data Transmission Systems

	Error probability, P_e
Coherent binary signaling:	
(a) Coherent PSK	$\frac{1}{2}\,\mathrm{erfc}(\sqrt{E_b/N_0})$
(b) Coherent detection of differentially encoded PSK	$\mathrm{erfc}(\sqrt{E_b/N_0}) - \frac{1}{2}\,\mathrm{erfc}^2(\sqrt{E_b/N_0})$
(c) Coherent FSK	$\frac{1}{2}\,\mathrm{erfc}(\sqrt{E_b/2N_0})$
Noncoherent binary signaling:	
(a) DPSK	$\frac{1}{2}\exp(-E_b/N_0)$
(b) Noncoherent FSK	$\frac{1}{2}\exp(-E_b/2N_0)$
Coherent quadrature signaling:	
(a) QPSK, (b) MSK	$\mathrm{erfc}(\sqrt{E_b/N_0}) - \frac{1}{4}\,\mathrm{erfc}^2(\sqrt{E_b/N_0})$

2. For any value of E_b/N_0, coherent PSK produces a smaller error rate than any of the other systems. Indeed, it may be shown that in the case of systems restricted to one-bit decoding, perturbed by additive white Gaussian noise, coherent PSK system is the optimum system for transmitting binary data in the sense that it achieves the minimum probability of symbol error for a given value of E_b/N_0.*
3. Coherent PSK and DPSK require an E_b/N_0 that is 3 dB less than the corresponding values for conventional coherent FSK and noncoherent FSK, respectively, to realize the same error rate.
4. At high values of E_b/N_0, DPSK and noncoherent FSK perform almost as well (to within about 1 dB) as coherent PSK and conventional coherent FSK, respectively, for the same bit rate and signal energy per bit.
5. In QPSK two orthogonal carriers $\sqrt{2/T}\cos(2\pi f_c t)$ and $\sqrt{2/T}\sin(2\pi f_c t)$ are used, where the carrier frequency f_c is an integral multiple of the symbol rate $1/T$, with the result that two independent bit streams can be transmitted and subsequently detected in the receiver. At high values of E_b/N_0, coherently detected binary PSK and QPSK have about the same error rate performance for the same value of E_b/N_0.
6. In MSK the two orthogonal carriers $\sqrt{2/T_b}\cos(2\pi f_c t)$ and $\sqrt{2/T_b}\sin(2\pi f_c t)$ are modulated by the two antipodal symbol shaping pulses $\cos(\pi t/2\,T_b)$ and $\sin(\pi t/2\,T_b)$, respectively, over $2T_b$ intervals, where T_b is the bit duration. Correspondingly, the receiver uses a coherent phase decoding process over two successive bit intervals to recover the original bit stream. We thus find that MSK has exactly the same error rate performance as QPSK.

* See Stein (1964).

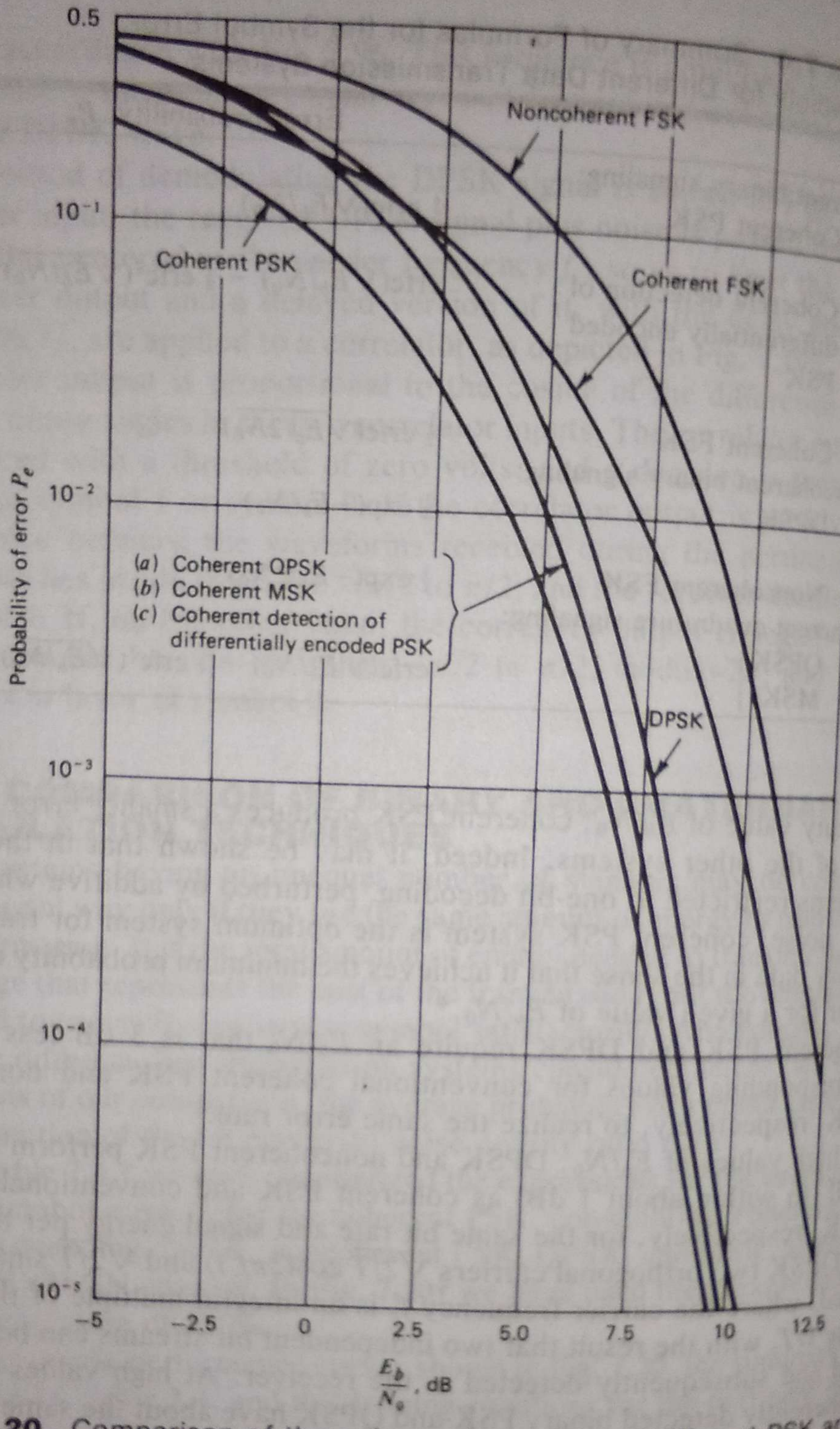


Figure 7.20 Comparison of the noise performances of different PSK and FSK schemes.

7. MSK scheme differs from the other signaling schemes in that its receiver has *memory*. In particular, MSK receiver makes decisions based on observations over two successive bit intervals. Thus, although the transmitted signal has a binary format represented by the transmission of two distinct frequencies, the presence of memory in the receiver makes it assume a form that has in-phase and quadrature paths as in QPSK.

* Quadrature phase shift keying.

In QPSK the phase of the carrier takes on one of four equally ~~ph~~ spaced value $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ & $\frac{7\pi}{4}$ and is given by

$$s_i(t) = \begin{cases} \sqrt{\frac{2E_b}{T}} \cos\left[2\pi f_c t + (2i-1)\frac{\pi}{4}\right] & 0 \le t \le T \\ 0 & \text{else} \end{cases} \quad \text{—} ①$$

$i = 1, 2, 3, 4$

each value corresponding to value is unique and called dbits

rewriting eqn ①

$$s_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos(2i-1)\frac{\pi}{4} \cos(2\pi f_c t) - \sqrt{\frac{2E}{T}} \sin(2i-1)\frac{\pi}{4} \sin(2\pi f_c t) & 0 \le t \le T \\ 0 & \text{else} \end{cases} \quad \text{—} ②$$

basis function $\phi_1(t)$ and $\phi_2(t)$

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos 2\pi f_c t \qquad 0 \le t \le T \quad \text{—} ③$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin 2\pi f_c t \qquad 0 \le t \le T \quad \text{—} ④$$

$\therefore$ eqn ② becomes :

$$S_i = \begin{cases} \sqrt{E}\cos(2i-1)\frac{\pi}{4}\,\phi_1(t) - \sqrt{E}\sin(2i-1)\frac{\pi}{4}\,\phi_2(t) & 0 \le t \le T \\ 0 & \text{else} \end{cases}$$

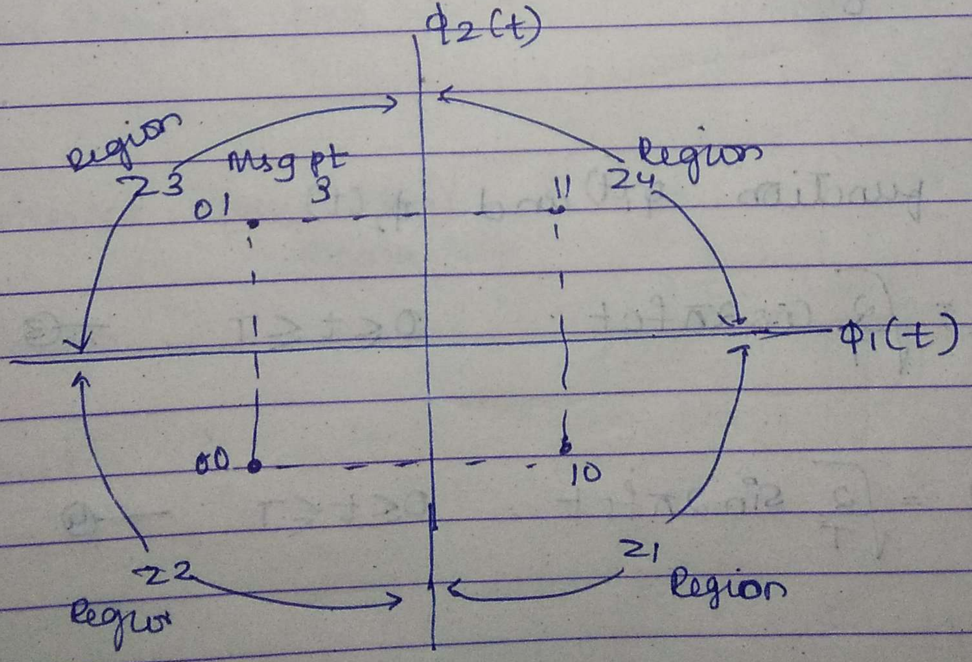
$\rightarrow$ $\therefore$ 4 msg points and associated signal vectors are given by

$$S_i = \begin{bmatrix} \sqrt{E}\cos(2i-1)\frac{\pi}{4} \\ \sqrt{E}\sin(2i-1)\frac{\pi}{4} \end{bmatrix} \quad i = 1,2,3,4$$

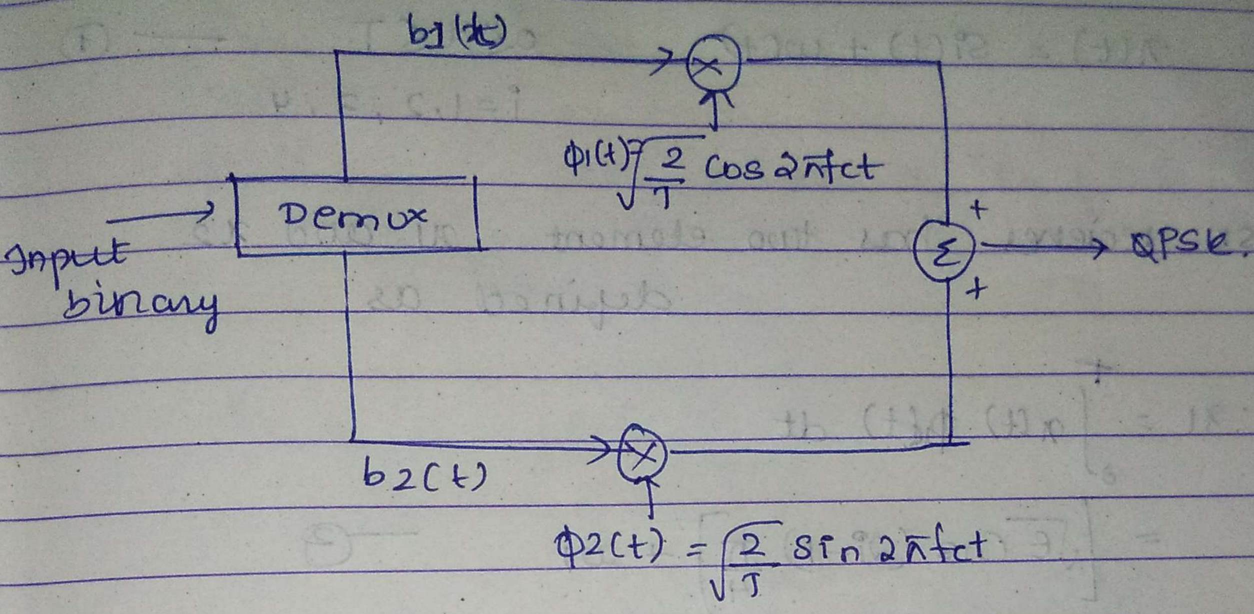
Constellation diagram

$N = 2 \rightarrow$ two dimensional

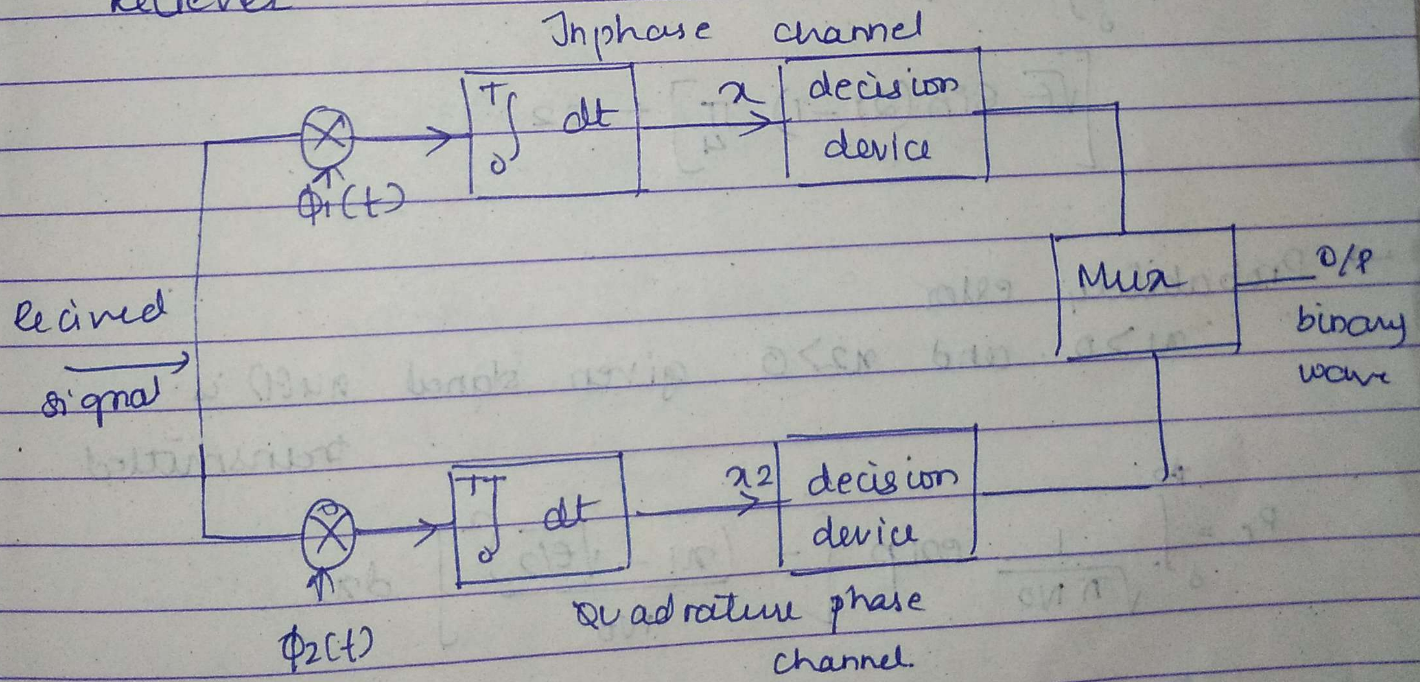
$M = 4 \rightarrow$ msg points



Transmitter



Reciever



The recieved signal $x(t)$ is given by

$$x(t) = s_i(t) + w(t) \qquad 0 \le t \le T \qquad \text{—— (1)}$$
$$i = 1, 2, 3, 4$$

QPSK recievers has two element x_1 and x_2 defined as

$$x_1 = \int_0^T x(t)\,\phi_1(t)\,dt$$

$$= \left[\sqrt{E}\cos\left((2i-1)\frac{\pi}{4}\right)\right] + w_1 \qquad \text{—— (2)}$$

$$x_2 = \int_0^T x(t)\,\phi_2(t)\,dt$$

$$\left[\sqrt{E}\sin\left((2i-1)\frac{\pi}{4}\right)\right] + w_2$$

→ Probability error

$x_1 > 0$ and $x_2 > 0$ given signal $x_4(t)$ is transmitted

$$P_c = \int_0^\infty \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(x_1 - \sqrt{E/2})^2}{N_0}\right] dx_1$$

$$\cdot \int_0^\infty \frac{1}{\pi} \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(x_2 - \sqrt{E/2})^2}{N_0}\right] dx_2$$

$$\frac{x_1 - \sqrt{E/2}}{\sqrt{N_0}} = \frac{x_2 - \sqrt{E/2}}{\sqrt{N_0}} = z$$

$$P_c = \int \frac{1}{\sqrt{\pi}} \int_{-\sqrt{E/2N_0}}^{\infty} \exp(-z^2)\, dz$$

$$P_c = \frac{1}{\sqrt{\pi}} \int_{-\sqrt{E/2N_0}}^{\infty} \exp(-z)^2 dz = 1 - \frac{1}{2} \operatorname{erfc}\sqrt{\frac{E}{2N_0}}$$

$$P_c = \left(1 - \frac{1}{2}\operatorname{erfc}\sqrt{\frac{E}{2N_0}}\right)^2$$

$$= 1 - \operatorname{erfc}\sqrt{\frac{E}{2N_0}} + \frac{1}{4}\operatorname{erfc}^2\sqrt{\frac{E}{2N_0}}$$

$$P_e = 1 - P_c$$
$$= \operatorname{erfc}\sqrt{\frac{E}{2N_0}} - \frac{1}{4}\operatorname{erfc}^2\sqrt{\frac{E}{2N_0}}$$

ignoring 2^{nd} term

$$P_e \simeq \operatorname{erfc}\sqrt{\frac{E}{2N_0}}$$

It is noted that $E = 2E_b$

$$P_e \simeq \operatorname{erf}\sqrt{\frac{E}{N_0}}$$

is characteristic of the type of modulation used.

In the sequel, we first study a quadrature-carrier signaling technique known as quadriphase-shift keying, which is an extension of binary PSK. Next, we consider minimum shift keying, which is a special form of continuous-phase frequency-shift keying (CPFSK). In this latter scheme, the receiver carries out the coherent detection in two successive bit intervals.

(1) Quadriphase-shift Keying

As with binary PSK, this modulation scheme is characterized by the fact that the information carried by the transmitted wave is contained in the phase. In particular, in *quadriphase-shift keying* (QPSK), the phase of the carrier takes on one of four equally spaced values, such as $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$, as shown by

$$s_i(t) = \begin{cases} \sqrt{\dfrac{2E}{T}} \cos\left[2\pi f_c t + (2i-1)\dfrac{\pi}{4}\right] & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases} \tag{7.32}$$

where $i = 1, 2, 3, 4$, and E is the transmitted signal energy per symbol, T is the symbol duration, and the carrier frequency f_c equals n_c/T for some fixed integer n_c. Each possible value of the phase corresponds to a unique pair of bits called a *dibit*. Thus, for example, we may choose the foregoing set of phase values to represent the Gray encoded set of dibits: 10,00,01, and 11.

Using a well-known trigonometric identity, we may rewrite Eq. 7.32 in the equivalent form:

$$s_i(t) = \begin{cases} \sqrt{\dfrac{2E}{T}} \cos\left[(2i-1)\dfrac{\pi}{4}\right] \cos(2\pi f_c t) & \\ \qquad - \sqrt{\dfrac{2E}{T}} \sin\left[(2i-1)\dfrac{\pi}{4}\right] \sin(2\pi f_c t) & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases} \tag{7.33}$$

where $i = 1, 2, 3, 4$. Based on this representation, we can make the following observations:

1. There are only two orthonormal basis functions, $\phi_1(t)$ and $\phi_2(t)$, contained in the expansion of $s_i(t)$. The appropriate forms for $\phi_1(t)$ and $\phi_2(t)$ are defined by

Table 7.1 Signal-Space Characterization of QPSK

Input dibit $0 \le t \le T$	Phase of QPSK signal (*radians*)	Coordinates of message points s_{i1}	s_{i2}
10	$\pi/4$	$+\sqrt{E/2}$	$-\sqrt{E/2}$
00	$3\pi/4$	$-\sqrt{E/2}$	$-\sqrt{E/2}$
01	$5\pi/4$	$-\sqrt{E/2}$	$+\sqrt{E/2}$
11	$7\pi/4$	$+\sqrt{E/2}$	$+\sqrt{E/2}$

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t) \qquad 0 \le t \le T \tag{7.34}$$

and

$$\phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t) \qquad 0 \le t \le T \tag{7.35}$$

2. There are four message points, and the associated signal vectors are defined by

$$\mathbf{s}_i = \begin{bmatrix} \sqrt{E}\cos\left((2i-1)\dfrac{\pi}{4}\right) \\ -\sqrt{E}\sin\left((2i-1)\dfrac{\pi}{4}\right) \end{bmatrix} \qquad i = 1, 2, 3, 4 \tag{7.36}$$

The elements of the signal vectors, namely, s_{i1} and s_{i2}, have their values summarized in Table 7.1. The first two columns of this table give the associated dibits and phase of the QPSK signal.

Accordingly, a QPSK signal is characterized by having a two-dimensional signal constellation (i.e., $N = 2$) and four message points (i.e., $M = 4$), as illustrated in Fig. 7.6.

EXAMPLE 1

Figure 7.7 illustrates the sequences and waveforms involved in the generation of a QPSK signal. The input binary sequence 01101000 is shown in Fig. 7.7*a*. This sequence is divided into two other sequences, consisting of odd- and even-numbered bits of the input sequence. These two sequences are shown in the top lines of Figs. 7.7*b* and 7.7*c*. The waveforms representing the in-phase and quadrature components of the QPSK signal are also shown in Figs. 7.7*b* and 7.7*c*, respectively. These two waveforms may individually be viewed as examples of a binary PSK signal. Adding them, we get the QPSK waveform shown in Fig. 7.7*d*.

To realize the decision rule for the detection of the transmitted data sequence, we partition the signal space into four regions, in accordance with Eq. 3.64, as described here:

1. The set of points closest to the message point associated with signal vector $\mathbf{s}_1$.
2. The set of points closest to the message point associated with signal vector $\mathbf{s}_2$.

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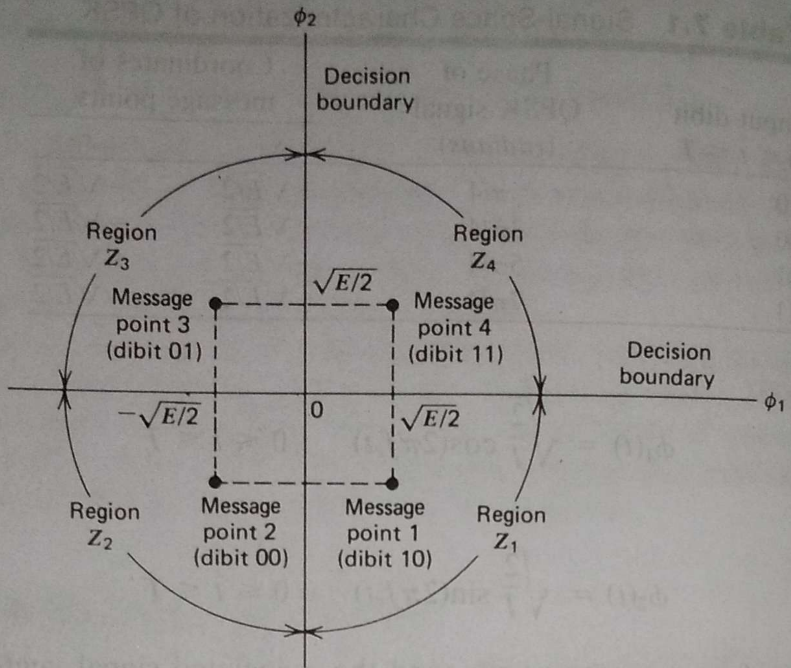

Figure 7.6 Signal space diagram for coherent QPSK system.

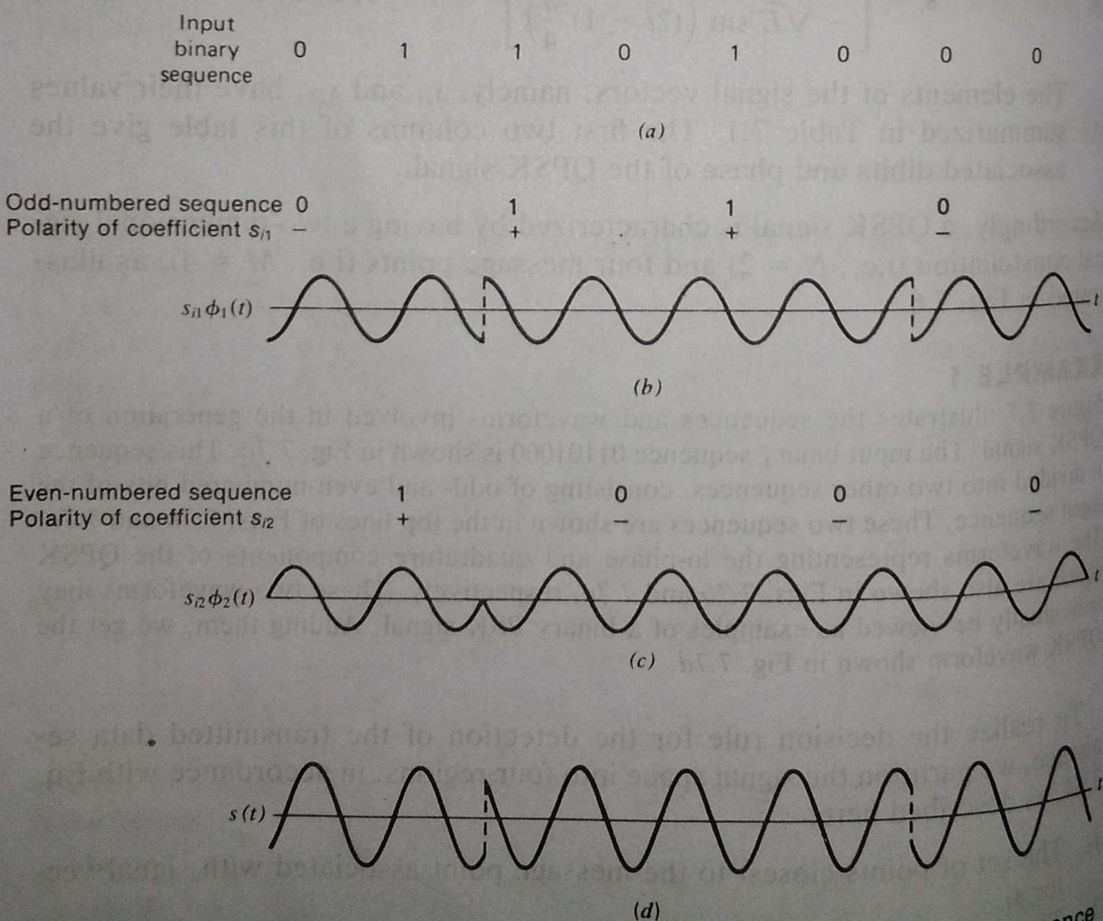

Figure 7.7 (*a*) Input binary sequence. (*b*) Odd-numbered bits of input sequence and associated binary PSK wave. (*c*) Even-numbered bits of input sequence and associated binary PSK wave. (*d*) QPSK waveform.

3. The set of points closest to the message point associated with signal vector $\mathbf{s}_3$.
4. The set of points closest to the message point associated with signal vector $\mathbf{s}_4$.

This is accomplished by constructing the perpendicular bisectors of the square formed by joining the four message points, and then marking off the appropriate regions. We thus find that the decision regions are quadrants whose vertices coincide with the origin. These regions are marked as Z_1, Z_2, Z_3, and Z_4, in Fig. 7.6, according to the message point about which they are constructed.

The received signal, $x(t)$, is defined by

$$x(t) = s_i(t) + w(t) \qquad \begin{array}{l} 0 \leq t \leq T \\ i = 1, 2, 3, 4 \end{array} \tag{7.37}$$

where $w(t)$ is the sample function of a white Gaussian noise process of zero mean and power spectral density $N_0/2$. The observation vector, $\mathbf{x}$, of a coherent QPSK receiver has two elements, x_1 and x_2, that are defined by

$$\begin{aligned} x_1 &= \int_0^T x(t)\phi_1(t)dt \\ &= \sqrt{E}\cos\left[(2i-1)\frac{\pi}{4}\right] + w_1 \end{aligned} \tag{7.38}$$

and

$$\begin{aligned} x_2 &= \int_0^T x(t)\phi_2(t)dt \\ &= -\sqrt{E}\sin\left[(2i-1)\frac{\pi}{4}\right] + w_2 \end{aligned} \tag{7.39}$$

where $i = 1, 2, 3, 4$.

Thus x_1 and x_2 are sample values of independent Gaussian random variables with mean values equal to $\sqrt{E}\cos[(2i-1)\pi/4]$ and $-\sqrt{E}\sin[(2i-1)\pi/4]$, respectively, and with a common variance equal to $N_0/2$.

The decision rule is now simply to guess $s_1(t)$ was transmitted if the received signal point associated with the observation vector $\mathbf{x}$ falls inside region Z_1, guess $s_2(t)$ was transmitted if the received signal point falls inside region Z_2, and so on. An erroneous decision will be made if, for example, signal $s_4(t)$ is transmitted but the noise $w(t)$ is such that the received signal point falls outside region Z_4.

We note that, owing to the symmetry of the decision regions, the probability of interpreting the received signal point correctly is the same regardless of which particular signal was actually transmitted. Suppose, for example, we know that signal $s_4(t)$ was transmitted. The receiver will then make a correct decision provided that the received signal point represented by the observation vector $\mathbf{x}$ lies inside region Z_4 of the signal space diagram in Fig. 7.6. Accordingly, for a correct decision when signal $s_4(t)$ is transmitted, the elements x_1 and x_2 of the observation vector $\mathbf{x}$ must be both positive, as illustrated in Fig. 7.8. This means that the *probability of a correct decision*, P_c, equals the conditional

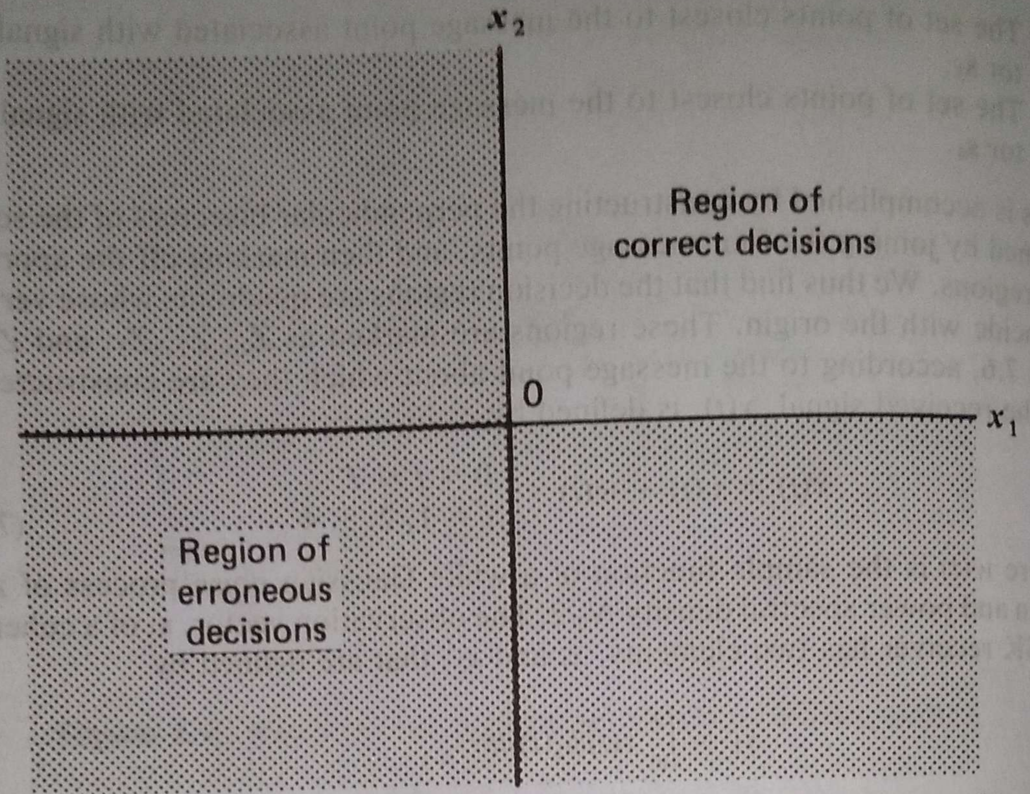

Figure 7.8 Illustrating the region of correct decisions and the region of erroneous decisions, given that signal $s_4(t)$ was transmitted.

probability of the joint event $x_1 > 0$ and $x_2 > 0$, given that signal $s_4(t)$ was transmitted. Since the random variables X_1 and X_2 (with sample values x_1 and x_2, respectively) are independent, P_c also equals the product of the conditional probabilities of the events $x_1 > 0$ and $x_2 > 0$, both given that signal $s_4(t)$ was transmitted. Furthermore, both X_1 and X_2 are Gaussian random variables with a conditional mean equal to $\sqrt{E/2}$ and a variance equal to $N_0/2$. Hence, we may write

$$P_c = \int_0^\infty \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(x_1 - \sqrt{E/2})^2}{N_0}\right] dx_1 \cdot \int_0^\infty \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(x_2 - \sqrt{E/2})^2}{N_0}\right] dx_2 \tag{7.40}$$

where the first integral on the right side is the conditional probability of the event $x_1 > 0$ and the second integral is the conditional probability of the event $x_2 > 0$, both given that signal $s_4(t)$ was transmitted. Let

$$\frac{x_1 - \sqrt{E/2}}{\sqrt{N_0}} = \frac{x_2 - \sqrt{E/2}}{\sqrt{N_0}} = z \tag{7.41}$$

Then, changing the variables of integration from x_1 and x_2 to z, we may rewrite Eq. 7.40 in the form

$$P_c = \left[\frac{1}{\sqrt{\pi}} \int_{-\sqrt{E/2N_0}}^{\infty} \exp(-z^2)\,dz\right]^2 \tag{7.42}$$

However, from the definition of the complementary error function, we find that

$$\frac{1}{\sqrt{\pi}}\int_{-\sqrt{E/2N_0}}^{\infty}\exp(-z^2)dz = 1 - \tfrac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) \tag{7.43}$$

Accordingly, we have

$$\begin{aligned} P_c &= \left[1 - \tfrac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)\right]^2 \\ &= 1 - \mathrm{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) + \tfrac{1}{4}\mathrm{erfc}^2\left(\sqrt{\frac{E}{2N_0}}\right) \end{aligned} \tag{7.44}$$

The average probability of symbol error for coherent QPSK is therefore

$$\begin{aligned} P_e &= 1 - P_c \\ &= \mathrm{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) - \tfrac{1}{4}\mathrm{erfc}^2\left(\sqrt{\frac{E}{2N_0}}\right) \end{aligned} \tag{7.45}$$

In the region where $(E/2N_0) \gg 1$, we may ignore the second term on the right side of Eq. 7.45, and so approximate the formula for the average probability of symbol error for coherent QPSK as

$$P_e \simeq \mathrm{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) \tag{7.46}$$

In a QPSK system, we note that there are two bits per symbol. This means that the transmitted signal energy per symbol is twice the signal energy per bit; that is,

$$E = 2E_b \tag{7.47}$$

Thus, expressing the average probability of symbol error in terms of the ratio E_b/N_0, we may write

$$P_e \simeq \mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) \tag{7.48}$$

Consider next the generation and demodulation of QPSK. Figure 7.9*a* shows the block diagram of a typical QPSK transmitter. The input binary sequence $b(t)$ is represented in polar form, with symbols 1 and 0 represented by $+\sqrt{E_b}$ and $-\sqrt{E_b}$ volts, respectively. This binary wave is divided by means of a *demultiplexer* into two separate binary waves consisting of the odd- and even-numbered input bits. These two binary waves are denoted by $b_1(t)$ and $b_2(t)$. We note that in any signaling interval, the amplitudes of $b_1(t)$ and $b_2(t)$ equal s_{i1} and s_{i2}, respectively, depending on the particular dibit that is being transmitted. The two binary waves $b_1(t)$ and $b_2(t)$ are used to modulate a pair of quadrature carriers or orthonormal basis functions: $\phi_1(t)$ equal to $\sqrt{2/T}\cos(2\pi f_c t)$ and $\phi_2(t)$ equal to $\sqrt{2/T}\sin(2\pi f_c t)$. The result is a pair of binary PSK waves, which may be detected independently due to the orthogonality of $\phi_1(t)$ and $\phi_2(t)$. Finally,

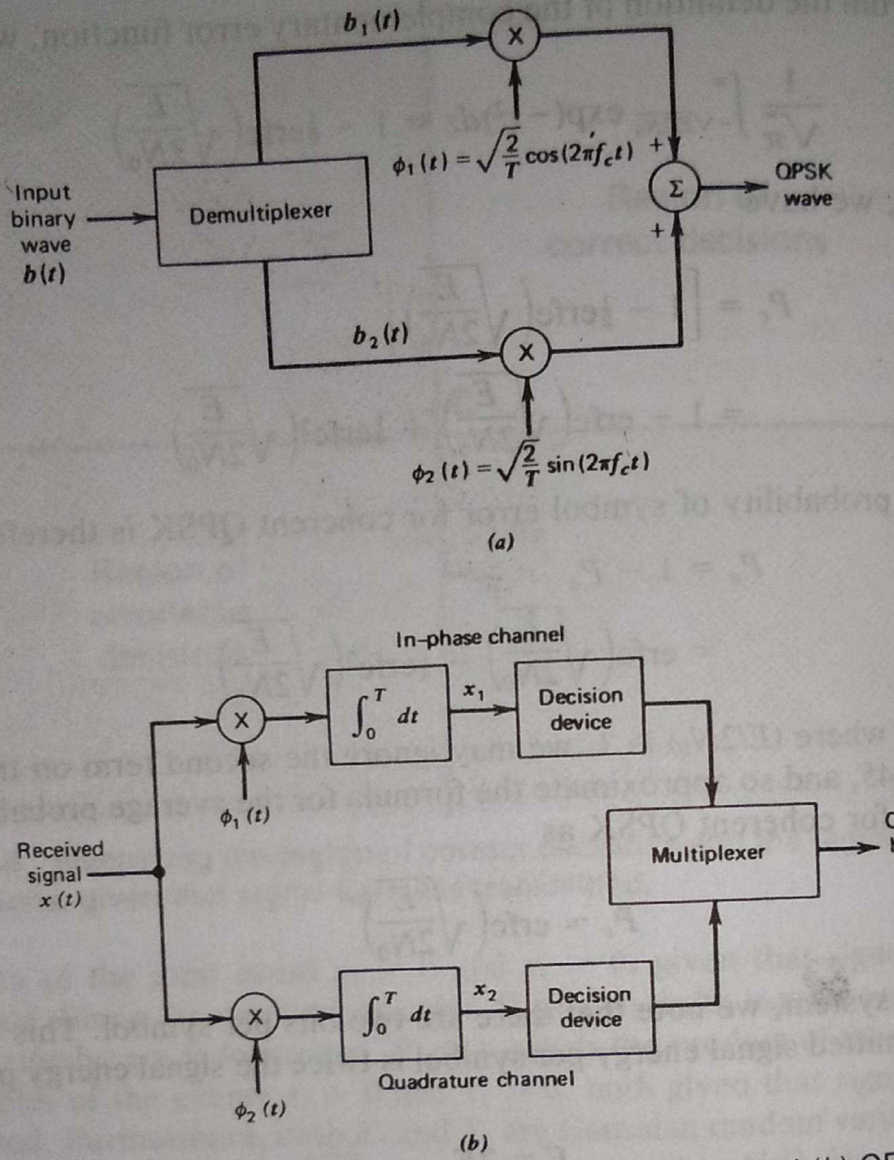

Figure 7.9 Block diagrams for (*a*) QPSK transmitter, and (*b*) QPSK receiver.

the two binary PSK waves are added to produce the desired QPSK wave. Note that the symbol duration, T, of a QPSK wave is twice as long as the bit duration, T_b, of the input binary wave. That is, for a given bit rate $1/T_b$, a QPSK wave requires half the transmission bandwidth of the corresponding binary PSK wave. Equivalently, for a given transmission bandwidth, a QPSK wave carries twice as many bits of information as the corresponding binary PSK wave.

The QPSK receiver consists of a pair of correlators with a common input and supplied with a locally generated pair of coherent reference signals $\phi_1(t)$ and $\phi_2(t)$, as in Fig. 7.9*b*. The correlator outputs, x_1 and x_2, are each compared with a threshold of zero volts. If $x_1 > 0$, a decision is made in favor of symbol 1 for the upper or in-phase channel output, but if $x_1 < 0$ a decision is made in favor of symbol 0. Similarly, if $x_2 > 0$, a decision is made in favor of symbol 1 for the lower or quadrature channel output, but if $x_2 < 0$, a decision is made in favor of symbol 0. Finally, these two binary sequences at the in-phase and quadrature channel outputs are combined in a *multiplexer* to reproduce the original binary sequence at the transmitter input with the minimum probability of symbol error.